# SCALING LIMITS FOR GRADIENT SYSTEMS IN RANDOM ENVIRONMENT

PATRÍCIA GONÇALVES AND MILTON JARA

ABSTRACT. It is well known that the hydrodynamic limit of an interacting particle system satisfying a gradient condition (such as the zero-range process or the symmetric simple exclusion process) is given by a possibly non-linear parabolic equation and the equilibrium fluctuations from this limit are given by a generalized Ornstein-Uhlenbeck process.

We prove that in the presence of a symmetric random environment, these scaling limits also hold for almost every choice of the random environment, with an homogenized diffusion coefficient that does not depend on the realization of the random environment.

## 1. INTRODUCTION

Consider a system of particles evolving on a multidimensional, periodic integer lattice of period $2N$. Each particle performs a continuous-time random walk with rates $p(x, y)$ that depend on both the position $x$ and the destination site $y$. These rates are chosen as a fixed realization of a random field, in such a way that the resulting single-particle random walk is reversible with respect to the counting measure on the lattice. We call these rates the *random environment*.

Particles interact between them only when they share a site, through an interaction function $g : \mathbb{N}_0 \to \mathbb{R}_+$. The dynamics for this system is the following. At each time $t$, let $\eta_t(x)$ denote the number of particles at the site $x$. For each pair of sites $\langle x, y \rangle$, after an exponential waiting time of rate $g(\eta_t(x))p(x, y)$ the particle at site $x$ jumps to site $y$. This is done independently for each pair $\langle x, y \rangle$ and after each jump, the exponential waiting time for each pair $\langle x, y \rangle$ starts afresh.

Such a system can be understood as a model for diffusion in heterogeneous media. The purpose of this article is to study the scaling limits of this system as $N \to \infty$ and mostly the influence of the randomness in this limit. As we will see, when the underlying random field is ergodic, stationary and satisfies an ellipticity condition, for any realization of the random environment the scaling limit depends on the randomness only through some constants which depend on the distribution of the random transition rates, but not on the particular realization of the random environment.

In this article we study two related scaling limits for this process: the hydrodynamic limit and the equilibrium fluctuations. The first one is a law of large numbers for the empirical distribution of particles when the process starts from a configuration of particles with macroscopic density close to some initial profile while the second one is a central limit theorem for the empirical distribution of particles when the system starts from an equilibrium measure.

The hydrodynamic limit has been obtained in the context of exclusion processes in [4] when the dimension $d \geq 3$ and extended in [14] to any dimension. In these references, it is not assumed the reversibility for the one-particle random walk with respect to the counting measure on the lattice, so in this sense their results are more general than ours. Their approach is based on the generalization of the non-gradient method of Varadhan [16], [13] for the case of random transition rates. In [3], [8] the one-dimensional simple exclusion process is considered.

In the reversible situation, we introduce the *corrected empirical process.* This process satisfies the gradient condition, which is a key property from which hydrodynamics and equilibrium fluctuations can be easily obtained like in the non-random situation [6], [2]. Therefore, our approach is simpler, does not require any mixing condition and can be generalized to situations in which the non-gradient method does not apply, like kinetically constrained particle systems, the zero-range process with bounded interaction rate and particle systems in non-homogeneous lattices [7].

The introduction of a corrected empirical measure can be understood as a version of Tartar's compensated compactness lemma in the context of particle systems. In this reversible situation the averaging due to the dynamics and the inhomogeneities introduced by the random media factorize after introducing the corrected empirical process, in such a way that we can average them separately. For the dynamic averaging, we use the entropy method of [6] to derive the hydrodynamic limit, while for the equilibrium fluctuations we adopt Chang's proof of the Boltzmann-Gibbs principle [2]; for the averaging of the random environment we use $\Gamma$-convergence. With this procedure, the scaling limits of the corrected empirical process are obtained. After this, we prove that in the limit as $N \to \infty$, the corrected empirical process and the original empirical process are close enough to recover the scaling limit for the original empirical distribution of particles.

In order to see how far can this picture be taken, we also prove the Boltzmann-Gibbs principle for functions that depend on both the particle configuration and the random environment. Notice that this more general version of the Boltzmann-Gibbs principle is not needed to obtain the equilibrium fluctuations for the empirical density of particles.

The Boltzmann-Gibbs principle states that non-conserved quantities oscillate faster than conserved quantities, and therefore when averaged in time, only the projections over the density field are observed. In consequence, the Boltzmann-Gibbs principle is interesting by its own. In order to give further motivations for the study of the Boltzmann-Gibbs principle for random functions, we present two applications at the end of the article.

The article is structured as follows. In Section 2 we describe the model and the main results. Section 3 is devoted to the proof of the hydrodynamic limit for this process and in the subsequent section we present the equilibrium fluctuations. The proof of the Boltzmann-Gibbs principle is referred to Section 5. For the reader's convenience, we include some well-known, but rather technical lemmas and definitions in the Appendix.

## 2. NOTATIONS AND RESULTS

2.1. **The zero-range process.** We define the zero-range process as a continuous-time Markov process $\eta_t$ with state space $\Omega_N^d = \{\eta : \mathbb{T}_N^d \to \mathbb{N}_0\}$, where $\mathbb{T}_N^d$ is the $d$-dimensional discrete torus $N^{-1}\mathbb{Z}^d/2N\mathbb{Z}^d$. We consider $\mathbb{T}_N^d$ as a subset of $U^d = [-1, 1]^d$ with periodic boundary conditions. This process has a generator whose action over local functions $f : \Omega_N^d \to \mathbb{R}$ is given by

$$\mathcal{L}_N f(\eta) = \sum_{x,y \in \mathbb{T}_N} p_N(x,y) g(\eta(x)) \big[ f(\eta^{xy}) - f(\eta) \big],$$

where $p_N : \mathbb{T}_N^d \times \mathbb{T}_N^d \to \mathbb{R}_+$ is the jump rate of a random walk in $\mathbb{T}_N^d$, $g : \mathbb{N}_0 \to \mathbb{R}_+$ is the interaction rate between the particles and $\eta^{xy} \in \Omega_N^d$ is given by

$$\eta^{xy}(z) = \begin{cases} \eta(x) - 1, z = x \\ \eta(y) + 1, z = y \\ \eta(z), z \neq x, y \end{cases} .$$

Notice that the dynamics of $\eta_t$ conserves the number of particles. In particular, the process $\eta_t$ is well defined for any initial configuration $\eta_0 \in \Omega_N^d$, since in that case the state space is finite.

We will assume that the interaction rate $g$ has linear growth:

$$\exists c_0 > 0 : c_0^{-1} n \leq g(n) \leq c_0 n \quad \forall n \in \mathbb{N}_0 \tag{2.1}$$

We will also assume that the motion of a single particle is a nearest-neighbor random walk, so we take $p_N(x,y) = 0$ if $|x - y| \neq 1/N$, where $|x - y| = \sum_{i \leq d} |x_i - y_i|$ is the sum norm in $\mathbb{R}^d$. This last hypothesis is not essential, but it simplifies the notation. We further assume that

$p_N(x,y) = p_N(y,x)$ for all $x, y \in \mathbb{T}_N^d$. This hypothesis will ensure the reversibility of the process $\eta_t$ with respect to the measures $\nu_\rho$ defined below, and the reversibility of the randomm walk generated by $p_N(x,y)$, which is crucial in what follows.

