ON THE MODULI SPACE OF QUASIHOMOGENEOUS FOLIATIONS ON $(\mathbb{C}^2, 0)$

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Abstract. We study the relationship between singular holomorphic foliations on $(\mathbb{C}^2, 0)$ and their separatrices. Under mild conditions we shall describe a complete set of analytic invariants characterizing foliations with quasihomogeneous separatrices.

Résumé. Nous étudions la relation entre les germes de feuilletages holomorphes singuliers dans $(\mathbb{C}^2, 0)$ e son séparatrices. Sur certains conditions génériques, nous décrivons une liste complet de invariantes analytiques caractérisant des germes de feuilletages avec des séparatrices quasi-homogènes.

1. Introduction

The problem of the local classification of differential equation of the form $Adx + Bdy = 0$ in two variables has been studied by various mathematicians — since the end of the nineteen century — as C. A. Briot, J. C. Bouquet, H. Dulac, H. Poincaré, I. Bendixson, G. D. Birkhoff, C. L. Siegel, A. D. Brjuno et Al. In the 1970s R. Thom restored the interest in this question proposing the following problem. Recall from [3] that every germ of singular holomorphic foliation on $(\mathbb{C}^2, 0)$ has an invariant curve through the origin called the separatrix set and denoted by $Sep(\mathcal{F})$.

Conjecture 1. *If $Sep(\mathcal{F})$ has a finite number of components, then the analytic type and the monodromy of $Sep(\mathcal{F})$ may determine $\mathcal{F}$ up to conjugacy class.*

In [13],[14], and [15] it is proved that the conjecture has an affirmative answer in case the linear part of the vector field defining the foliation is non-nilpotent. But in [17] it is proved that the conjecture is not true in general, by introducing an analytic invariant called vanishing holonomy. Since this time such question is known as Thom's problem. In [6] the results of [17] are generalized, classifying a Zariski open subset of the nilpotent singularities in terms of the vanishing holonomy (now called projective holonomy). Other contributions are given by many authors such as [7], [16], [2], [21], etc. From a quite different point of view in [12] J.-F. Mattei studied this problem, and as a consequence of his rigidity theorem (cf. Theorem 1) he describes, under generic conditions, the local moduli space of such singularities, i.e. invariants that characterize deformations of a given singular foliation. Roughly speaking, they are: the resolution tree up to first order, the projective holonomy, and the analytic type of $Sep(\mathcal{F})$. The link between these two approaches is that the separatrices of generic nilpotent singularities are particular instances of quasi-homogeneous curves (cf. definition below). The difference in this case is that we

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a have a global classification based on the same invariants. Here we tight these two approaches together, showing (under mild conditions over $\mathcal{F}$) that the global invariants of a germ of foliation with a quasihomogeneous separatrix are quite the same.

2. Statements: recent and new

2.1. **Basic definitions and notations.** A germ of singular foliation $(\mathcal{F} : \omega = 0)$ on $(\mathbb{C}^2, 0)$ of codimension 1 is, roughly speaking, the set of integral curves of a given germ of 1-form $\omega \in \Omega^1(\mathbb{C}^2, 0)$, which may be assumed to have just an isolated singularity at the origin. Let $Diff(\mathbb{C}^k, 0)$ be the group of germs of analytic diffeomorphisms of $(\mathbb{C}^k, 0)$ fixing the origin. We say that two germs of foliations $(\mathcal{F}_j : \omega_j = 0)$ on $(\mathbb{C}^2, 0)$, $j = 1, 2$, are analytically conjugate if there is $\Phi \in Diff(\mathbb{C}^2, 0)$ such that Φ sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. We say that $h_1, h_2 \in Diff(\mathbb{C}, 0)$ are analytically conjugate if there is $\phi \in Diff(\mathbb{C}, 0)$ such that $Ad(\phi)(h_1) := \phi \circ h_1 \circ \phi^{-1} = h_2$. Let us denote the *Hopf bundle of Chern class* $-k$ by $p_{(k)} : \mathcal{H}(-k) \to D$, with $D \simeq \mathbb{CP}(1)$, or just by its total space $\mathbb{H}(-k)$,recall from the theory of algebraic curves that if $\pi : (\widetilde{X}, D) \longrightarrow (\mathbb{C}^2, 0)$ is a map resulting from the iteration of finite number of blowing-ups with exceptional curve $D = \pi^{-1}(0)$ and whose irreducible components are D_j, $j = 1, \ldots, n$, with self-intersection number equal to $-k_j$, then a suitable neighborhood of D in $\widetilde{X}$ results from pasting together suitable neighborhoods of the zero sections of $\mathbb{H}(-k_j)$. Denote by $\widetilde{\mathcal{F}}$ the unique extension of $\pi^*(\mathcal{F})$ whose singular set has codimension greater or equal to 2 (cf. [13]). For each Hopf bundle component $p_j : \mathcal{H}_j \to D_j$ of a given resolution, we shall denote by $\widetilde{\mathcal{F}}_j$ the germ of foliation on $(\mathcal{H}_j, D_j)$ induced by the restriction of $\widetilde{\mathcal{F}}$, and call it the j^{th} **Hopf component** of the resolution. We shall denote by $\widetilde{\mathcal{F}}_{ij}$ the restriction of $\widetilde{\mathcal{F}}$ to a neighborhood of the corner $t_{ij} := D_i \cap D_j$. The "strict transform" of $Sep(\mathcal{F})$ at $D_j \subset \mathcal{H}_j$, i.e. the set of local separatrices of $\widetilde{\mathcal{F}}_j$, namely $Sep(\widetilde{\mathcal{F}}_j) = \overline{(\pi^* Sep(\mathcal{F})) \mid_{\mathcal{H}_j} \backslash D_j}$, will be called the j^{th} **Hopf component** of $\pi^*(Sep(\mathcal{F}))$.

Let $\mathbb{H} : \big(p : \mathcal{H} \to D \simeq \mathbb{CP}^1\big)$ be a Hopf bundle, and $\mathcal{F}$ a germ of foliation defined on $(\mathcal{H}, D)$. Then we say that $\mathcal{F}$ is **non-dicritical** if D is an invariant set of $\mathcal{F}$, and **dicritical** otherwise. In the former case, the holonomy of $\mathcal{F}$ with respect to D evaluated at the transversal section Σ is called the **projective holonomy** of $\mathcal{F}$ and denoted by $Hol_\Sigma(\mathcal{F}, D)$. Furthermore, we say that $\mathcal{F}$ is **resolved** if it has just **reduced** singularities (cf. [13]). Let $\widetilde{\mathcal{F}}^1, \widetilde{\mathcal{F}}^2$ be two germs of singular non-dicritical foliation at $D \subset \mathcal{H}$ without saddle-nodes, and $\varphi \in PGL(2, \mathbb{C})$ be an isomorphism between their sets of singular points $\{t_j^i\}_{j=1}^n$, i.e. $\varphi(t_j^1) = t_j^2$. Further, let $t_0^1 \in D$ be a regular point of $\widetilde{\mathcal{F}}^1$, $t_0^2 = \varphi(t_0^1)$, and denote by h_γ^i the holonomy of a path $\gamma \in \pi_1(D \backslash \{t_j^i\}_{j=1}^n, t_0^i)$ with respect to sections Σ_i, $i = 1, 2$, transversal to D. Then we say that the projective holonomies of these foliations have an analytic **conjugacy subordinated to** φ if there is $\phi \in Diff(\mathbb{C}, 0)$ such that $\phi_*(h_\gamma^1) = h_{\varphi_*\gamma}^2$, for every $\gamma \in \pi_1(D \backslash \{t_j^1\}_{j=1}^n, t_0^1)$.

