GROWTH OF THE NUMBER OF GEODESICS BETWEEN POINTS AND INSECURITY FOR RIEMANNIAN MANIFOLDS

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ABSTRACT. A Riemannian manifold is said to be uniformly secure if there is a finite number s such that all geodesics connecting an arbitrary pair of points in the manifold can be blocked by s point obstacles. We prove that the number of geodesics with length $\leq T$ between every pair of points in a uniformly secure manifold grows polynomially as $T \to \infty$. By results of Gromov and Mañé, the fundamental group of such a manifold is virtually nilpotent, and the topological entropy of its geodesic flow is zero. We derive from this that a compact Riemannian manifold with no conjugate points whose geodesic flow has positive topological entropy is totally insecure: the geodesics between any pair of points cannot be blocked by a finite number of point obstacles.

1. INTRODUCTION

intro

This paper explores interactions between two approaches to the study of geodesics in a Riemannian manifold. Both approaches study geodesic segments between points of the manifold. One of them relates the growth as $T \to \infty$ of the number of these geodesics with length $\leq T$ to the topological entropy of the geodesic flow. The other studies whether or not these geodesics can be blocked by a finite number of point obstacles.

Let M be a compact Riemannian manifold that is uniformly secure with security threshold s. This means that the geodesics between any two points in M can be blocked by s obstacles. Our main technical result concerns the growth rate of joining geodesics in such a manifold. We show that there are constants C and d, which depend only on s and the injectivity radius, such that the number of geodesics with length $\leq T$ between any pair of points in M is bounded from above by CT^d. See Proposition 4.1 and Theorem 4.2. This bound has strong

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consequences for the geometry and topology of M. In particular, its fundamental group is virtually nilpotent. See Theorem 4.3.

Our methods yield strong results for manifolds with no conjugate points. In particular, a compact manifold with nonpositive curvature satisfies the following dichotomy. Either such a manifold is flat and the geodesics between every pair of points are blocked by a finite number of obstacles; or the manifold is not flat and removing a finite number of points from the manifold cannot block the geodesics connecting any pair of points. See Corollary 4.7.

We thank J.F. Lafont and B. Schmidt for pointing out a simplification to our original proof of Theorem 4.5. They have independently studied connections between security and the geometry of a manifold [11]. In particular, they have obtained Theorem 4.5 and the lemmas in Section 3.

2. Background and Previous Results

back&prev

Throughout this paper by a (Riemannian) manifold we will mean a complete, connected, infinitely differentiable (Riemannian) manifold. Geodesics will always be parametrized by arclength. Unless otherwise specified, a geodesic will be a segment of finite, positive length. Let M be a Riemannian manifold. If $[a, b]$ is a closed bounded interval and $\gamma : [a, b] \to M$ is a geodesic, the *endpoints* of γ are its *initial point* $\gamma(a)$ and its *final point* $\gamma(b)$. The points $\gamma(s)$, where $a < s < b$, are the *interior points* of γ. We will say γ *passes through* z to indicate that z is an interior point of γ. Note that a point $z \in M$ may be both an interior point and an endpoint of γ.

A *configuration* in M is an ordered pair of points in M. The points in question may coincide.

connect_def **Definition 2.1.** Let (x, y) be a configuration in M. A geodesic γ *joins* x to y if x is its initial point and y is its final point. A geodesic γ *connects* x and y if it joins x and y and does not pass through either x or y.

We denote the set of geodesics joining x and y by $G(x, y)$ and the set of geodesics connecting x and y by $\Gamma(x, y)$. Of course $\Gamma(x, y) \subset G(x, y)$. The subsets of $G(x, y)$ and $\Gamma(x, y)$ consisting of geodesics with length $\leq T$ will be denoted by $G_T(x, y)$ and $\Gamma_T(x, y)$ respectively. Let $m_T(x, y)$ be the number of geodesics in $\Gamma_T(x, y)$ and let $n_T(x, y)$ be the number of geodesics in $G_T(x, y)$. If one of the sets is infinite, the corresponding number will be ∞.