For each $\alpha \geq 0$, let $\bar{\nu}_\alpha$ be the product measure in $\Omega_N^d$ whose marginals are given by

$$\bar{\nu}_\alpha\{\eta; \eta(x) = k\} = \frac{1}{Z(\alpha)} \frac{\alpha^k}{g(k)!},$$

where $g(k)! = g(1) \cdots g(k)$ for $k \geq 1$, $g(0) = 1$ and $Z(\alpha)$ is the normalizing constant for which $\bar{\nu}_\alpha(\Omega_N^d) = 1$. By the linear growth of $g$ (2.1), $\bar{\nu}_\alpha$ is well defined for all $\alpha \geq 0$.

Define $\rho = \rho(\alpha)$ as the density of particles with respect to $\bar{\nu}_\alpha$, namely:

$$\rho(\alpha) = E_{\bar{\nu}_\alpha}\big[\eta(x)\big] = \sum_{k \geq 0} \frac{1}{Z(\alpha)} \frac{k\alpha^k}{g(k)!} = \frac{\alpha Z'(\alpha)}{Z(\alpha)}.$$

Again by the linear growth of $g$, $\alpha \mapsto \rho(\alpha)$ is an homeomorphism from $[0, \infty)$ to $[0, \infty)$ and the inverse function $\alpha = \alpha(\rho)$ is well defined for all $\rho \in [0, \infty)$. We define $\nu_\rho = \bar{\nu}_{\alpha(\rho)}$ and $\phi(\rho) = E_{\nu_\rho}[g(\eta(0))]$. Due to the symmetry of $p_N(x,y)$, the measure $\nu_\rho$ is invariant and reversible for this process.

2.2. **The random environment.** Now we discuss the choice of the jump rates $p_N(x,y)$. Let $(\mathcal{X}, \mathcal{F}, P)$ be a probability space and take a family $\{\theta_x; x \in \mathbb{Z}^d\}$ of $\mathcal{F}$-measurable mappings $\theta_x : \mathcal{X} \to \mathcal{X}$ such that

    i) $P(\theta_x^{-1}A) = P(A)$ for all $A \in \mathcal{F}$, $x \in \mathbb{Z}^d$.
    ii) $\theta_z \theta_{z'} = \theta_{z+z'}$ for all $z, z' \in \mathbb{Z}^d$.
    iii) If $\theta_z A = A$ for all $z \in \mathbb{Z}^d$, then $P(A) = 0$ or $1$.

In this case we say that the family $\{\theta_x\}_{x \in \mathbb{Z}^d}$ is invariant and ergodic under $P$. Let $a = (a_1, ..., a_d) : \mathcal{X} \to \mathbb{R}^d$ be a $\mathcal{F}$-measurable function such that there exists $\epsilon_0 > 0$ with

$$\epsilon_0 \leq a_i(\omega) \leq \epsilon_0^{-1} \text{ for all } \omega \in \mathcal{X} \text{ and } i = 1, ..., d. \tag{2.2}$$

Fix $\omega \in \mathcal{X}$. For each $x \in \{-1 + 1/N, -1 + 2/N, ..., 1\}^d$ and $i = 1, ..., d$, define

$$p_N(x, x + e_i/N) = p_N(x + e_i/N, x) = N^2 a_i(\theta_{Nx}\omega) \tag{2.3}$$

to which we call the *random environment*.

For each $G : \mathbb{T}_N^d \to \mathbb{R}$, define the operator $L_N G$ by

$$L_N G(x) = \sum_{y \in \mathbb{T}_N^d} p_N(x,y)\big[G(y) - G(x)\big].$$

In the space of functions $l_N(\mathbb{T}_N^d) = \{f : \mathbb{T}_N^d \to \mathbb{R}\}$, define the following norms:

$$||f||_{0,N}^2 = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} f(x)^2$$

and

$$||f||_{1,N}^2 = ||f||_{0,N}^2 + \frac{1}{N^d} \sum_{\substack{x,y \in \mathbb{T}_N^d \\ |x-y|=1/N}} N^2 \big[f(y) - f(x)\big]^2.$$

We denote by $\mathcal{L}_N^2$ the space of functions $l_N(\mathbb{T}_N^d)$ endowed with the norm $|| \cdot ||_{0,N}$ and by $\langle \cdot, \cdot \rangle_N$ the inner product in $\mathcal{L}_N^2$. Define $\mathcal{H}_{1,N}$ as the space of functions in $l_N(\mathbb{T}_N^d)$ endowed with the norm $|| \cdot ||_{1,N}$.

Denote by $\mathcal{L}^2(U^d)$ the space of square integrable functions in $U^d$ with respect to the Lebesgue measure and by $|| \cdot ||_0$ the corresponding norm in $\mathcal{L}^2(U^d)$. For each $k \geq 1$, denote by $\mathcal{H}_k(U^d)$ the Sobolev space in $U^d$ defined as the completion of $\mathcal{C}_\infty(U^d)$ under the norm

$$||f||_k^2 = \sum_{|\alpha| \leq k} ||\partial^\alpha f||_0^2,$$

where $|\alpha|$ denotes the order of the multi-index $\alpha$ and $\partial^\alpha$ is the partial derivative of order $\alpha$.

The definition of convergence of a sequence $f_N \in \mathcal{H}_{1,N}$ (or $\mathcal{L}_N^2$) to $f \in \mathcal{H}_1(U^d)$ (or $\mathcal{L}^2(U^d)$) is given in Appendix B.1.

From the homogenization theory the following holds:

**Proposition 2.1.** *Fix a typical realization of $p_N(\cdot, \cdot)$ and $\lambda > 0$. There exists a positive defined matrix $\mathcal{A}$ that depends only on the distribution of $a = (a_1, ..., a_d)$ such that for any $f_N$ and $f$ such that $f_N \in \mathcal{H}_{-1,N}$ converges strongly to $f \in \mathcal{H}_{-1}(U^d)$, $u_N$ converges weakly in $\mathcal{H}_{1,N}$ to $u$, where $u_N$ is defined as the solution of the equation*

$$\lambda u_N(x) - L_N u_N(x) = f_N(x)$$

*and $u$ is the solution of the equation*

$$\lambda u(x) - \nabla \cdot \mathcal{A} \nabla u(x) = f(x).$$

A proof of this proposition can be found in [12]. Notice that the statement of this proposition makes sense for any choice of the jump rate $p_N(x, y)$.

In order to prove the hydrodynamic limit we need this property on the jump rates $p_N(x, y)$ and for this reason we introduce the following definition.