Recall that a **generalized curve** is a germ of singular foliation on $(\mathbb{C}^2, 0)$ which has no saddle-node or dicritical components along its minimal resolution ([4]). A germ of holomorphic function $f \in \mathbb{C}\{x, y\}$ is said to be **quasihomogeneous** if there is a local system of coordinates such that f can be represented by a quasihomogeneous polynomial, i.e. $f(x,y) = \sum_{ai+bj=d} a_{ij}x^i y^j$ where $a, b, d \in \mathbb{N}$. The separatrix set of a germ of foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ is said to be quasihomogeneous if $Sep(\mathcal{F}) = f^{-1}(0)$ where f is a quasihomogeneous function. The set of germs of foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ with quasihomogeneous separatrix is denoted by $\mathcal{QHS}$. We shall say that $\mathcal{F}$ is a **generic** $\mathcal{QHS}$ foliation if the corners of $\widetilde{\mathcal{F}}$ are in the Poincaré-Dulac or Siegel domain (cf. [1]).

A **tree of projective lines** is an embedding of a connected and simply connected chain of projective lines intersecting transversely in a complex surface (two dimensional complex analytic manifold), with two projective lines in each intersection, which consists of the pasting of Hopf bundles whose zero sections are the projective lines themselves. A **tree of points** is any tree of projective lines in which are discriminated a finite number of points. Finally a **tree of singularities** is a tree of points such that in each point is specified a germ of singular foliation.

Note that the above nomenclature has a natural motivation. In fact, as is well know, we can assign to each projective line a point and to each intersection an edge in other to form a tree (as in graph theory).

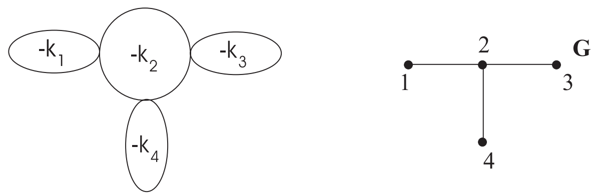


Figure 2

Hence we shall use the following notations for a tree of projective lines, a tree of points, and a tree of singularities respectively: $\Gamma_{PL} = \{G, \{k_i\}_{i=1}^k\}$, $\Gamma_p = \{G, \{k_i\}_{i=1}^k, \{t_{ij}\}_{j=1}^{l_i}\}$, $\Gamma_s = \{G, \{k_i\}_{i=1}^k, \{t_{ij}\}_{j=1}^{l_i}, \{\omega_{ij}\}_{j=1}^{l_i}\}$, where G is a tree, $k_i \in \mathbb{N}$, $t_{ij} \in D_i$, D_i is the zero section of the Hopf bundle $\mathbb{H}_i$ with Chern class $-k_i$, and $\omega_{ij} = 0$ is a germ of singular holomorphic foliation at t_{ij}.

Two trees of projective lines $\Gamma_{PL} = \left\{G, \{k_i\}_{i=1}^k\right\}, \Gamma'_{PL} = \{G', \{k'_i\}_{i=1}^{k'}\}$, are isomorphic if G, G' are isomorphic (as graphs) and $k_i = {k_i}'$. Moreover, two trees of points $\Gamma_p = \{G, \{k_i\}_{i=1}^k, \{t_{ij}\}_{j=1}^{l_i}\}, \Gamma'_p = \{G', \{k'_i\}_{i=1}^{k'}, \{t'_{ij}\}_{j=1}^{l'_i}\}$ are analytically componentwise isomorphic if their trees of projective lines are isomorphic, $l_i = l'_i$ and there are $\varphi_i \in Diff(D_i)$ such that $\varphi_i(t_{ij}) = t'_{ij}$. Finally, two trees of singularities $\Gamma_s = \{G, \{k_i\}_{i=1}^k, \{t_{ij}\}_{j=1}^{l_i}, \{\omega_{ij}\}_{j=1}^{l_i}\}, \Gamma'_s = \{G', \{k'_i\}_{i=1}^{k'}, \{t'_{ij}\}_{j=1}^{l'_i}, \{\omega'_{ij}\}_{j=1}^{l'_i}\}$ are analytically componentwise isomorphic up to first order if their trees of points are analytically componentwise isomorphic and $J^1(\omega_{ij}) = J^1(\omega'_{ij})$, i.e. $\omega_{ij}, \omega'_{ij}$ have the same linear part. We shall denote the minimal resolution tree of $\mathcal{F}$ by $\Gamma(\widetilde{\mathcal{F}})$. Recall from [20] that any germ of holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ has a minimal resolution denoted by $\widetilde{\mathcal{F}}$. Therefore, if $Sep(\widetilde{\mathcal{F}}) = \cup_j Sep(\widetilde{\mathcal{F}}_j)$ intersects just one projective line of $\Gamma(\widetilde{\mathcal{F}})$, then the later is called the **principal projective line** of $\widetilde{\mathcal{F}}$.

2.2. **Statements.** Recall that a germ of holomorphic function $f \in \mathbb{C}\{x, y\}$ is quasihomogeneous if there is a local system of coordinates such that f can be represented by the a quasihomogeneous polynomial, i.e. $f(x, y) = \sum_{ai+bj=d} a_{ij}x^i y^j$ where $a, b, d \in \mathbb{N}$. Equivalently, we may say that its Newton polygon is contained in a line. From [19] we know that such a germ of function can be characterized by the following algebraic property: $f \in (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, i.e. f belongs to its Jacobian ideal. Another characterization of a quasihomogeneous polynomial may be given in terms of deformations. One can prove that a germ of function $f \in \mathcal{O}_{(\mathbb{C}^2,0)}$ is quasihomogeneous if and only if any topologically trivial deformation $F \in \mathcal{O}_{(\mathbb{C}^{p+2},0)}$ of f satisfies the following property: *F is analytically trivial if and only if the family of curves $(F_t^{-1}(0))_{t\in(\mathbb{C}^p,0)}$ are analytically trivial.* In [12] Mattei extended this concept for foliations by the following definition. A germ of foliation $(\mathcal{F} : \omega = 0)$ on $(\mathbb{C}^2, 0)$ is **topologically quasihomogeneous** if each topologically trivial deformation of $\mathcal{F}$, say $(\mathcal{G} : \eta = 0)$ where $\eta(x, y, t) = A(x, y, t)dx + B(x, y, t)dy$, satisfies the following property: *$\mathcal{G}$ is analytically trivial if and only if $Sep(\eta_t)$ is analytically trivial.* Now recall that a germ of singular foliation $(\mathcal{F} : \omega = 0)$ is said to be **quasihyperbolic generic** if it is non-dicritical, all the singularities of its minimal resolution $\widetilde{\mathcal{F}}$ have linear part given by $ydx + \lambda xdy$ with $\lambda \in \mathbb{C} - \mathbb{R}_{\leq 0}$, and at least one of its Hopf components have non-solvable projective holonomy. Under this hypothesis we have the following characterization of quasihomogeneous foliations.