SSn_T

2.1. **Growth of $n_T(x, y)$.** The behavior of $n_T(x, y)$ for compact manifolds has been extensively studied beginning with Berger and Bott's 1962 paper [1]. They observed that for each $T > 0$ the function $n_T(x, y)$ is finite and locally constant on an open subset of $M \times M$ of full Riemannian measure. We will use the notation $d\mu(x)$ for the density of this measure. Denote by $B(\widetilde{x}, T)$ the ball of radius T in the universal cover of M around a lift $\widetilde{x}$ of x. By Proposition 4.3 in [1],

ber_bot_eq
$$(1) \qquad \int_M n_T(x, y) \, d\mu(y) \geq \operatorname{Vol} B(\widetilde{x}, T)$$

for any $x \in M$.

Let $h = h(M)$ be the topological entropy of the geodesic flow. Manning showed in [14] that

$$h \geq \lim_{T \to \infty} \frac{1}{T} \log \operatorname{Vol} B(\widetilde{x}, T);$$

the limit on the right hand side exists and is the same for all points $\widetilde{x}$ in the universal cover. Mañé improved this estimate in [13] by showing that

mane_eq
$$(2) \qquad h = \lim_{T \to \infty} \frac{1}{T} \log \int_{M \times M} n_T(x, y) \, d\mu(x) d\mu(y)$$

for any compact Riemannian manifold.

A Riemannian manifold has no conjugate points if the exponential map at any point is a local diffeomorphism. An equivalent characterization is that any two points in the universal cover are joined by exactly one geodesic. The preceding relations are even stronger for compact manifolds with no conjugate points. In this case $n_T(x, y)$ is always finite, and for any $x, y \in M$ we have

rate
$$(3) \qquad h = \lim_{T \to \infty} \frac{1}{T} \log n_T(x, y).$$

See for example Corollary 1.2 in [13].

Equation (3) fails, in general, if M has conjugate points. There are examples where the growth of $n_T(x, x)$ is arbitrarily large for some exceptional points $x \in M$ [3]. There are also examples in which the limit in equation (3) is smaller than the topological entropy for an open set of configurations [4].

SSsec

2.2. **Security and insecurity.** A set B is a *blocking set* for a collection of geodesics if every geodesic belonging to the collection passes through a point of B.

secur_def **Definition 2.2.** A configuration (x, y) is *secure* if the collection $\Gamma(x, y)$ of connecting geodesics has a finite blocking set. Otherwise the configuration is *insecure*.

A blocking set for $\Gamma(x, y)$ must lie in $M \setminus \{x, y\}$. It is also a blocking set for $G(x, y)$.

If the configuration (x, y) is secure, its *security threshold* $\sigma(x, y)$ is the minimal number of points needed to block $\Gamma(x, y)$.

securmfld_def **Definition 2.3.** A Riemannian manifold is *secure* if every configuration is secure, *insecure* if some configurations are insecure, and *totally insecure* if every configuration is insecure. It is *uniformly secure* if there is $s < \infty$ such that $\sigma(x, y) \leq s$ for all $x, y \in M$. The smallest such s is the *security threshold* of the manifold.

In a geometric optics interpretation, a configuration is secure if one of the points can be shaded from the light emanating from the other by a finite number of point screens. Another obvious interpretation of Definition 2.2 suggested the name "security": all geodesic paths connecting elements in a secure configuration can be monitored from a finite number of observation spots. Since the security of configurations concerns the global properties of geodesics, it is instructive to compare spaces from this viewpoint.

The framework of security arose in the context of polygonal billiards. (The geodesics are the billiard orbits.) A polygon $P \subset \mathbb{R}^2$ is *rational* if the angles of P are commensurable with π. The billiard orbits in a rational polygon P correspond to the geodesics in the associated *translation surface* $S = S(P)$. These surfaces are endowed with flat metrics with special cone singularities. Definitions 2.2 and 2.3 extend to polygons and translation surfaces.