**Definition 2.2.** *We say that a family of jump rates $\{p_N : \mathbb{T}_N^d \times \mathbb{T}_N^d \to \mathbb{R}_+\}_N$ admits homogenization, if there exist a constant $\epsilon_0 > 0$ such that $\epsilon_0 N^2 \leq p_N(x, y) \leq \epsilon_0^{-1} N^2$ and a matrix $\mathcal{A}$ such that for any $f \in \mathcal{H}_{-1}(U^d)$ smooth enough there exists a sequence $f_N$ converging strongly in $\mathcal{H}_{-1,N}$ to $f$ such that the solution $u_N \in \mathcal{H}_{1,N}$ of the equation*

$$\lambda u_N(x) - L_N u_N(x) = f_N(x)$$

*converges weakly in $\mathcal{H}_{1,N}$ to the solution $u \in \mathcal{H}_1(U^d)$ of*

$$\lambda u(x) - \nabla \cdot \mathcal{A} \nabla u(x) = f(x).$$

*In this case, we say that the matrix $\mathcal{A}$ is the $\Gamma$-limit of $L_N$.*

For our purposes, $f$ will be smooth enough if it is three times countinuously differentiable.

**Remark 2.3.** *By the theory of $\Gamma$-convergence, the matrix $\mathcal{A}$ satisfies the coerciveness assumption $\epsilon_0 |\xi|^2 \leq \sum_{ij} \xi_i \xi_j \mathcal{A}_{ij} \leq \epsilon_0^{-1} |\xi|^2$ for all vectors $\xi \in \mathbb{R}^d$. In the previous definition, nothing excludes the possibility of the matrix $\mathcal{A}$ to be a function of the position $x \in U^d$. See [11] for a one-dimensional example on which the $\Gamma$-limit of $L_N$ is not constant in space.*

2.3. **Hydrodynamic limit.** Fix a function $\rho_0 : U^d \to \mathbb{R}_+$. A family of measures $\{\mu_N\}_{N \geq 1}$ in $\Omega_N^d$ is said to be associated to the profile $\rho_0$ if for any function $G \in \mathcal{C}(U^d)$ and any $\epsilon > 0$,

$$\lim_{N \to \infty} \mu_N \left( \eta \in \Omega_N^d; \left| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) G(x) - \int \rho_0(x) G(x) dx \right| > \epsilon \right) = 0.$$

Here and in the sequel, denote by $E_{\mu_N}$ the expectation with respect to $\mu_N$ and by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$, the distribution of the process $\eta_t$ starting from $\mu_N$ in $\mathcal{D}([0, T], \Omega_N^d)$. We follow the evolution of the process $\eta_t$ in a finite time interval $[0, T]$ in order to avoid uninteresting complications due to the lack of compactness of $[0, \infty)$.

Let $\rho > 0$ be a fixed density. The entropy of $\mu_N$ with respect to $\nu_\rho$ is defined by

$$H_N(\mu_N | \nu_\rho) = \begin{cases} \int \frac{d\mu_N}{d\nu_\rho} \log \frac{d\mu_N}{d\nu_\rho} d\nu_\rho, \text{ if } \mu_N \ll \nu_\rho \\ +\infty, \text{ otherwise} \end{cases},$$

where for two measures $\mu$ and $\nu$, $\mu \ll \nu$ means that the measure $\nu$ is absolutely continuous with respect to $\mu$.

We introduce a partial order $\preceq$ in $\Omega_N^d$ as follows. For $\eta$, $\eta'$ in $\Omega_N^d$, we say that $\eta \preceq \eta'$ if $\eta(x) \leq \eta'(x)$ for every $x \in \mathbb{T}_N^d$. Once there is a partial order in the space state $\Omega_N^d$, we can introduce a partial order in the space of measures in $\Omega_N^d$. We say that $\mu_N$ is stochastically dominated by $\nu_\rho$ (also denoted by $\mu_N \preceq \nu_\rho$) if there exists a measure $\bar{\mu}$ in $\Omega_N^d \times \Omega_N^d$ such that:

i) For all $\eta \in \Omega_N$, $\bar{\mu}(\eta, \Omega_N) = \mu_N(\eta)$.
ii) For all $\eta \in \Omega_N$, $\bar{\mu}(\Omega_N, \eta) = \nu_\rho(\eta)$.
iii) The set $\{(\eta, \eta'); \eta \preceq \eta'\}$ has full measure under $\bar{\mu}$.
In this case we say that $\bar{\mu}$ is a *coupling* of $\mu_N$ and $\nu_{\bar{\rho}}$.

**Theorem 2.4.** *Let $\rho_0 : U^d \to \mathbb{R}$ be a bounded profile, and let $\{\mu_N\}_{N \geq 1}$ be a sequence of measures in $\Omega_N^d$ associated to the profile $\rho_0$. Assume that the interaction rate $g(\cdot)$ is non-decreasing and has linear growth (see Section 2.1). Suppose that there exist constants $K_0$ and $\bar{\rho}$ such that $H(\mu_N | \nu_{\bar{\rho}}) \leq K_0 N^d$ and $\mu_N \preceq \nu_{\bar{\rho}}$ for every $N$ large enough. Suppose also that the jump rates $p_N(x, y)$ admit homogenization with homogenized matrix $\mathcal{A}$.*

*Then, for every $t \leq T$, every continuous function $G : U^d \to \mathbb{R}$ and every $\delta > 0$,*

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \left| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} G(x) \eta_t(x) - \int G(u) \rho(t, u) du \right| > \delta \right] = 0,$$

*where $\rho(t, u)$ is the unique weak solution of the hydrodynamic equation*

$$\begin{cases} \partial_t \rho = \nabla \cdot \left( \mathcal{A} \nabla \phi(\rho) \right) \\ \rho(0, \cdot) = \rho_0(\cdot). \end{cases} \tag{2.4}$$

In the sake of completeness we introduce the definition of weak solutions of equation (2.4).

**Definition 2.5.** *Fix a bounded profile $\rho_0 : U^d \to \mathbb{R}$. A bounded function $\rho : [0, T] \times U^d \to \mathbb{R}$ is a weak solution of equation (2.4) if for every function $G : [0, T] \times U^d \to \mathbb{R}$ of class $C^{1,2}([0, T] \times U^d)$,*

$$\int_0^t \int_{U^d} \left\{ \rho(s, u) \partial_s G(s, u) + \phi(\rho(s, u)) \nabla \cdot \mathcal{A} \nabla G(s, u) \right\} du ds$$

$$+ \int_{U^d} \rho_0(u) G(0, u) du = \int_{U^d} \rho(T, u) G(T, u) du. \tag{2.5}$$

Let $\mathcal{M}_+$ be the set of positive Radon measures in $U^d$. The empirical measure $\pi_t^N$ is defined as the process in $\mathcal{D}([0, T], \mathcal{M}_+)$ given by

$$\pi_t^N(du) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x) \delta_x(du),$$

where $\delta_x$ is the Dirac distribution at $x$.