Theorem 1 (Mattei)**.** *Let $(\mathcal{F} : \omega = 0)$ be quasihyperbolic generic with $\omega(x, y) = a(x, y)dx + b(x, y)dy$, and let $f \in \mathcal{O}_{(\mathbb{C}^2,0)}$ be a reduced equation for $Sep(\mathcal{F})$. Then the following statements are equivalent:*

(1) *$\mathcal{F}$ is topologically quasihomogeneous;*
(2) *$f \in (a, b) \subset \mathcal{O}_{(\mathbb{C}^2,0)}$;*
(3) *df is topologically quasihomogeneous;*
(4) *$f \in (f_x, f_y) \subset \mathcal{O}_{(\mathbb{C}^2,0)}$;*
(5) *There is a coordinate system (u, v) for $\mathbb{C}^2$, $g, h \in \mathcal{O}_{(\mathbb{C}^2,0)}$ with $u(0) = v(0) = 0$, $g(0) \neq 0$, and $\alpha, \beta, d \in \mathbb{N}$, such that $f(u, v) = \sum_{\alpha i+\beta j=d} a_{ij}u^iv^j \in \mathbb{C}[[u, v]]$ and $g\omega = df + h(\beta vdu - \alpha udv)$.*

In view of Thom's conjecture, its natural to ask under what conditions can we assure that the projective holonomies, the minimal resolution tree up to first order, and $Sep(\mathcal{F})$ determine the analytic type of $\mathcal{F}$. Theorem 1 tells us (at least under generic conditions) that we have to ask $Sep(\mathcal{F})$ to be a quasihomogenous curve. Here we shall answer this question under generic conditions over $\mathcal{F}$.

Theorem 2. *Let $\mathcal{F}, \mathcal{G}$ be two generic $\mathcal{QHS}$ foliations, and let $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}$ their minimal resolution. Then $\mathcal{F}, \mathcal{G}$ are analytically equivalent if and only if:*

(1) *$\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}$ have isomorphic trees of resolution up to first order, say $\varphi : \Gamma(\widetilde{\mathcal{F}}) \longrightarrow \Gamma(\widetilde{\mathcal{G}})$;*
(2) *There is a conjugacy between $Hol_{\Sigma}(\mathcal{F}, D), Hol_{\Sigma'}(\mathcal{F}', D')$ subordinated to φ;*
(3) *$Sep(\widetilde{\mathcal{F}}), Sep(\widetilde{\mathcal{G}})$ are analytically equivalent;*

3. Quasihomogeneous polynomials

3.1. **Normal forms.** Recall that $f \in \mathbb{C}[x, y]$ is a **commode** quasihomogeneous polynomial if its Newton polygon intersects both coordinate axis. Now notice that a polynomial in two variables $P \in \mathbb{C}[x, y]$ can be considered as a polynomial in the variable y with coefficients in $\mathbb{C}[x]$, i.e. $P \in (\mathbb{C}[x])[y]$. Then ord_yP is the order of P as a polynomial in $(\mathbb{C}[x])[y]$. Similarly we define ord_xP as the order of P as an element of $(\mathbb{C}[y])[x]$. Therefore, a quasihomogeneous polynomial $P \in \mathbb{C}[x, y]$ is commode if and only if $ord_x P = ord_y P = 0$. Now we deescribe the general behavior of a quasihomogeneous polynomial.

Lemma 1. *Let $P \in \mathbb{C}[x, y]$ is a quasihomogeneous polynomial, then it has a unique decomposition in the form*

$$P(x, y) = x^m y^n P_0(x, y)$$

where $m, n \in \mathbb{N}$, $\lambda \in \mathbb{C}, P_0$ is a commode quasihomogeneous polynomial.

Proof. Let $m := ord_x P, n := ord_y P$, then clearly x^m, y^n divide P, thus it can be written in the form $P(x, y) = \sum_{ai+bj=d} a_{ij} x^i y^j$ where $i \geq m, j \geq n$. Therefore $P(x, y) = x^m y^n P_0(x, y)$ where $P_0(x, y) = \sum_{ai'+bj'=d'} a_{i'+m,j'+n} x^{i'} y^{j'}, d' := d - am - bn$. Finally, since $m := ord_x P$ and $n := ord_y P$, then $ord_x P_0 = 0 = ord_y P_0$. The result then follows directly from the above remark. $\square$

Lemma 2. *Let $P(x, y) = \prod_{l=1}^{k} (y^p - \lambda_j x^q)$ with $\lambda_j \in \mathbb{C}^*$, $j = 1, \ldots, k$. Then P is a commode quasihomogeneous polynomial given by*

$$P(x, y) = \sum_{pi+qj=pqk}^{k} \sigma_{\frac{i}{q}}(\lambda_1, \cdots, \lambda_k) x^i y^j$$

where $\sigma_0 := 1$ and σ_l is the l^{th} elementary symmetric polynomial in k variables, for $l = 1, \ldots, k$.

Proof. A straightforward calculation shows that

$$P(x, y) = \prod_{l=1}^{k} (y^p - \lambda_j x^q) = \sum_{l=0}^{k} \sigma_l(\lambda_1, \cdots, \lambda_k) x^{ql} y^{pk-pl} \tag{3.1}$$

Now, if we let $i := ql$ and $j := pk - pl$, then $pi + qi = pqk$; and the result follows. $\square$

Definition 1. *We shall say that a commode polynomial $P \in \mathbb{C}[x, y]$ is* ***monic*** *in y if its a monic polynomial in $(\mathbb{C}[x])[y]$.*

Theorem 3. *Let $P \in \mathbb{C}[x, y]$ be a commode quasihomogeneous polynomial which is monic in y. Then P can be written uniquely as*

$$P(x, y) = \prod_{l=1}^{k} (y^p - \lambda_j x^q)$$

where $g.c.d.(p, q) = 1$.

Proof. First remark that any quasihomogeneous polynomial can be written in the form $P(x, y) = \sum_{ai+bj=d} a_{ij} x^i y^j$ where $a, b, d \in \mathbb{N}$ and $g.c.d.(a, b) = 1$. Since P is commode, then there are $i_0, j_0 \in \mathbb{N}$ such that $bj_0 = d$ and $ai_0 = d$; in particular $k := d/ab \in \mathbb{N}$. Hence, if we let $p := a$ and $q := b$, then $ai + bj = d$ can be rewritten as $pi + qj = pqk$. Since $g.c.d.(p, q) = 1$, then q divides i and p divides j. Now if we let $i = qi'$ and $j = pj'$, then $pqi' + qpj' = pqk$ and thus $i' + j' = k$. Now let $l := i'$, then $P(x, y) = \sum_{l=0}^{k} a_{ql,p(k-l)} x^{ql} y^{p(k-l)}$. If $\{\lambda_j\}_{j=1}^{k}$ are the roots of $p(t) = \sum_{l=0}^{k} a_{ql,p(k-l)} t^{k-l}$, then we shall have from elementary algebra that $P(x, y) = \sum_{l=0}^{k} \sigma_l(\lambda_1, \cdots, \lambda_k) x^{ql} y^{pk-pl}$. Thus the result follows by (3.1). $\square$

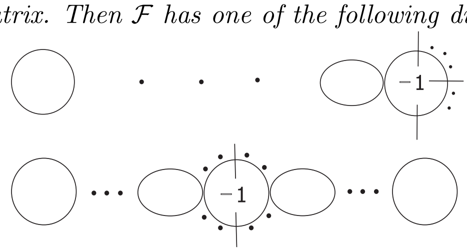
Corollary 1. *Let $P \in \mathbb{C}[x,y]$ be a quasihomogeneous polynomial. Then P can be written uniquely in the form*

$$P(x,y) = \mu x^m y^n \prod_{l=1}^{k} (y^p - \lambda_j x^q)$$

Proof. In view of Lemma 1 and Theorem 3, we only have to remark that any commode quasihomogeneous polynomial $P \in \mathbb{C}[x,y]$ can be written uniquely as $P = \mu P_0$, where P_0 is monic in y. □

3.2. **Resolution.** We investigate the geometry of the exceptional divisor of the minimal resolution of a germ of foliation $\mathcal{F}$ with quasihomogeneous separatrix.