The results on security in this context are limited up to now to *lattice polygons* and *lattice translation surfaces*. The lattice condition was introduced by Veech; the class of lattice translation surfaces partitions into two subclasses: arithmetic and nonarithmetic [8]. From the security standpoint, these subclasses complement each other: An arithmetic translation surface is uniformly secure; almost all configurations in a nonarithmetic translation surface are insecure [7]. This dichotomy holds for billiard orbits in lattice polygons as well [7].

Gutkin and Schroeder investigated the security of compact, locally symmetric spaces in [9]. Let M be such a space. Then $M = S/\Gamma$, where S is a simply connected symmetric space, and Γ is a discrete, cocompact group of isometries freely acting on S. The space S uniquely decomposes into a product $S = S_0 \times S_- \times S_+$ of simply connected

symmetric spaces of *euclidean type, noncompact type, and compact type* respectively. If $M = S/\Gamma$ where S belongs to one of the three types, we say that M is a compact, locally symmetric space of that type.

We summarize the results of [9] in the following proposition.

`gut_schr_prop`

Proposition 2.4. *Let M be a compact, locally symmetric space. Then the following statements hold:*
a) If M is of noncompact type then it is totally insecure.
b) If M is of compact type then it has an open, dense, full measure set of secure configurations; it always has insecure configurations as well.
c) If M is of euclidean type then it is uniformly secure, and its security threshold is bounded in terms of $\dim(M)$.
d) If M is secure, then it is of euclidean type.

3. Key Lemmas

`keysect`

The main results of this paper, which are presented in the next section, are based on the following lemmas. They have been obtained independently by Lafont and Schmidt [11].

`m&n`

Lemma 3.1. *Let M be a compact Riemannian manifold, and let δ be its injectivity radius. Then for any $x, y \in M$ we have*

$$m_T(x, y) \leq n_T(x, y) \leq \left(\frac{T}{2\delta}\right)^2 m_T(x, y).$$

Proof. The left hand inequality is obvious because $\Gamma_T(x, y) \subset G_T(x, y)$. We prove the other one by constructing a surjective map $\lambda : G_T(x, y) \to \Gamma_T(x, y)$ that is at most $(T/2\delta)^2 : 1$.

Observe firstly that if t and t' are consecutive times when a geodesic passes through a point, z, then $2\delta \leq |t - t'|$. Indeed, the geodesic must travel (at least) from the center of the ball $B(z, \delta)$ to its boundary, and then back. An immediate corollary is that the set of times when a geodesic passes through z is discrete.

Let $\gamma : [a, b] \to M$ be a geodesic that belongs to $G_T(x, y)$. This means that $\gamma(a) = x$ and $\gamma(b) = y$. Let b' be the leftmost element of $(a, b]$ such that $\gamma(b') = y$. Let a' be the rightmost element of $[a, b']$ such that $\gamma(a') = x$. Observe that the restriction of γ to $[a', b']$ belongs to $\Gamma_T(x, y)$ and that $[a', b']$ is the leftmost subinterval of $[a, b]$ with this property. Furthermore, if γ already belongs to $\Gamma_T(x, y)$, we will have $a' = a$ and $b' = b$.

Define $\lambda(\gamma)$ to be the restriction of γ to $[a', b']$. The resulting map $\lambda : G_T(x, y) \to \Gamma_T(x, y)$ is surjective because it is the identity on $\Gamma_T(x, y)$.