For $G : U^d \to \mathbb{R}$ continuous, define $\pi_t^N(G) = \int G(u) \pi_t^N(du)$. The statement of Theorem 2.4 is equivalent to say that under $\mathbb{P}_{\mu_N}$ the random variables $\pi_t^N(G)$ converge in probability to $\int G(u) \rho(t, u) du$ for every $G$ continuous and every $t \in [0, T]$. We will prove a stronger result for $\pi_t^N$:

**Theorem 2.6.** *Under the hypothesis of Theorem 2.4, $\pi_t^N$ converges in distribution in $\mathcal{D}([0, T], \mathcal{M}_+)$ to the trajectory $\rho(t, u) du$.*

**Remark 2.7.** *Since $\rho(t, u) du$ is a deterministic element of $\mathcal{D}([0, T], \mathcal{M}_+)$, the convergence in distribution of $\pi_t^N$ implies its convergence in probability, from which Theorem 2.4 follows.*

2.4. **Equilibrium fluctuations.** Now we state a central limit theorem for the empirical measure, starting from an equilibrium measure $\nu_\rho$. Fix $\rho > 0$ and denote by $\mathcal{S}(U^d)$ the Schwartz space of infinitely differentiable functions in $U^d$.

Denote by $\mathcal{Y}^N$ the density fluctuation field, a linear functional acting on functions $G \in \mathcal{S}(U^d)$ as

$$\mathcal{Y}_t^N(G) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} G(x)(\eta_t(x) - \rho). \tag{2.6}$$

Notice that

$$\mathcal{Y}_t^N(G) = N^{d/2} \left( \int G(u) \pi_t^N(du) - \rho \int G(u) du \right).$$

In this way we have defined a process in $\mathcal{D}([0,T], \mathcal{S}'(U^d))$, where $\mathcal{S}'(U^d)$ is the space of tempered distributions, which corresponds to the dual of the Schwartz space $\mathcal{S}(U^d)$.

**Theorem 2.8.** *Consider the fluctuation field $\mathcal{Y}^N_\cdot$ defined above. Assume that the interaction rate $g(\cdot)$ has linear growth and that the jump rates admit homogenization with homogenized matrix $\mathcal{A}$.*

*Then, for every $t_1, ..., t_k \in [0,T]$ and every $G_1, ..., G_k \in \mathcal{S}(U^d)$, the vector $(\mathcal{Y}^N_{t_1}(G_1), ..., \mathcal{Y}^N_{t_k}(G_k))$ converges in distribution to $(\mathcal{Y}_{t_1}(G_1), ..., \mathcal{Y}_{t_k}(G_k))$, where $\mathcal{Y}_t$ is the generalized Ornstein-Uhlenbeck process of characteristics $\phi'(\rho)\nabla \cdot \mathcal{A}\nabla$ and $\sqrt{\phi(\rho)\mathcal{A}}\nabla$.*

## 3. Proof of Theorem 2.4

By remark 2.7, in order to prove Theorem 2.4 it is enough to prove Theorem 2.6. The proof of Theorem 2.6 follows the standard lines of the proof of hydrodynamic limit by the entropy method for interacting particle systems. The route to proceed is the following:

First we show that the distributions of $\pi^N_t$ in $\mathcal{D}([0,T], \mathcal{M}_+)$ form a tight sequence. Then we prove that the limit points of $\pi^N_t$ are concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure in $U^d$ with a bounded density. Finally, we prove that these limit points are concentrated on weak solutions of the hydrodynamic equation (2.4). By the uniqueness of these weak solutions on the space of bounded functions we conclude that $\pi^N_t$ has a unique limit point, concentrated on the trajectory with density $\rho(t,u)$, where $\rho(t,u)$ is the weak solution of equation (2.4). Since the topology of convergence in distribution is metrizable, we conclude that the whole sequence $\pi^N_t$ converges to $\rho(t,u)du$.

Unfortunately, this plan cannot be accomplished directly for $\pi^N_t$, but for another auxiliary process, the *corrected empirical measure*, that we define below.

Let $\lambda > 0$ be fixed. A function $G : U^d \to \mathbb{R}$ is said to be regular if the function $f_N \in \mathcal{L}^2_N$ defined by $f_N(x) = \lambda G(x) - \nabla \cdot \mathcal{A}\nabla G(x)$ converges strongly in $\mathcal{H}_{-1,N}$ to $\lambda G - \nabla \cdot \mathcal{A}\nabla$. Notice that a sufficient condition for $G$ to be regular, is $G \in \mathcal{C}^3(U^d)$, where $\mathcal{C}^3(U^d)$ denotes the space of three times continuously differentiable functions on $U^d$.

Let $G : U^d \to \mathbb{R}$ be regular. For each $N \geq 1$, define $R^\lambda G(x) = \lambda G(x) - \nabla \cdot \mathcal{A}\nabla G(x)$ and $G^\lambda_N : \mathbb{T}^d_N \to \mathbb{R}$ as the solution of

$$\lambda G^\lambda_N(x) - L_N G^\lambda_N(x) = R^\lambda G(x). \tag{3.1}$$

By Lemma B.1, the following estimates hold:

$$||G^\lambda_N||_{0,N} \leq \lambda^{-1}||R^\lambda G||_{0,N}, \tag{3.2}$$

$$\frac{1}{N^d} \sum_{x,y \in \mathbb{T}_N} p_N(x,y)[G^\lambda_N(y) - G^\lambda_N(x)]^2 \leq \lambda^{-1}||R^\lambda G||^2_{0,N} \tag{3.3}$$

and

$$||G^\lambda_N||_{\infty,N} \leq \lambda^{-1}||R^\lambda G||_{\infty,N}. \tag{3.4}$$

We define the corrected empirical measure $\pi^{N,\lambda}_t$ by

$$\pi^{N,\lambda}_t(G) = \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta_t(x)G^\lambda_N(x).$$

Notice that $\pi^{N,\lambda}_t(G)$ is defined only for $G$ regular, so $\pi^{N,\lambda}_t$ is not a well defined process in $\mathcal{D}([0,T], \mathcal{M}_+)$. Lemma B.1 shows that $\pi^{N,\lambda}_t$ is a well defined process in the Sobolev space $\mathcal{H}^{-k}(U)$ for $k \geq 3$. However, this point will not be relevant for our proof of Theorem 2.6.

Since $\mathcal{M}_+$ is separable and the vague topology in $\mathcal{M}_+$ is metrizable, in order to prove tightness of $\pi^N_t$ in $\mathcal{D}([0,T], \mathcal{M}_+)$, it is enough to show tightness of $\pi^N_t(G)$ in $\mathcal{D}([0,T], \mathbb{R})$ for $G$ in a dense subset of the set $\mathcal{C}(U^d)$ of continuous functions in $U^d$. Therefore, it is enough to prove tightness of $\pi^N_t(G)$ for $G$ regular.