Lemma 3. *Let $\mathcal{F}$ be a generalized curve with a commode quasihomogeneous separatrix. Then $\mathcal{F}$ has one of the following diagrams of resolution:*

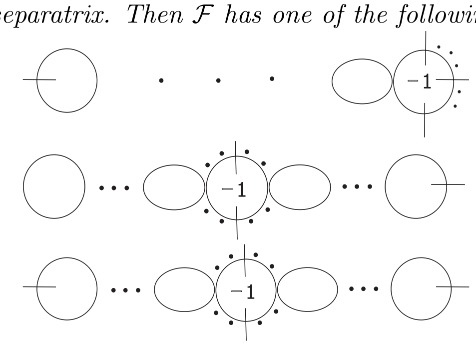


Proof. Recall that a generalized curve is resolved together with its separatrices ([4]). Further recall, from Theorem 3, that there is a local system of coordinates (x,y) such that $f(x,y) = \prod_{l=1}^{k}(y^p - \lambda_j x^q)$, with $p < q$. Since each irreducible curve $y^p - \lambda_j x^q = 0$ is a generic fiber of the fibration $\frac{y^p}{x^q} \equiv const$, then it is resolved together with the fibration. After one blow-up we obtain:

$$t^p/x^{q-p} \equiv const,$$
$$u^q y^{q-p} \equiv const.$$

Since $p < q$, we have a singularity with holomorphic first integral at infinity and a meromorphic first integral at the origin (as before). Going through with this process, Euclid's algorithm assures that the resolution ends when we blow-up a radial foliation. In particular, if $p = 1$, then it is easy to see that the principal projective line is transversal to just one projective line of the divisor. Otherwise (i.e. in case $p \neq 1$) the singularity with meromorphic first integral "moves" to the "infinity", i.e. it will appear in a corner singularity. Then the principal projective line intersects exactly two projective lines of the divisor. □

Lemma 4. *Let $\mathcal{F}$ be a generalized curve with a non-commode quasihomogeneous separatrix. Then $\mathcal{F}$ has one of the following diagrams of resolution:*



Proof. Recall from Corollary 1 that there is a local system of coordinates (x, y) such that $f(x, y) = \mu x^m y^n \prod_{l=1}^{k}(y^p - \lambda_j x^q)$, with $p < q$. As $\mu x^m y^n$ is resolved after one blow-up, then $f(x, y)$ is resolved together with the fibration $\frac{y^p}{x^q} \equiv const$, as before. Since a generalized curve is resolved together with its separatrices ([4]), then the result follows from Lemma 3. □

4. Hopf bundles and projective holonomy

We describe the invariants that determine the analytic type of resolved and "rectified" singular foliations without saddle-nodes, defined on a neighborhood of the zero section of a Hopf bundle.

Given two analytically conjugate singularities, then they have isomorphic trees of singular points along their minimal resolution. Thus, if we consider isomorphic Hopf components, it is clear that isomorphic points have their local holonomy generators conjugate by a global conjugacy. To clarify the ideas we need the following:

Definition 2. *Let $\mathbb{H} : (p : \mathcal{H} \to D)$ be the Hopf bundle, $\mathcal{F}$ a germ of resolved and non-dicritical singular foliation on $(\mathcal{H}, D)$ without saddle nodes. Then we say that a germ of holomorphic map $f : (\mathcal{H}, D) \to (D, Id|_D)$ is a* ***transversal fibration*** *to $\mathcal{F}$, if it satisfies:*

1. *f is a retraction, i.e. f is a submersion and $f|_D = Id|_D$;*
2. *the fiber $f^{-1}(t_j)$ is a separatrix of $\mathcal{F}$ for each $t_j \in Sing(\mathcal{F})$;*
3. *$f^{-1}(t)$ is transversal to $\mathcal{F}$ for every point $t \in \mathcal{F} - Sing(\mathcal{F})$.*

Consider a Hopf bundle $\mathbb{H} : (p : \mathcal{H} \to D)$, a germ of resolved singular foliation $\mathcal{F}$ on $(\mathcal{H}, D)$ without saddle nodes, a transversal fibration to $\mathcal{F}$, namely $f : (\mathcal{H}, D) \to D$, and $t_0 \in D - Sing(\mathcal{F})$ a regular point of $\mathcal{F}$. Hence, by the path lifting construction, the projective holonomy $Hol(\mathcal{F}, f^{-1}(t))$ is completely determined by $Hol(\mathcal{F}, f^{-1}(t_0))$ with $t, t_0 \in D - Sing(\mathcal{F})$. Such a holonomy will be called the **projective holonomy** of $\mathcal{F}$ with respect to f. If there is no doubt about the fibration, we only talk about the projective holonomy of the foliation and denote it by $Hol(\mathcal{F}, D)$.

Definition 3. *Let $\mathbb{H} : (p : \mathcal{H} \to D)$ be a Hopf bundle, $\mathcal{F}$,$\mathcal{F}_o$ germs of singular resolved non-dicritical foliations on $(\mathcal{H}, D)$ without saddle-nodes, and φ an isomorphism between their trees of singularities. Then we set*

$$Diff_{\mathcal{F},\mathcal{F}_o}(\mathcal{H}, D) := \{\Phi \in Diff(\mathcal{H}, D) : \Phi_*(\mathcal{F}) = \mathcal{F}_o, \Phi|_{Sing(\mathcal{F})} = \varphi|_{Sing(\mathcal{F})}\}$$

and call

$$Iso(\mathcal{F}_o) := \{\Phi \in Diff_{\mathcal{F}_o,\mathcal{F}_o}(\mathcal{H}, D) : \Phi|_{Sing(\mathcal{F})} = Id\}$$

the ***isotropy group*** *of the foliation $\mathcal{F}_o$.*

Definition 4. *Let $\mathbb{H} : (p : \mathcal{H} \to D)$ be a Hopf bundle, $\mathcal{F}_1, \mathcal{F}_2$ two germs of resolved and non-dicritical singular foliations without saddle nodes on $(\mathcal{H}, D)$, whose trees of singular points are given by $\{t^i_j\}_{j=1}^n$, $i = 1, 2$, and $\varphi \in PSL(2, \mathbb{C})$ an isomorphism between those trees, i.e. $\varphi(t^1_j) = t^2_j$. Let f_1, f_2 be two analytic fibrations such that f_i is transversal to $\mathcal{F}_i$. Further, let $t^1_0 \in D$ be a regular point of $\mathcal{F}_1$, $t^2_0 = \varphi(t^1_0)$ and denote by h^i_γ the holonomy of a path $\gamma_i \in \pi_1(D \backslash \{t^i_j\}_{j=1}^n, t^i_0)$ with respect to f_i ($i = 1, 2$). Then we say that the projective holonomies of these foliations have a conjugacy* ***subordinated*** *to φ if there is $\phi \in Diff(\mathbb{C}, 0)$ such that $\phi_*(h^1_\gamma) = h^2_{\varphi_*\gamma}$ for every $\gamma \in \pi_1(D \backslash \{t^1_j\}_{j=1}^n, t^1_0)$.*