Let $\gamma' : [a', b'] \rightarrow M$ belong to $\Gamma_T(x, y)$. We will investigate the preimage $\lambda^{-1}(\gamma') \subset G_T(x, y)$. Let $\gamma : \mathbb{R} \rightarrow M$ be the extension of γ' to a complete geodesic. The geodesics in $\lambda^{-1}(\gamma')$ are the restrictions of γ to intervals of the form $[a, b]$ such that $\gamma(a) = x$ and $\gamma(b) = y$. We must also have $a \leq a' < b' \leq b$, and $|b - a| \leq T$. These inequalities ensure that $a \in [b' - T, a']$ and $b \in [b', a' + T]$; both of these intervals have length $\leq T$. We see from the remarks at the beginning of the proof that there can be at most $T/2\delta$ possible choices for $a \in [b' - T, a']$ with $\gamma(a) = x$. Similarly there can be at most $T/2\delta$ possible choices for $b \in [b', a' + T]$ with $\gamma(b) = y$. It follows that γ' has at most $(T/2\delta)^2$ preimages under λ, as desired. ∎

basiclem **Lemma 3.2.** *Let M be a compact Riemannian manifold. Let $\Gamma_T(x, y; z)$ denote the subset of geodesics in $\Gamma_T(x, y)$ that pass through $z \neq x, y$. Let $m_T(x, y; z)$ be the number of geodesics in $\Gamma_T(x, y; z)$. Then*

$$m_T(x, y; z) \leq m_{T/2}(x, z) + m_{T/2}(z, y).$$

Proof. We construct a 1:1 map $\mu : \Gamma_T(x, y; z) \rightarrow \Gamma_{T/2}(x, z) \cup \Gamma_{T/2}(z, y)$.

Suppose $\gamma : [a, b] \rightarrow M$ is a geodesic in $\Gamma_T(x, y; z)$. Let γ_1 be the initial segment of γ that ends at the first time when γ passes through z; let γ_2 be the terminal segment of γ that starts at the last time when γ passes through z. Observe that γ_1 and γ_2 both belong to $\Gamma(x, z) \cup \Gamma(z, y)$, and their domains are subintervals of $[a, b]$ that intersect in at most one point.

Since the total length of γ_1 and γ_2 is at most T, at least one of them has length $\leq T/2$. Set $\mu(\gamma) = \gamma_1$ if γ_1 has length $\leq T/2$ and set $\mu(\gamma) = \gamma_2$ otherwise. We will now show that μ is 1:1.

A geodesic in $\Gamma(x, z)$ can be extended past z to obtain a geodesic in $\Gamma(x, y)$ in at most one way: we must stop at the first time (if there is one) when the geodesic reaches y. Similarly, a geodesic in $\Gamma(z, y)$ can be extended past z in at most one way to obtain a geodesic in $\Gamma(x, y)$.

Thus each $\gamma_1 \in \Gamma_{T/2}(x, z)$ can be the intial segment for at most one $\gamma \in \Gamma(x, y)$. Similary, each $\gamma_2 \in \Gamma_{T/2}(z, y))$ can be the terminal segment for at most one $\gamma \in \Gamma(x, y)$. Therefore the mapping $\mu : \Gamma_T(x, y; z) \rightarrow \Gamma_{T/2}(x, z) \cup \Gamma_{T/2}(z, y)$ is injective. ∎

4. Main Results

mainsect

Our main technical result relates the security threshold of a manifold with the growth rate of joining geodesics.

main_prop **Proposition 4.1.** *Let M be a compact Riemannian manifold, and let δ be its injectivity radius. Suppose that M is uniformly secure, and let*

$s = s(M)$ be the security threshold. Then for any $x, y \in M$ we have

$$n_T(x, y) < \frac{s}{2} \cdot \left(\frac{T}{\delta}\right)^{3+\log_2 s}.$$

Proof. By the uniform security assumption, any configuration $(p, q) \in M \times M$ has a blocking set $B(p, q) \subset M \setminus \{p, q\}$ containing at most s points. From Lemma 3.2, we have

reduc_eq (4) $$m_T(x, y) \leq \sum_{z \in B(x,y)} \left(m_{T/2}(x, z) + m_{T/2}(z, y)\right).$$

The right hand side of this inequality has the form $\sum m_{T/2}(p_i, q_i)$, where the (p_i, q_i) run through a finite set, $\mathcal{P}_1$, of pairs of points in M with at most $2s$ elements.