By Dynkin's formula,

$$\mathcal{M}^N_t(G) = \pi^{N,\lambda}_t(G) - \pi^{N,\lambda}_0(G) - \int_0^t \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} g(\eta_s(x))L_N G^\lambda_N(x)ds \tag{3.5}$$

is a martingale of quadratic variation given by

$$\langle \mathcal{M}_t^N(G) \rangle = \int_0^t \frac{1}{N^{2d}} \sum_{x,y \in \mathbb{T}_N^d} g\big(\eta_s(x)\big) p_N(x,y) \big[G_N^\lambda(y) - G_N^\lambda(x)\big]^2 ds.$$

We claim that $\mathcal{M}_t^N(G)$ goes to 0 as $N \to \infty$ in $\mathcal{L}^2(\mathbb{P}_{\mu_N})$. In fact,

$$\mathbb{E}_{\mu_N}\big[\mathcal{M}_t^N(G)^2\big] = \mathbb{E}_{\mu_N}\big[\langle \mathcal{M}_t^N(G) \rangle\big]$$

$$= \int_0^t \frac{1}{N^{2d}} \sum_{x,y \in \mathbb{T}_N^d} \mathbb{E}_{\mu_N}\big[g(\eta_s(x))\big] p_N(x,y) \big[G_N^\lambda(y) - G_N^\lambda(x)\big]^2 ds$$

$$\leq \frac{t\phi(\bar{\rho})}{N^d} \lambda^{-1} ||R^\lambda G||_{0,N}^2 \xrightarrow{N \to \infty} 0.$$

In order to obtain this last bound, we have used the estimate (3.3), the fact that $\mu_N$ is stochastically dominated by $\nu_{\bar{\rho}}$ and Proposition A.1.

To prove tightness for the martingale $\mathcal{M}_t^N(G)$, we use the following criterion, due to Aldous:

**Proposition 3.1.** *A sequence of probability measures $\{P_N\}_N$ in $\mathcal{D}([0,T],\mathbb{R})$ is tight if*

(i) *For all $0 \leq t \leq T$ and for all $\epsilon > 0$ there exists a finite constant $A$ such that $\sup_N P_N(|x_t| > A) < \epsilon$,*

(ii) *For all $\epsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \sup_{\substack{\tau \in \mathcal{T} \\ \beta \leq \delta}} P_N(|x_{\tau+\beta} - x_\tau| > \delta) = 0 \,,$$

*where $\mathcal{T}$ is the set of stopping times with respect to the canonical filtration, bounded by $T$.*

A proof of this lemma can be found in [9]. In our case, condition $i$) follows from the fact that $\mathcal{M}_t^N(G)$ converges to 0 in $\mathcal{L}^2(\mathbb{P}_{\mu_N})$ and Tchebyshev's inequality. On the other hand, by Doob's optimal sampling theorem, we have that

$$\mathbb{P}_{\mu_N}\big[|\mathcal{M}_{\tau+\beta}^N(G) - \mathcal{M}_\tau^N(G)| > \epsilon\big] \leq$$

$$\leq \frac{1}{\epsilon^2} \mathbb{E}_{\mu_N}\Big[\langle \mathcal{M}_{\tau+\beta}^N(G) \rangle - \langle \mathcal{M}_\tau^N(G) \rangle\Big]$$

$$\leq \frac{1}{\epsilon^2} \mathbb{E}_{\mu_N}\Big[\int_\tau^{\tau+\beta} \frac{1}{N^{2d}} \sum_{x,y \in \mathbb{T}_N^d} g\big(\eta_s(x)\big) p_N(x,y) \big[G_N^\lambda(y) - G_N^\lambda(x)\big]^2 ds\Big]$$

$$\leq \frac{\beta C(G, c_0, \lambda, \epsilon_0)}{N^{d-2}} E_{\mu_N}\Big[\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)\Big].$$

In this last bound we have used the conservation of the number of particles, the estimate (3.3), and the uniform bound for $p_N(x,y)$. Since the expected initial density of particles is bounded by $\bar{\rho}$, condition $ii$) follows.

Notice that the integral term in (3.5) can be written as

$$\int_0^t \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} g\big(\eta_s(x)\big) \big[\lambda G_N^\lambda(x) - R^\lambda G(x)\big] ds. \tag{3.6}$$

We see that

$$\mathbb{E}_{\mu_N}\Big[\sup_{|s-t| \leq \delta} \Big|\int_s^t \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} g\big(\eta_{t'}(x)\big) \big[\lambda G_N^\lambda(x) - R^\lambda G(x)\big] dt'\Big|^2\Big] \leq$$

$$\leq \delta C(G,g) \mathbb{E}_{\mu_N}\Big[\sup_{t \in [0,T]} ||\eta_t||_{0,N}^2\Big],$$

that goes to 0 as $\delta \to 0$, uniformly in $N$ by Lemma A.2. Therefore, by Arzelà-Ascoli criterion, the integral terms in (3.5) form a tight sequence in $\mathcal{D}([0,T],\mathbb{R})$ and their limit points are concentrated on continuous trajectories. By equation (3.5) the sequence $\pi_t^{N,\lambda}(G)$ is tight in $\mathcal{D}([0,T],\mathbb{R})$. On the other hand, since $\mathcal{M}_t^N(G)$ goes to 0 in $\mathcal{L}^2(\mathbb{P}_{\mu_N})$, any limit point of $\mathcal{M}_t^N(G)$ has null

finite-dimensional distributions. Therefore, $\mathcal{M}_t^N(G)$ converges to 0 in distribution as a process in $\mathcal{D}([0,T],\mathbb{R})$. Consequently, the limit points of $\pi_t^{N,\lambda}(G)$ are concentrated on continuous trajectories.

Notice now that

$$\mathbb{E}_{\mu_N}\Big[\sup_{t\in[0,T]}\big|\pi_t^{N,\lambda}(G)-\pi_t^N(G)\big|^2\Big]\leq ||G_N^\lambda-G||_{0,N}^2\mathbb{E}_{\mu_N}\Big[\sup_{t\in[0,T]}||\eta_t||_{0,N}^2\Big].$$

By Proposition 2.1, $||G_N^\lambda-G||_{0,N}$ converges to 0 as $N\to\infty$, and by Lemma A.2 $\mathbb{E}_{\mu_N}\big[\sup_t||\eta_t||_{0,N}^2\big]$ is bounded in $N$. Therefore, $\sup_t|\pi_t^{N,\lambda}(G)-\pi_t^N(G)|\to 0$ in $\mathcal{L}^2(\mathbb{P}_{\mu_N})$. A simple $\varepsilon/3$ argument allows us to obtain from this result that $\pi_t^N(G)$ is also tight in $\mathcal{D}([0,T],\mathbb{R})$ and that $\pi_t^N(G)$ and $\pi_t^{N,\lambda}$ have the same limit points. Since the set of regular functions is dense in $\mathcal{C}(U^d)$, this ends the proof of tightness for $\pi_t^N$ in $\mathcal{D}([0,T],\mathcal{M}_+)$.

Let $\pi_t$ be a limit point of $\pi_t^N$, and let $Q$ be its distribution in $\mathcal{D}([0,T],\mathcal{M}_+)$. For any positive function $G\in\mathcal{C}(U^d)$,

$$\begin{aligned}Q(\pi_t(G)>M)&\leq\liminf_{N\to\infty}Q(\pi_t^N(G)>M)\\&=\liminf_{N\to\infty}\mu_N\big(N^{-d}\sum_{x\in\mathbb{T}_N^d}\eta_t(x)G(x)>M\big)\\&\leq\liminf_{N\to\infty}\nu_{\bar\rho}\big(N^{-d}\sum_{x\in\mathbb{T}_N^d}\eta(x)G(x)>M\big)\\&\leq\mathbf{1}\Big(\int G(u)du>M/\bar\rho\Big).\end{aligned}$$

Here we have used once more, the fact of $\mu_N$ being stochastically dominated by an invariant measure $\nu_{\bar\rho}$ and Proposition A.1.