Proposition 1. *Let $\mathbb{H} : (p : \mathcal{H} \to D)$ be a Hopf bundle, $\mathcal{F}^1, \mathcal{F}^2$ be two germs of foliations defined on $(\mathcal{H}, D)$ as in the above definition. Then $\mathcal{F}^1, \mathcal{F}^2$ are analytically conjugate if and only if their projective holonomies have a conjugacy subordinated to $\varphi \in Diff(D)$.*

Proof. As we already remarked, the necessary part is straightforward. Let us treat the sufficient part; consider the trees of singular points $\{t^i_j\}_{j=1}^n$ and the regular points $t^i_0 \in D$, as in the above definition, and let us suppose that there is $\phi \in Diff(\mathbb{C}, 0)$ such that $\phi \circ (h^1_j) \circ \phi^{-1} = h^2_j$ for all $j = 1, \ldots, n$. Then we define the map $\Phi : \mathcal{F} \setminus \bigcup_{j=1}^n f_1^{-1}(t^1_j) \longrightarrow \mathcal{F}' \setminus \bigcup_{j=1}^n f_2^{-1}(t^2_j)$ by

$$\Phi(t, x) := \Phi_t(x) := h^2_t \circ \phi \circ (h^1_t)^{-1}(x),$$

where $x \in f_1^{-1}(t)$ and $h^i_t : f_i^{-1}(t_0) \longrightarrow f_i^{-1}(t)$ are the holonomy maps obtained by path lifting a curve connecting t_0 to t in the leaves of $\mathcal{F}^i$. Note that this map does not depend on the chosen base curves, since ϕ conjugates the elements of the respective projective holonomies of $\mathcal{F}^1, \mathcal{F}^2$. Moreover, by (complex) ODE theory and by Hartogs's theorem, Φ is holomorphic since it is holomorphic in each variable separately. Further, by [15], [13], we can extend the diffeomorphisms to the local separatrices on a neighborhood of D, as desired. □

5. Analytic invariants

We consider $\mathcal{QHS}$ foliations and use the trees of singularities of their minimal resolutions up to first order, and the projective holonomies of the Hopf components of these resolutions, in order to identify some analytical cocycles that appear as obstructions to extend componentwise conjugacies. Finally, we relate these obstructions with their analytic classification.

5.1. Componentwise isomorphisms and realization.

5.1. Componentwise isomorphisms and realization. We find conditions to determine whether two componentwise isomorphic up to first order $\mathcal{QHS}$ foliations are componentwise isomorphic (for the minimal resolution of $(\mathcal{F} : \omega = 0)$). Finally we verify the uniqueness (up to biholomorphisms) of the ambient surface for the minimal resolution of any element of $\mathcal{QHS}^{c,1}_\omega$ (see definition below).

Definition 5. *we shall denoted by $\mathcal{QHS}^{c,1}_\omega$ (respect. $\mathcal{QHS}^{c,1}_\omega$) the set of $\mathcal{QHS}$ foliations whose minimal resolutions are componentwise analytically equivalent (respect. up to first order) to the minimal resolution of $(\mathcal{F} : \omega = 0)$. Further, we denote by $\mathcal{QHS}^{c,1}_{\omega,f}$ the subset of $\mathcal{QHS}^{c,1}_\omega$ whose separatrix set has the same analytic type of the curve $f^{-1}(0)$.*

We determine now the moduli space $\mathcal{QHS}^{c,1}_{\omega,f} \Big/ \mathcal{QHS}^{c}_{\omega,f}$. The following result is a straightforward consequence of Proposition 1.

Proposition 2. *Let $\mathcal{F}, \mathcal{F}'$ belong to the same class in $\mathcal{QHS}^{c,1}_{\omega,f}$. Then they belong to the same class in $\mathcal{QHS}^{c}_{\omega,f}$ if and only if the projective holonomies of the Hopf components of their minimal resolutions are analytically conjugate.*

Given two germs of foliations in $\mathcal{QHS}^{c}_\omega$, we want to verify under what conditions they are in fact globally holomorphically conjugate. For this purpose, we need first to verify that the minimal resolutions of elements of $\mathcal{QHS}^{c}_\omega$ which have equivalent trees are in fact defined on the same ambient surface (up to biholomorphism).

Definition 6. *We say that a complex surface is resolution-like if it is obtained by a holomorphic pasting of Hopf bundles with negative Chern classes, in such a way that the union of their zero sections become a tree of projective lines isomorphic to the exceptional divisor of the composition of a finite numbers of blowing-up process applied to* $(\mathbb{C}^2, 0)$.

Clearly this definition is given in such a way that every resolution surface of some singularity is automatically resolution-like. In fact, any resolution-like surface is biholomorphic to the resolution surface of some singularity.

Proposition 3. *Let* M_1, M_2 *be two resolution-like surfaces with isomorphic trees of projective lines* D_1, D_2 *respectively. Then* (M_1, D_1) *is biholomorphic to* (M_2, D_2).

In order to prove the proposition, we need the following results about complex line bundles.

Theorem 4 (Grauert [9])**.** *Let* S *be a complex surface, and* $C \subset S$ *be a rational curve with negative self-intersection number. Then there are neighborhoods* U, V *of* C *respectively in* S *and* $N(C; S)$ *(the normal bundle of* C *in* S*) and a biholomorphism* $\Psi : U \to V$ *sending* C *in the zero section of* $N(C; S)$.

Theorem 5 (Grothendieck [10])**.** *Two complex line bundles over the Riemann sphere have the same Chern class if and only if they are biholomorphic.*

Proof of Proposition 3. The proof is done by induction on the number of projective lines in the chains. If the chains are composed by just one projective line, then the result follows by the theorems of Grauert and Grothendieck. Now suppose that the result is true for all chains composed by $n \geq 1$ projective lines and let D_j have $n + 1$ projective lines. Then, by hypothesis, D_j has two intersecting projective lines, namely C_j^1 and C_j^2, with self-intersection numbers given respectively by -1 and -2. Hence, applying Grauert's and Grothendieck's theorems, one obtains that a neighborhood of each curve is biholomorphic to a neighborhood of the zero section of the Hopf bundle with the Chern classes given by their self-intersection numbers. Therefore we can blow-down a neighborhood of the curve C_j^1 obtaining yet an analytic surface defined on a neighborhood of a Riemann sphere, say $\pi(C_j^2)$ — where π stands for the blow-down (see Figure 6). In fact, $\pi(C_j^2)$ has Chern class -1. For, recall that a neighborhood of C_2 in the surface is biholomorphic to the Hopf bundle of class -2. Hence one can construct a global meromorphic section for this line bundle without zeros an with just one pole of order two at the corner (for instance, in affine coordinates one may have $x = 1$ and $u = \frac{1}{u^2}$ where $u = 0$ is the corner). Now if we consider the affine charts $\mathcal{A} = \{(t, x), (u, y) : u = 1/t, y = tx\}$ of a neighborhood of C_1 in the surface (with the corner at $t = 0$), then the meromorphic section for C_2 is locally given by:

$$t = \frac{a_{-2}}{x^2} + \frac{a_{-1}}{x} + a_0 + a_1 x + \cdots$$

where $a_{-2} \neq 0$. Hence, after blowing-down C_1 (see Figure 6), we obtain a global meromorphic section for $\pi(C_2)$ without zeros and with just one simple pole given by

$$y = \frac{a_{-2}}{x} + a_{-1} + a_0 x + \cdots$$

Thus $\pi(C_2)$ has Chern class -1, as claimed.