Iterating the estimate above k times, we obtain a sequence, $\mathcal{P}_k$, of subsets of $M \times M$, such that $\mathcal{P}_k$ has at most $(2s)^k$ elements and

induc_eq (5) $$m_T(x, y) \leq \sum_{(p_i, q_i) \in \mathcal{P}_k} m_{T/2^k}(p_i, q_i).$$

Observe that for any $L > 0$ we have $m_L(p_i, q_i) \leq \sup_{p,q \in M} m_L(p, q)$. Combining this with inequality (5) yields

estim_eq (6) $$m_T(x, y) \leq (2s)^k \sup_{p,q \in M} m_{T/2^k}(p, q).$$

Now choose $k \in \mathbb{N}$ so that $\log_2(\frac{T}{\delta}) < k \leq 1 + \log_2(\frac{T}{\delta})$, or equivalently

$$\frac{\delta}{2} \leq \frac{T}{2^k} < \delta.$$

Since $T/2^k < \delta$, we have $m_{T/2^k}(p, q) \leq 1$ for any $p, q \in M$. Substituting this into inequality (6), we obtain

final_eq (7) $$m_T(x, y) < 2s \left(\frac{T}{\delta}\right)^{(1+\log_2 s)}.$$

Combining this inequality with Lemma 3.1 yields the claim. ∎

Proposition 4.1 directly implies that a uniformly secure manifold has uniform polynomial growth of geodesics. The following theorem states this precisely. Recall that a positive function $f(T)$ has polynomial growth if there are constants C and d such that $f(T) \leq CT^d$ as $T \to \infty$. The minimum possible d is the degree of the polynomial growth.

polin_thm **Theorem 4.2.** *Let M be a compact Riemannian manifold. If M is uniformly secure, then there are positive constants C and d such that for any pair $x, y \in M$ we have*

$$n_T(x, y) \leq CT^d.$$

This inequality yields strong consequences for the geometry and topology of secure manifolds.

`entrop_thm` **Theorem 4.3.** *Let M be a compact Riemannian manifold that is uniformly secure. Then the topological entropy of the geodesic flow for M is zero, and the fundamental group of M is virtually nilpotent.*

If, in addition, M has no conjugate points, then M is flat.

Proof. Our first claim is immediate from equation (2) and Theorem 4.2. Set

$$N(x, T) = \int_M n_T(x, y)\, dy.$$

By Theorem 4.2, $N(x, T)$ has polynomial growth as $T \to \infty$ for any $x \in M$. By equation (1), this implies polynomial growth as $T \to \infty$ of the volume of the ball $B(\widetilde{x}, T) \subset \widetilde{M}$ of radius T in the universal cover of M, which in turn immediately implies polynomial growth of the fundamental group of M. The latter means that for a finite set S of generators for $\pi_1(M)$ the number of elements of $\pi_1(M)$ expressible as words of length $\leq n$ in the alphabet S has polynomial growth. Polynomial growth for one finite set of generators implies polynomial growth of the same degree for any finite set of generators.

By a famous theorem of Gromov, a finitely generated group with polynomial growth is virtually nilpotent [5], i.e. it has a finite index nilpotent subgroup. This proves our second claim. Our last claim now follows from a recent result of Lebedeva [12]: if the fundamental group of a compact Riemannian manifold with no conjugate points has polynomial growth, then the manifold is flat. ∎

Theorem 4.3 has an immediate corollary.

`admit_cor` **Corollary 4.4.** *Let M be a compact manifold. If $\pi_1(M)$ is not virtually nilpotent, then M does not admit a uniformly secure Riemannian metric.*

Our next result concerns manifolds with no conjugate points. It has been obtained independently by Lafont and Schmidt [11].