Therefore, if $0\leq G\leq 1$ then $Q(\pi_t(G)>2^d\bar\rho)=0$. By the dominated convergence theorem, for every closed $B\subseteq U^d$ it holds that $Q(\pi_t(B)>\bar\rho\Lambda(B))=0$, where $\Lambda$ denotes the Lebesgue measure in $U^d$. In particular, the process $\pi_t$ is concentrated on measures absolutely continuous with respect to $\Lambda$.

Let $\pi(t,u)$ be the density of $\pi_t$ with respect to $\Lambda$. The same estimates prove that $\pi(t,u)$ is bounded by $\bar\rho$ in $[0,T]\times U^d$.

Notice that $R^\lambda G$ is a smooth function, but $\lambda G_N^\lambda(x)$ it is not smooth. However,

$$\mathbb{E}_{\nu_N}\Big[\big|\int_0^t\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}g\big(\eta_s(x)\big)\big[G_N^\lambda(x)-G(x)\big]ds\big|^2\Big]\leq$$

$$\begin{aligned}&\leq c_0^{-2}t\int_0^t\mathbb{E}_{\mu_N}\Big[\Big(\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\eta_s(x)\big|G_N^\lambda(x)-G(x)\big|\Big)^2\Big]ds\\&\leq c_0^{-2}t^2\int\Big(\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}\eta(x)\big|G_N^\lambda(x)-G(x)\big|\Big)^2d\nu_{\bar\rho}(\eta)\\&\leq c_0^{-2}t^2\int\eta(0)^2d\nu_{\bar\rho}||G_N^\lambda-G||_{0,N}^2\xrightarrow{N\to\infty}0.\end{aligned}$$

In the previous we used Schwarz inequality together with the translation invariance of $\nu_{\bar\rho}$. As a consequence,

$$\mathcal{M}_t^N(G)=\pi_t^{N,\lambda}(G)-\pi_0^{N,\lambda}(G)-\int_0^t\frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d}g\big(\eta_s(x)\big)\nabla\cdot\mathcal{A}\nabla G(x)ds\qquad(3.7)$$

plus a rest vanishing in $\mathcal{L}^2(\mathbb{P}_{\mu_N})$ as $N\to\infty$. The next result will allow us to write the integral term (3.6) as a function of $\pi_t^N$ plus a vanishing term as $N\to\infty$.

**Proposition 3.2.** *(Replacement Lemma)*

*For every $\delta > 0$,*

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \mathbb{P}_{\mu_N} \Big[ \int_0^T \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} V_{\varepsilon N}(\eta_s, x) ds > \delta \Big] = 0,$$

*where*

$$V_l(\eta, x) = \Big| \frac{1}{(2l+1)^d} \sum_{|y| \le l} g\big(\eta(x+y)\big) - \phi\big(\eta^l(x)\big) \Big|$$

*and*

$$\eta^l(x) = \frac{1}{(2l+1)^d} \sum_{|y| \le l} \eta(x+y).$$

The proof of this proposition is the same as the one presented in Chapter 5 of [9], so we omit it. Using this proposition, we see that for any continuous function $G : U^d \to \mathbb{R}$,

$$\int_0^t \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \Big\{ g\big(\eta_s(x)\big) - \phi\big(\eta_s^{\varepsilon N}(x)\big) \Big\} G(x) ds \to 0$$

in $\mathbb{P}_{\mu_N}$-probability as $N \to \infty$ and then $\varepsilon \to 0$. On the other hand, since $\eta_s^{\varepsilon N}(x) = \pi_s^N(\mathbf{1}(|u-x| \le \varepsilon))$, we conclude that

$$\int_0^t \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} g\big(\eta_s(x)\big) G(x) ds \to \int_0^t ds \int \phi(\pi(s,u)) G(u) du$$

in $\mathbb{P}_{\mu_N}$-probability. Since $\mathcal{M}_t^N(G)$ converges to 0, taking $N \to \infty$ in equation (3.7) we obtain that

$$0 = \int \pi(t,u) G(u) du - \int \rho_0(u) G(u) du - \int_0^t \int \phi(\pi(s,u)) \nabla \cdot \mathcal{A} \nabla G(u) du ds \qquad (3.8)$$

for every $G$ regular. Approximating a twice-differentiable function $G$ by regular functions $G_n$ in the uniform topology, we extend this identity to functions $G \in \mathcal{C}^2(U^d)$.

Let $G : [0,T] \times U^d \to \mathbb{R}$ be of class $\mathcal{C}^{1,2}$. Take the partition $\{t_i = Ti/n; i = 0, ..., n\}$ of the interval $[0,T]$ and define $G_n : [0,T] \times U^d \to \mathbb{R}$ by

$$G_n(t,u) = \frac{n(t_i - t_{i-1})}{T} G(t_{i-1}, u) + \frac{n(t_i - t)}{T} G(t_i, u),$$

for $t \in [t_{i-1}, t_i]$. In general, for a piecewise-differentiable path $G : [0,T] \to \mathcal{L}_N^2$,

$$\pi_t^N(G_t) - \pi_t^N(G_0) - \int_0^t \Big\{ \pi_s^N(\partial_s G_s) + \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} g\big(\eta_s(x)\big) L_N G_s(x) \Big\} ds$$

is a martingale of quadratic variation

$$\int_0^t \frac{1}{N^{2d}} \sum_{x,y \in \mathbb{T}_N^d} p_N(x,y) g\big(\eta_s(x)\big) \big[ G_s(y) - G_s(x) \big]^2 ds.$$

Repeating the arguments in the proof of equation (3.8) for $G_n$, we conclude that

$$0 = \int \pi(t,u) G_n(t,u) du - \int \rho_0(u) G_n(0,u) du$$
$$- \int_0^t \int \Big\{ \pi(s,u) \partial_s G_n(s,u) + \phi(\pi(s,u)) \nabla \cdot \mathcal{A} \nabla G_n(s,u) du ds.$$

Taking the limit as $n$ goes to $\infty$, we obtain that

$$0 = \int \pi(t,u) G(t,u) du - \int \rho_0(u) G(0,u) du -$$
$$- \int_0^t \int \Big\{ \pi(s,u) \partial_s G(s,u) + \phi(\pi(s,u)) \nabla \cdot \mathcal{A} \nabla G(s,u) du ds$$

for every $G : [0,T] \times U^d \to \mathbb{R}$ of class $\mathcal{C}^{1,2}$. This is the weak form of the hydrodynamic equation (2.4), see (2.5). Since equation (2.4) has at most one weak solution, we conclude that $\pi(t,u) = \rho(t,u)$ $Q - a.s.$, which ends the proof of Theorem 2.6.

## 4. Proof of Theorem 2.8

Denote by $\mathcal{Q}^N$ the distribution in $\mathcal{D}([0,T], \mathcal{S}'(U^d))$ induced by the process $\mathcal{Y}_t^N$ and $\nu_\rho$. The standard proof of equilibrium fluctuations cannot be accomplished for the density field $\mathcal{Y}_\cdot^N$. In order to overcome this problem we introduce as before, the corrected density fluctuation field defined on functions $G \in \mathcal{S}(U^d)$ by

$$\mathcal{Y}_t^{N,\lambda}(G) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_{N^d}} G_N^\lambda(x)(\eta_t(x) - \rho),$$

where $G_N^\lambda$ is the solution of equation (3.1).