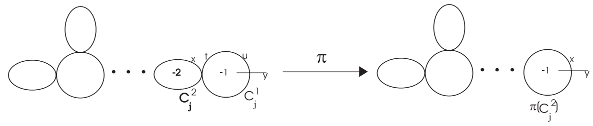


Figure 6

By induction we have a biholomorphism $\Phi : (\pi(M_1), \pi(D_1)) \longrightarrow (\pi(M_2), \pi(D_2))$ taking $\pi(C_1^1)$ in $\pi(C_2^1)$. Finally, by Riemann's extension theorem, the map Φ can be lifted to a biholomorphism $\widetilde{\Phi} : (M_1, D_1) \longrightarrow (M_2, D_2)$. □

5.2. **Analytic cocycles.** We determine a characteristic class for generalized curves with quasihomogeneous separatrices, which appears as obstruction for the global pasting of componentwise isomorphisms. In order to do this, we shall strongly use the concept of non-abelian cohomology. For the reader which is not familiar with this subject, we refer to [8], [14].

Definition 7. *We denote by* $Iso(\mathcal{F})$ *the isotropy group of the germ of foliation* $(\mathcal{F} : \omega = 0)$ *on* $(\mathbb{C}^2, 0)$ *given by*

$$Iso(\mathcal{F}) = \{\phi \in Diff(\mathbb{C}^2, 0) : \phi^*\omega \wedge \omega = 0\}$$

Let $(\mathcal{F} : \omega = 0)$ be a generalized curves with quasihomogeneous separatrix and pick $\mathcal{F}^{\mathrm{o}}$, componentwise isomorphic to $\mathcal{F}$, such that, in the coordinates given by its minimal resolution, $Sep(\widetilde{\mathcal{F}}_j^{\mathrm{o}})$ is contained in the fibers of $\mathbb{H}_j$ (this resolution does exist by [5]). $\mathcal{F}^{\mathrm{o}}$ shall be called a **fixed model** for $\mathcal{F}$. Now consider the elements $\Phi_j \in Diff(\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}_j^{\mathrm{o}})$ which shall be called a **projective chart** for the j^{th} component of the fixed model. Then it is straightforward that:

Lemma 5. *For each* $\widetilde{\mathcal{F}}_j = \widetilde{\mathcal{F}}\Big|_{\mathcal{H}_j, D_j}$ *and each fixed model* $\widetilde{\mathcal{F}}_j^o$ *there exists only one projective chart up to left composition by an element of* $Iso(\widetilde{\mathcal{F}}_j^o)$.

Now consider the sheaf of non-abelian groups $\Lambda^{\mathrm{o}} := Iso(\widetilde{\mathcal{F}}^{\mathrm{o}})$, then we say that $\mathcal{U} := \cup U_j$ is a **good covering** for Λ^{o} if U_j are simply-connected neighborhoods of $D_j \subset \mathcal{H}_j$ whose intersections are simply-connected. Then consider the first cohomology set $H^1(\mathcal{U}, \Lambda^{\mathrm{o}})$ associated with the good covering $\mathcal{U}$, and set $H^1(D, \Lambda^{\mathrm{o}})$ as the direct limit of $H^1(\mathcal{U}, \Lambda^{\mathrm{o}})$ for the good coverings of $\widetilde{\mathcal{F}}$ associated with $D = \cup D_j$ (the exceptional divisor of the given minimal resolution of $\mathcal{F}$). Hence, by Proposition 3, the map

$$\begin{array}{ccc} \mathcal{QHS}_\omega^{\mathrm{c}} & \stackrel{\Theta}{\longrightarrow} & Z^1(D, \Lambda^{\mathrm{o}}) \\ \mathcal{F} & \mapsto & (\Phi_{i,j}) := \Phi_i \circ \Phi_j^{-1} \end{array}$$

is well defined and onto $H^1(D, \Lambda^{\mathrm{o}})$. Note that Θ does not depend on the fixed models, up to componentwise conjugacy class. By the definition of the fixed model we have $\widetilde{\omega}_j^{\mathrm{o}}(u_j, y_j) = \widetilde{\omega}_{j+1}^{\mathrm{o}}(t_{j+1}, x_{j+1})$ where $(\widetilde{\mathcal{F}}_j^{\mathrm{o}} : \widetilde{\omega}_j^{\mathrm{o}} = 0)$ for $j = 1, \ldots, k$.

Proposition 4. *Two generalized curves with quasihomogeneous separatrices* $\mathcal{F}, \mathcal{G} \in \mathcal{QHS}_\omega^c$ *are analytically equivalent if and only if* $[\Theta(\mathcal{F})] = [\Theta(\mathcal{G})] \in H^1(D, \Lambda^o)$.

Proof. Let $\Theta(\mathcal{F}) = (\Phi_1 \circ \Phi_2^{-1}, \cdots, \Phi_{k-1} \circ \Phi_k^{-1})$, and $\Theta(\mathcal{G}) = (\Psi_1 \circ \Psi_2^{-1}, \cdots, \Psi_{k-1} \circ \Psi_k^{-1})$. First let us verify the necessary part. Suppose that H is a global conjugation between $\mathcal{F}$ and $\mathcal{G}$, i.e. $H^*(\mathcal{G}) = \mathcal{F}$. Then, by Lemma 5, we have $\Psi_j = \alpha_j \circ \Phi_j \circ H$ for some $\alpha_j \in Iso(\widetilde{\mathcal{F}}_j^{\text{o}})$. Therefore,

$$\begin{aligned}\Psi_{j-1} \circ \Psi_j^{-1} &= \alpha_{j-1} \circ \Phi_{j-1} \circ H \circ H^{-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1} \\ &= \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1}.\end{aligned}$$

Now let us verify the sufficient part. By hypothesis, $\mathcal{F}$ and $\mathcal{G}$ have the same fixed model. Hence, if $[(\Phi_1 \circ \Phi_2^{-1}, \cdots, \Phi_{k-1} \circ \Phi_k^{-1})] = [\Theta(\mathcal{F})] = [\Theta(\mathcal{G})] = [(\Psi_1 \circ \Psi_2^{-1}, \cdots, \Psi_{k-1} \circ \Psi_k^{-1})]$, then there is a collection $(\alpha_j) \subset Iso(\widetilde{\mathcal{F}}_j^{\text{o}})$ such that $\Psi_{j-1} \circ \Psi_j^{-1} = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1}$. Henceforth, $(\alpha_{j-1} \circ \Phi_{j-1})^{-1} \circ \Psi_{j-1} = (\alpha_j \circ \Phi_j)^{-1} \circ \Psi_j$. Thus we can define a global conjugacy between them, just by leting $H := (\alpha_j \circ \Phi_j)^{-1} \circ \Psi_j$ for all $j = 1, \ldots, k$. □

6. Quasihomogeneous polynomials and vanishing cocycles

In order to prove Theorem 2, we have to state some notation and preliminary results. Recall that $\mathcal{F} \in \mathcal{QHS}_{\omega,f}^{c,1}$ with $f(x,y) = \mu y^m x^n \prod_{j=1}^{d} (y^p - \lambda_j x^q)$, $1 \le p < q, m, n \in \mathbb{N}^*$, $g.c.d.(p,q) = 1$, and $\lambda_j, \mu \in \mathbb{C}^*$. Then we order the first projective line to arise in the course of the resolution process with 1, and increasingly with 2 the next one to intersect it — in the minimal resolution — and so on (see Lemma 3 and Figure 8). In particular we denote the principal projective line by D_l. From Lemmas 3 and 4 we have that $\widetilde{\mathcal{F}}_j$ has at most two singularities for all $j \neq l$. Since $\mathcal{F}$ is generic, then the index theorem guarantees that the singularities of $\widetilde{\mathcal{F}}_j$ are (simultaneously) linearizable for all for $j \neq l$ (cf. [22]).