`nocp_thm` **Theorem 4.5.** *Let M be a compact Riemannian manifold with no conjugate points whose geodesic flow has positive topological entropy. Then M is totally insecure.*

Proof. Let $p, q \in M$ be arbitrary. From equation (3) and Lemma 3.1, we have

`con_entr` (8)
$$\lim_{T \to \infty} \frac{1}{T} \log m_T(p, q) = h.$$

This is where we use the no conjugate points property.

Suppose now that the claim is false. Then there are two points $x, y \in M$ such that any geodesic from x to y is blocked by one of a finite number of points, say $z_1, \ldots, z_n$. Lemma 3.2 tells us that

T&T/2 (9)
$$m_T(x, y) = \sum_{i=1}^{n} \left(m_{T/2}(x, z_i) + m_{T/2}(z_i, y) \right)$$

for any $T > 0$. Consider a number $\alpha > 1$. By (8), there is a $T_0 = T_0(\alpha)$ such that for any $T \geq T_0$ and any $p, q \in \{x, y, z_1, \ldots, z_n\}$ we have

$$h/\alpha < \frac{1}{T} \log m_T(p, q) < \alpha h,$$

which is equivalent to

$$e^{hT/\alpha} < m_T(p, q) < e^{hT\alpha}.$$

Combining this with equation (9), we obtain

$$e^{hT/\alpha} < 2n e^{hT\alpha/2}$$

for all large enough T. Now choose $\alpha < \sqrt{2}$. Then the exponent on the left is larger than the exponent on the right, and the inequality is absurd for large T. ∎

Theorem 4.5 and the variational principle for entropy [6] immediately imply the following proposition.

pos_ent_cor **Corollary 4.6.** *Let M be a compact Riemannian manifold with no conjugate points. If the geodesic flow of M has positive metric entropy, then the manifold M is totally insecure.*

Any compact, locally symmetric space of noncompact type satisfies the hypotheses of Theorem 4.5. This gives a new proof of case (a) in Proposition 2.4. The original proof in [9] used an entirely different approach.

For manifolds of nonpositive curvature we obtain a dichotomy along the lines security/insecurity.

neg_curv_cor **Corollary 4.7.** *A compact Riemannian manifold of nonpositive curvature is either totally insecure or uniformly secure. In the latter case the manifold is flat; its security threshold is bounded above in terms of the dimension of manifold.*

Proof. Let M be a compact Riemannian manifold of nonpositive curvature. By Pesin's formula (Corollary 3 in [15]), the following dichotomy holds: either i) the geodesic flow for M has positive entropy, or ii) M is flat.

If i) holds, then M satisfies the hypotheses of Corollary 4.6; thus, M is totally insecure. In case ii), M is a compact, locally symmetric space of euclidean type. The claim follows from case (c) of Proposition 2.4. ∎

5. Examples, conjectures and open problems

We begin by producing examples of insecure manifolds with conjugate points.

conj&probs

prod_lem

Lemma 5.1. *A configuration in a Riemannian product $M \times M'$ is secure if and only if its projections to M and M' are secure.*

Proof. Geodesics in $M \times M'$ are products of geodesics in M and geodesics in M'. More precisely, if $\gamma : [a, b] \to M$ and $\gamma' : [a', b'] \to M'$ are unit speed geodesics, then the unit speed geodesic $\gamma \times \gamma'$ in M of length $L = \sqrt{(b - a)^2 + (b' - a')^2}$ is defined by

$$(\gamma \times \gamma')(t) = (\gamma(a + (b - a)t/L), \gamma'(a' + (b' - a')t/L)$$

for $0 \leq t \leq L$. All geodesics in $M \times M'$ arise in this way.