For $t \geq 0$, let $\mathcal{F}_t$ be the $\sigma$-algebra on $\mathcal{D}([0,T], \mathcal{S}'(U^d))$ generated by $\mathcal{Y}_s(H)$ for $s \leq t$ and $H$ in $\mathcal{S}(U^d)$ and set $\mathcal{F} = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$. Denote by $\mathcal{Q}_N^\lambda$ the distribution on $\mathcal{D}([0,T], \mathcal{S}'(U^d))$ induced by the corrected density fluctuation field $\mathcal{Y}_\cdot^{N,\lambda}$ and $\nu_\rho$.

We make use of the following result, which permits to identify the limiting process:

**Proposition 4.1.**

*There exists a unique process $\mathcal{Y}_t$ in $\mathcal{C}([0,T], \mathcal{S}'(U^d))$ such that:*

i) *For every function $G \in \mathcal{S}(U^d)$,*

$$M_t(G) = \mathcal{Y}_t(G) - \mathcal{Y}_0(G) - \int_0^t \mathcal{Y}_s(\phi'(\rho)\nabla \cdot \mathcal{A}\nabla G)ds$$

   *and*

$$(M_t(G))^2 - \phi(\rho)t \int_{U^d} \nabla G(u) \cdot \mathcal{A}\nabla G(u)du$$

   *are $\mathcal{F}_t$-martingales.*

ii) *$\mathcal{Y}_0$ is a Gaussian field of mean zero and covariance given by*

$$E[\mathcal{Y}_0(G)\mathcal{Y}_0(H)] = \chi(\rho) \int_{U^d} G(u)H(u)du, \tag{4.1}$$

*where $\chi(\rho) = \boldsymbol{Var}(\eta(0), \nu_\rho)$ and $G$, $H \in \mathcal{S}(U^d)$. The process $\mathcal{Y}_t$ is called the generalized Ornstein-Ulenbeck process of mean zero and characteristics $\phi'(\rho)\nabla \cdot \mathcal{A}\nabla$, $\sqrt{\phi(\rho)\mathcal{A}}\nabla$.*

Theorem 2.8 is a consequence of the following result about the corrected fluctuation field.

**Theorem 4.2.** *Let $\mathcal{Q}$ be the probability measure on $\mathcal{C}([0,T], \mathcal{S}'(U^d))$ corresponding to the stationary generalized Ornstein-Uhlenbeck process of mean zero and characteristics $\phi'(\rho)\nabla \cdot \mathcal{A}\nabla$, $\sqrt{\phi(\rho)\mathcal{A}}\nabla$. Then the sequence $\{\mathcal{Q}_N^\lambda\}_{N \geq 1}$ converges weakly to the probability measure $\mathcal{Q}$.*

Before we enter into the proof of this theorem, we prove Theorem 2.8 from it. In fact, it is enough to show that

$$\lim_{N \to \infty} \mathbb{E}_{\nu_\rho}[(\mathcal{Y}_t^N(G) - \mathcal{Y}_t^{N,\lambda}(G))^2] = 0 \tag{4.2}$$

for any $t \in [0,T]$, $G \in \mathcal{S}(U^d)$. But this is immediate from the fact that $G_N^\lambda$ converges to $G$ in $\mathcal{L}_N^2$ and the independence of $\eta(x)$, $\eta(y)$ for $x \neq y$ under the invariant measure $\nu_\rho$.

In order to prove Theorem 4.2, we need to verify that the sequence of probability measures $\{\mathcal{Q}_N^\lambda\}_{N \geq 1}$ is tight and to characterize the limit field. Then we show that the limit field is equal in distribution to $\mathcal{Y}_t$ using its characterization in terms of the martingale problem (Proposition 4.1).

Fix a smooth function $G \in \mathcal{S}(U^d)$. By Dynkin's formula,

$$M_t^{N,\lambda}(G) = \mathcal{Y}_t^{N,\lambda}(G) - \mathcal{Y}_0^{N,\lambda}(G) - \int_0^t \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} g(\eta_s(x))L_N G_N^\lambda(x)ds \tag{4.3}$$

is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(\eta_s, s \leq t)$ whose quadratic variation is given by

$$\langle M_t^{N,\lambda}(G)\rangle = \int_0^t \frac{1}{N^d} \sum_{x,y\in\mathbb{T}_N^d} g(\eta_s(x))p_N(x,y)\big[G_N^\lambda(y) - G_N^\lambda(x)\big]^2 ds.$$

At first, we establish the limit of the quadratic variation. Notice that in the previous formula we can replace $g(\eta_s(x))$ by $\phi(\rho)$, since

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \frac{1}{N^d} \sum_{x,y\in\mathbb{T}_N^d} \{g(\eta_s(x)) - \phi(\rho)\}p_N(x,y)\big[G_N^\lambda(y) - G_N^\lambda(x)\big]^2 ds\Big)^2\Big] \le$$

$$\le \frac{t^2}{N^{2d}}\mathbf{Var}(g,\nu_\rho) \sum_{x,y\in\mathbb{T}_N^d} p_N(x,y)\big[G_N^\lambda(y) - G_N^\lambda(x)\big]^2 \times$$

$$\times \sup_{x\in\mathbb{T}_N^d} \sum_{y\in\mathbb{T}_N^d} p_N(x,y)\big[G_N^\lambda(y) - G_N^\lambda(x)\big]^2$$

$$\le \frac{Ct^2}{N^{d-2}}\|R^\lambda G\|_{0,N}^2 \|R^\lambda\|_{\infty,N}.$$

For dimension $d \ge 3$, this last expression goes to 0 as $N \to \infty$. In order to cover the case $d = 2$, we can use Theorem (1.31) of [15], expression (1.32) with $t = s$ and $\alpha = 1/N$ and take the Laplace transform of equation (1.32), to obtain a sharper estimate for $G_N^\lambda(x) - G_N^\lambda(y)$. In this case, we obtain that the last line is bounded by $N^{-(d-2+2\sigma)}$, for some $\sigma > 0$. As a consequence, for any $d \ge 2$, the quadratic variation can be written as

$$\int_0^t \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \phi(\rho) \sum_{y\in\mathbb{T}_N^d} p_N(x,y)\big[G_N^\lambda(y) - G_N^\lambda(x)\big]^2 ds,$$

plus a vanishing term in the $\mathcal{L}^2(\mathbb{P}_{\nu_\rho})$-norm. Using the convergence of $G_N^\lambda$ in $\mathcal{L}_N^2$ and the resolvent estimates in the proof of Lemma B.1, this last integral converges to

$$t\phi(\rho)\int_U \nabla G(u) \cdot \mathcal{A}\nabla G(u)du,$$

as $N$ goes to $\infty$.