Now fix the projective models $(\widetilde{\mathcal{F}}_j^{\text{o}} : \widetilde{\omega}_j^{\text{o}} = 0)$ given by the (global) multivalued first integral:

$$\begin{cases} \widetilde{\omega}_j^{\text{o}}(\tau, \xi) \wedge d(\tau^{p_j} \xi^{q_j}) = 0, \\ \widetilde{\omega}_j^{\text{o}}(\upsilon, \zeta) \wedge d(\upsilon^{k_j q_j - p_j} \zeta^{q_j}) = 0. \end{cases} \tag{6.1}$$

where $\mathcal{A}_j = \{(\tau, \xi), (\upsilon, \zeta) : \upsilon = 1/\tau, \zeta = \tau\xi\}$ are affine charts for the Hopf bundle $\mathbb{H}_j(-k_j)$ and $\nu_j, \mu_j \in \mathbb{C}$ are non-resonant for all $j \neq l$. Then, by the index theorem, we have that $\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}_j^{\text{o}}$ are isomorphic up to first order. In fact, as the projective holonomy of $\widetilde{\mathcal{F}}_2$ is linearizable, then Proposition 1 guarantees that they are indeed isomorphic for all $j \neq l$. Since the coordinate system (τ, ξ), (υ, ζ) — as in (6.1) — satisfies

$$\begin{aligned} \tau &= t \cdot U_1(t,x) & \xi &= x \cdot U_2(t,x) \\ \upsilon &= u \cdot V_1(u,y) & \zeta &= y \cdot V_2(u,y) \end{aligned}$$

where $U_j, V_j \in \mathcal{O}^*(\mathbb{C}, \mathbb{D}_\epsilon)$ ($\mathbb{D}_\epsilon$ is the disk of radius ϵ centered at the origin) and $U_j(t,0) = V_j(u,0) = 1$. Then we have $V_1(u,y) = 1/U_1(1/u, u^{k_j} y)$ and $V_2(u,y) = U_1(1/u, u^{k_j} y) \cdot U_2(1/u, u^{k_j} y)$. Thus the zero and polar sets of U_1 do not intersect $D_j \subset \mathcal{H}_j$. Hence $U_1(t,x) = \sum_{m < \frac{n}{k_j}} a_{m,n} t^m x^n$ and $U_2(t,x) = \sum_{m < \frac{n}{k_j}} b_{m,n} t^m x^n$. Therefore $\tau^{\nu_j} \xi^{\mu_j} = t^{\nu_j} x^{\mu_j} U(t,x)$ and $\upsilon^{k_j \mu_j - \nu_j} \zeta^{\mu_j} = u^{k_j \mu_j - \nu_j} y^{\mu_j} V(u,y)$, where $U(t,x) = \sum_{m < \frac{n}{k_j}} c_{m,n} t^m x^n \in \mathcal{O}^*(\mathbb{C}, \mathbb{D}_\epsilon)$ and $V(u,y) := U(1/u, u^{k_j} y) \in \mathcal{O}^*(\mathbb{C}, \mathbb{D}_\epsilon)$.

Thus the Hopf component $\widetilde{\mathcal{F}}_j$ has a (global) multivalued first integral given in affine charts (t,x) and (u,y) respectively by $t^{\nu_j} x^{\mu_j} U(t,x)$, $u^{k_j \mu_j - \nu_j} y^{\mu_j} V(u,y)$ for all $j \neq l$.

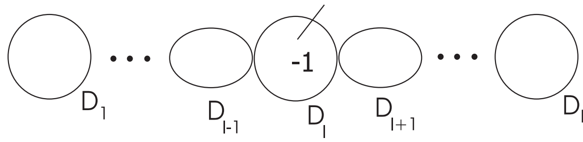


Figure 8: The principal projective line for $p \neq 1$.

By the argument used in the proof of Lemma 3, $\mathcal{F}$ is resolved together with any "generic" fiber of the "companion" fibration $\frac{y^p}{x^q} \equiv const$, i.e. $(\mathcal{G} : \eta = 0)$ where $\eta(x,y) = pxdy - qydx$. In other words, $\mathcal{F}$ and $\mathcal{G}$ are resolved by the same sequence of blowing-ups. In particular, the minimal resolution of $\mathcal{G}$ has the same tree of projective lines of the minimal resolution of any element of $\mathcal{QHS}^{c,1}_{\omega,f}$, and contains its separatrices as fibers. Furthermore, for each $j \neq l$ the foliation $\widetilde{\mathcal{G}}_j$ has a (global) holomorphic first integral of the form

$$\begin{cases} \widetilde{\eta}(t,x) = d(t^{r_j}x^{s_j}), \\ \widetilde{\eta}(u,y) = d(u^{k_j s_j - r_j}y^{s_j}). \end{cases}$$

where $\mathcal{B}_j = \{(t,x),(u,y) : u = 1/t, y = tx\}$ are affine charts for the Hopf bundle $\mathbb{H}_j(-k_j)$, and r_j, $s_j \in \mathbb{N}$ are relatively prime. On the other hand, since $\widetilde{\mathcal{G}}_l$ is a radial fibration, then $\widetilde{\mathcal{G}}_{l-1}$ has just one singularity (cf. Figure 7). Therefore, comparing $\widetilde{\mathcal{F}}_j$ and $\widetilde{\mathcal{G}}_j$ (starting from $l-1$ to 1) in view of index theorem, it follows that $\nu_j s_j - \mu_j r_j \neq 0$ for all $j \neq l$.

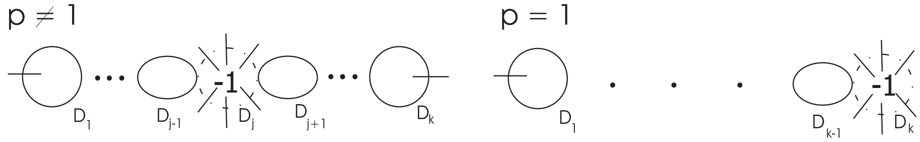


Figure 7: The resolution tree of $\mathcal{G} : (\frac{x^p}{y^q} = const.)$

Remark 1. *In case $Sep(\mathcal{F})$ is commode then Lemma 3 and the index theorem assures that the generic conditions are automatically satisfied, since in this case the multivalued first integrals turns out to be holomorphic.*

Now let $(\mathcal{F} : \omega = 0)$ be a germ of singular holomorphic foliation on $(\mathbb{C}^2, 0)$, $\widetilde{\mathcal{F}}$ its minimal resolution, and $Fix(\widetilde{\mathcal{F}})$ the subset of $Iso(\widetilde{\mathcal{F}})$ given by the germs automorphisms of $\widetilde{\mathcal{F}}$ fixing the leaves of $\widetilde{\mathcal{F}}$, and consider the subsheaf of Λ^{o} given by $\Gamma^{o} := Fix(\widetilde{\mathcal{F}}^{o})$. Then, from the geometry of the divisor of any element of $\mathcal{QHS}^{c,1}_{\omega}$, the holonomy of the non-principal Hopf components coincides with the holonomy of their corners. Therefore, we may suppose, without loss of generality, that $(\phi_{ij}) \in Z^1(\mathcal{U}, \Gamma^{o})$.