Let $\xi = (x, x')$ and $\eta = (y, y')$ be two points in $M \times M'$. The set $G(\xi, \eta)$ consists of the products of geodesics in $G(x, y)$ and $G(x', y')$. Let $\Gamma_0(\xi, \eta)$ be the subset of $G(\xi, \eta)$ formed by the products of geodesics in $\Gamma(x, y)$ and $\Gamma(x', y')$. Then $\Gamma_0(\xi, \eta) \subset \Gamma(\xi, \eta)$. (There may be more geodesics in $\Gamma(\xi, \eta)$, for example products of geodesics in $\Gamma(x, y)$ and in $G(x', y') \setminus \Gamma(x', y')$.)

Suppose now that x and y form a secure configuration in M with blocking set $B \subset M \setminus \{x, y\}$, and that x' and y' form a secure configuration in M' with blocking set $B' \subset M' \setminus \{x', y'\}$. Then B and B' also block all the geodesics in $G(x, y)$ and $G(x', y')$ respectively. If $x \neq y$ and $x' \neq y'$, the set $B \times B'$ will block all geodesics in $G(\xi, \eta)$. If $x = y$, we will also use the blocking points (x, b'), where $b' \in B'$, in order to block geodesics from ξ to η that have constant projection to the first factor. Similarly if $x' = y'$, we include the points (b, x'), where $b \in B$, in our blocking set. In all cases, we produce a finite set $Z \subset M \times M'$, such that $B \times B' \subset Z \subset M \times M' \setminus \{\xi, \eta\}$, blocking all geodesics in $G(\xi, \eta)$.

Conversely, if ξ and η form a secure configuration, there is a finite set of points $(z_1, z_1'), \ldots, (z_n, z_n')$ in $M \times M'$ which block all geodesics in $\Gamma_0(\xi, \eta)$. Then $\{z_1, \ldots, z_n\}$ is a blocking set for $\Gamma(x, y)$ and $\{z_1', \ldots, z_n'\}$ is a blocking set for $\Gamma(x', y')$. ∎

The following criteria for security and insecurity follow immediately from the preceding lemma and its proof.

riem_prod_prop **Proposition 5.2.** *Let M and M' be Riemannian manifolds. Then the following statements hold.*
1) If one of M and M' is (totally) insecure, then $M \times M'$ is (totally) insecure.
2. $M \times M'$ is (uniformly) secure if and only if both M and M' are (uniformly) secure.

cover_rem **Remark 5.3.** Lemma 5.1 and Proposition 5.2 are closely related to Proposition 5 and Corollary 4 in [9]. They extend to fibre bundles such that each point of the base has a neighborhood, O, over which the fibration is the Riemannian product of O and the fibre. We leave the details to the reader.

Combining Proposition 5.2 with Theorem 4.5 immediately gives the following result.

nocp_corol **Corollary 5.4.** *Let M be a compact Riemannian manifold with no conjugate points whose geodesic flow has positive topological entropy. Let M' be a Riemannian manifold. Then the Riemannian product $M \times M'$ is totally insecure.*

This proposition provides us with examples of totally insecure manifolds that have conjugate points.

The hypothesis of no conjugate points in Theorem 4.5 is used only to ensure that equation (3) holds. One can strengthen equation (3) by showing that the convergence to the limit is uniform for all $p, q \in M$. The proof of Theorem 4.5 does not require this uniformity.

In fact, in order to prove Theorem 4.5, we need considerably less than equation (3). It suffices to have positive constants h_1 and h_2 such that $h_2 < 2h_1$, and such that for arbitrary pairs $p, q \in M$ and all large enough T we have the inequality

up_low_eq (10) $$e^{h_1 T} \leq n_T(p, q) \leq e^{h_2 T}.$$

Suppose that inequality (10) holds. Then all configurations in M are insecure, whether there are conjugate points or not. We believe that many Riemannian manifolds satisfy this condition, but the only examples that we know presently are the manifolds with no conjugate points.

Following another approach, we can replace inequality (10) by

limits_eq (11) $$\sup_{x,y \in M} \limsup_{T \to \infty} \frac{1}{T} \log n_T(x, y) < 2 \liminf_{T \to \infty} \frac{1}{T} \log n_T(p, q) < \infty.$$

Similarly, inequality (11) implies that the pair $p, q \in M$ is insecure. We believe that inequality (11) frequently holds, but do not know any examples outside the realm of manifolds with no conjugate points.

Examples show that one cannot simply discard the hypothesis of no conjugate points from Theorem 4.5. In fact, the claim does not extend to all manifolds whose geodesic flows have positive topological entropy. There are compact Riemannian manifolds with positive topological entropy that contain secure configurations.

We outline such an example; it was used for a different purpose in [4]. It is a special case of a construction due to Weinstein. See [2], Appendix C. Choose a Riemannian metric $\widehat{g}$ on S^2 whose geodesic flow has positive topological entropy. Such metrics exist; the first was given by Knieper and Weiss [10]. Now choose a smooth family of metrics g_t, $-\pi/2 \leq t \leq \pi/2$, such that $g_t = \widehat{g}$ when $|t| \leq \pi/6$, and g_t is the standard metric on the unit sphere in $\mathbb{R}^3$ when $\pi/3 \leq |t| \leq \pi/2$. Choose also a scaling function $\rho : [-\pi/2, \pi/2] \to \mathbb{R}$ such that $\rho(t) = 1$ when $|t| \leq \pi/6$, $\rho(t) > 0$ when $\pi/6 \leq |t| \leq \pi/3$, and $\rho(t) = \cos t$ when $\pi/3 \leq |t| \leq \pi/2$.

Define a metric on $[-\pi/2, \pi/2] \times S^2$ that is degenerate on the boundary as follows. Let $t \in [-\pi/2, \pi/2]$ denote the first coordinate. We require:

(1) $\|\partial/\partial t\| \equiv 1$;
(2) $\partial/\partial t$ is everywhere orthogonal to $\{t\} \times S^2$;
(3) the restriction of the metric to the vectors tangent to $\{t\} \times S^2$ is $\rho(t)g_t$.

The result is a smooth metric g on S^3. The spheres $\{\pm\pi/2\} \times S^2$ collapse to points, p and p', which we call poles. The metric g coincides with the standard round metric on S^3 near the poles.

The spheres $\{t\} \times S^2$ for $|t| \leq \pi/6$ are totally geodesically isometrically embedded copies of $(S^2, \widehat{g})$; this ensures that the geodesic flow for the metric g on S^3 has positive entropy. The geodesics which pass through the poles are circles of length 2π and are orthogonal to the "spheres of latitude". They are meridians just as in the usual round metric on S^3; the only difference is that the spheres of latitude have an unusual geometry away from the poles. All meridians that leave one pole focus at the opposite pole after distance π. The pairs (p, p) and (p', p') are secure configurations because their connecting geodesics are blocked by the opposite pole.

We conclude with some conjectures related to the results of this paper.

Presently, the only examples of compact Riemannian manifolds that are secure are the flat manifolds. We believe that this is not an accident and Theorems 4.2 and 4.3 are evidence for:

secur_flat_conj

Conjecture 5.5. *A compact Riemannian manifold is secure if and only if it is flat.*

Our next conjecture concerns classical Riemannian geometry but is motivated by security questions, as we explain below.

noconju_flat_conj

Conjecture 5.6. *Let M be a compact Riemannian manifold with no conjugate points. Then either M is flat or its geodesic flow has positive topological entropy.*

If Conjecture 5.6 is true, then the alternative for manifolds with nonpositive curvature stated in Corollary 4.7 would extend to compact Riemannian manifolds with no conjugate points. More precisely, the following dichotomy would hold: A compact manifold with no conjugate points is either
i) flat, uniformly secure, and has zero topological entropy for its geodesic flow; or
ii) not flat, totally insecure, and has positive topological entropy for its geodesic flow.

Finally, we suspect that there are many examples of totally insecure manifolds other than those described in this paper. In fact, we believe that "most" Riemannian manifolds are totally insecure.

More conjectures related to security in Riemanian manifolds can be found in [11].

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