Lemma 6. *For all $j = 1, \ldots, k$ there is $\Phi_j \in Diff_{\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}^{o}_j}(\mathcal{H}_j, D_j)$ such that $\Phi_j \in Fix(\widetilde{\mathcal{G}}_j)$. In particular, there are affine coordinates (t_1, x_1), (u_1, y_1) for $\mathbb{H}_j$ such that the first integrals of $\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{G}}_j$ are given respectively by $t_1^{\nu_j}x_1^{\mu_j}$, $u_1^{k_j\mu_j - \nu_j}y_1^{\mu_j}$ and $t_1^{r_j}x_1^{s_j}$, $u_1^{k_j s_j - r_j}y_1^{s_j}$.*

Proof. In case $j = l$ the statement results from Proposition 1. Now let us consider the case $j \neq l$. First notice that $\widetilde{\mathcal{F}}_j$ has first integrals of the form $t^{\nu_j}x^{\mu_j}U(t,x)$ where $U \in \mathcal{O}^*_2$. Hence, if $\Phi_j^{-1}(t,x) = (a_j(t,x), b_j(t,x))$, then we have to find a solution for the system of equations

$$\begin{cases} a_j(t,x)^{\nu_j} b_j(t,x)_j{}^{\mu_j} = t^{\nu_j} x^{\mu_j} U(t,x) \\ a_j(t,x)^{r_j} b_j(t,x)^{s_j} = t^{r_j} x^{s_j} \end{cases}$$

which can be given in the affine chart (t, x) by

$$\begin{cases} a_j(t,x) = tU(t,x)^{\frac{s_j}{\nu_j s_j - \mu_j r_j}}, \\ b_j(t,x) = xU(t,x)^{\frac{r_j}{\mu_j r_j - \nu_j s_j}}. \end{cases}$$

A straightforward calculation shows that the expression of Φ_j^{-1} in the affine chart (u, y) is given by

$$\Phi_j^{-1}(u,y) = (uV(u,y)^{\frac{s_j}{\mu_j r_j - \nu_j s_j}}, yV(u,y)^{\frac{r_j - k_j s_j}{\mu_j r_j - \nu_j s_j}})$$

where $V(u, y) := U(1/u, u^{k_j} y) \in \mathcal{O}^*(\mathbb{C}, \mathbb{D}_\epsilon)$. Finally, by construction and the implicit function theorem, $\Phi_j^{-1} \in Diff(\mathcal{H}_j, D_j)$ carry leaves of $\widetilde{\mathcal{F}}_j$ into leaves of $\widetilde{\mathcal{F}}_j^{\text{o}}$. □

Recall $\widetilde{\mathcal{F}}_{j,j+1}, \widetilde{\mathcal{G}}_{j,j+1}$ denote the germs of foliations defined on a neighborhood of the corner $t_{j,j+1} = D_i \cap D_j$ by the restrictions of $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}$ respectively.

Lemma 7. *Let $\Phi_{j-1,j} \in Fix(\widetilde{\mathcal{F}}_{j-1,j}^o) \cap Fix(\widetilde{\mathcal{G}}_{j-1,j})$ for $j = 2, \ldots, k$. Then $\Phi_{j-1,j}$ has a unique extension to $\Phi_j \in Fix(\widetilde{\mathcal{F}}_j^o)$*

Proof. Notice that the uniqueness is a consequence of the identity theorem. Then, assuming that the corner is in the origin of the affine (u, y) of $\mathcal{H}_j$, we shall have $\Phi_{j-1,j}(t, x) = (a(t, x), b(t, x)) \in \mathcal{O}(\mathbb{D}^* \times \mathbb{D})$ where $\mathbb{D}$ is the unity disk. Since $\Phi_{j-1,j} \in Fix(\widetilde{\mathcal{F}}_{j-1,j}^{\text{o}}) \cap Fix(\widetilde{\mathcal{G}}_{j-1,j})$, then it satisfies the following system of equations

$$\begin{cases} a(t,x)^{\nu_j} b(t,x)^{\mu_j} = t^{\nu_j} x^{\mu_j} \\ a(t,x)^{r_j} b(t,x)^{s_j} = t^{r_j} x^{s_j} \end{cases}$$

whose solutions are $a_j(t, x) = \alpha t$ and $b_j(t, x) = \beta x$ where $\alpha, \frac{1}{\beta}$ are $(\nu_j s_j - \mu_j r_j)$-roots of unity. □

Proof of Theorem 2. For simplicity, we prove the statement just in case the principal projective line is in the "edge" of the resolution, i.e. it intersects just one projective line. Let $\mathcal{F}, \mathcal{F}' \in \mathcal{QHS}_{\omega,f}^{c,1}$ with projective charts $\Phi_j \in Diff_{\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}_j^{\text{o}}}(\mathcal{H}_j, D_j)$ and $\Phi'_j \in Diff_{\widetilde{\mathcal{F}}'_j, \widetilde{\mathcal{F}}_j^{\text{o}}}(\mathcal{H}_j, D_j)$ respectively for all $j = 1, \ldots, l-1$. Then, by Lemma 6, we may suppose that $\Phi_j, \Phi'_j \in Fix(\widetilde{\mathcal{G}}_j)$. Hence we construct (by a decreasing induction) a collection (α_j) with $\alpha_j \in Fix(\widetilde{\mathcal{F}}_j^{\text{o}}) \cap Fix(\widetilde{\mathcal{G}}_{j,j+1})$ such that $\Phi'_{j-1} \circ \Phi_j'^{-1} = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j{}^{-1}$. First let $\alpha_l = Id|_{Iso(\widetilde{\mathcal{F}}_l^{\text{o}})}$ (here l stands for the principal projective line) and suppose that α_j $(j < l)$ is already defined. Then let $\alpha_{j-1,j} := \Phi'_{j-1} \circ \Phi_j'^{-1} \circ \alpha_j \circ (\Phi_{j-1} \circ \Phi_j^{-1})^{-1}$. By construction $\alpha_{j-1,j} \in Fix(\widetilde{\mathcal{F}}_{j-1,j}^{\text{o}}) \cap Fix(\widetilde{\mathcal{G}}_{j-1,j})$. Thus, by Lemma 7, it has an unique extension, namely $\alpha_{j-1} \in Fix(\widetilde{\mathcal{F}}_{j-1}^{\text{o}})$. Then we have that $\Phi'_{j-1} \circ \Phi_j'^{-1} = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1}$ for all $j = 1, \ldots, l-1$. The other case can be treated similarly, just differing by the induction arguments. □

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