Now we study the limit of the martingale $M_t^{N,\lambda}(G)$, see expression (4.3). Since $\sum_{x\in\mathbb{T}_N^d} L_N G_N^\lambda(x) = 0$, we can rewrite the integral part of the martingale as

$$\int_0^t \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} \{g(\eta_s(x)) - \phi(\rho)\}L_N G_N^\lambda(x)ds.$$

On the other hand, since $G_N^\lambda$ is the solution of equation (3.1), the last integral can be written as

$$\int_0^t \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} \{g(\eta_s(x)) - \phi(\rho)\}\Big\{\lambda G_N^\lambda(x) - \lambda G(x) + \nabla \cdot \mathcal{A}\nabla G(x)\Big\}ds.$$

Our aim now consists in showing that it is possible to write the integral part of the martingale as the integral of a function of the density fluctuation field plus a term that goes to zero in $\mathcal{L}^2(\mathbb{P}_{\nu_\rho})$. The first result needed to proceed in that direction is the following:

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} \{g(\eta_s(x)) - \phi(\rho)\}\big[G_N^\lambda(x) - G(x)\big]ds\Big)^2\Big] \le$$

$$\le C\mathbf{Var}(g,\nu_\rho)\|G_N^\lambda - G\|_{0,N}^2 \xrightarrow{N\to\infty} 0.$$

The second one is known as the Boltzmann-Gibbs principle. Here we have the need to introduce some definitions. Take a function $f : \chi \times \Omega_N^d \to \mathbb{R}$. For each $\omega \in \chi$ and each $x \in \mathbb{T}_N^d$, define

$$f(x,\eta) = f(x,\eta,\omega) =: f(\theta_{xN}\omega, \tau_x\eta),$$

where $\tau_x\eta$ is the shift of $\eta$ to $x$: $\tau_x\eta(y) = \eta(x+y)$. Notice that we do not include explicitly the dependence of $f(x,\eta)$ in $\omega$, since in our setting $\omega$ is fixed.

**Definition 4.3.** *We say that $f$ is local if there exists $R > 0$ such that $f(\omega, \eta)$ depends only on the values of $\eta(y)$ for $|y| \leq R$. In this case, we can consider $f$ as defined in all the spaces $\chi \times \Omega_N^d$ for $N \geq R$.*

**Definition 4.4.** *We say that $f$ is Lipschitz if there exists $c = c(\omega) > 0$ such that for all $x$, $|f(\omega, \eta) - f(\omega, \eta')| \leq c|\eta(x) - \eta'(x)|$ for any $\eta$, $\eta'$ such that $\eta(y) = \eta'(y)$ for any $y \neq x$. If the constant $c$ can be chosen independently of $\omega$, we say that $f$ is uniformly Lipschitz.*

**Theorem 4.5.** *(Boltzmann-Gibbs principle)*
*For every $G \in \mathcal{S}(U^d)$, every $t > 0$ and every local, uniformly Lipschitz function $f : \chi \times \Omega_N^d \to \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\rho}\Big[\int_0^t \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} G(x)V_f(x,\eta_s)ds\Big]^2 = 0 \tag{4.4}$$

*where*

$$V_f(x,\eta) = f(x,\eta) - E_{\nu_\rho}\big[f(x,\eta)\big] - \partial_\rho E\Big[\int f(x,\eta)d\nu_\rho(\eta)\Big]\big(\eta(x) - \rho\big).$$

*Here $E$ denotes the expectation with respect to $P$, the random environment.*

In order to simplify the exposition, the proof of this last result is postponed to the next section. As we need to write the integral part of the martingale $M_t^{N,\lambda}(G)$ in terms of the density fluctuation field, by using the first result stated above we are able to write the integral part of the martingale as

$$\int_0^t \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} \big\{g\big(\eta_s(x)\big) - \phi(\rho)\big\}\nabla \cdot \mathcal{A}\nabla G(x)ds$$

plus a term that converges to $0$ in the $\mathcal{L}^2(\mathbb{P}_{\nu_\rho})$-norm. The replacement of the function $g(\eta_s) - \phi(\rho)$ by $\phi'(\rho)[\eta_s(x) - \alpha]$ in the last integral, is possible thanks to the Boltzmann-Gibbs principle. Doing so, the integral part of the martingale can be written as

$$M_t^{N,\lambda}(G) = \mathcal{Y}_t^{N,\lambda}(G) - \mathcal{Y}_0^{N,\lambda}(G) - \int_0^t \frac{1}{N^{d/2}} \sum_{x\in\mathbb{T}_N^d} \phi'(\rho)\nabla \cdot \mathcal{A}\nabla G(x)\big(\eta_s(x) - \rho\big)ds$$

plus a term that vanishes in $\mathcal{L}^2(\mathbb{P}_{\nu_\rho})$ as $N \to \infty$. Notice that the integrand in the previous expression is a function of the density fluctuation field $\mathcal{Y}_t^N$, see (2.6). By (4.2), we can replace inside the integral of last expression the density fluctuation field $\mathcal{Y}_t^N$ by the corrected density fluctuation field $\mathcal{Y}_t^{N,\lambda}$.

Suppose that the sequence $\{\mathcal{Q}_N^\lambda\}_{N\geq 1}$ is tight and let $\mathcal{Q}^\lambda$ be a limit point of it. Denote by $\mathcal{Y}_t$ the process in $\mathcal{D}([0,T], \mathcal{S}'(U^d))$ induced by the canonical projections under $\mathcal{Q}^\lambda$. Taking the limit as $N \to \infty$ under an appropriate subsequence in expression (4.3), we obtain that

$$M_t^\lambda(G) = \mathcal{Y}_t(G) - \mathcal{Y}_0(G) - \int_0^t \mathcal{Y}_s(\phi'(\rho)\nabla \cdot \mathcal{A}\nabla G)ds$$

is a martingale of quadratic variation

$$t\phi(\rho)\int_{U^d} \nabla G(u) \cdot \mathcal{A}\nabla G(u)du.$$

On the other hand, it is not hard to show that $\mathcal{Y}_0$ is a Gaussian field with covariance given by (4.1). Therefore, $\mathcal{Q}^\lambda$ is equal to the probability distribution $\mathcal{Q}$ of a generalized Ornstein-Uhlenbeck process in $\mathcal{C}([0,T], \mathcal{S}'(U^d))$ (and it does not depend on $\lambda$). As a consequence, the sequence $\{\mathcal{Q}_N^\lambda\}_{N\geq 1}$ has at most one limit point and Theorem 4.2 shall follow if we prove tightness for $\{\mathcal{Q}_N^\lambda\}_{N\geq 1}$.

Lastly, it remains to treat the problem of tightness of the sequence $\{Q_N^\lambda\}_{N\geq 1}$. For that we use a criterion due to Mitoma [10] (see also [5]), which allows to conclude that the sequence is tight and that any weak limit is supported in $C([0,T], S'(U^d))$, since the following estimates hold:

a) For every $T > 0$ and $G \in \mathcal{S}(U^d)$,

$$\sup_N \sup_{t \in [0,T]} \mathbb{E}_{\nu_\rho} \left[ \mathcal{Y}_t^{N,\lambda}(G) \right]^2 < \infty.$$

$$\sup_N \sup_{t \in [0,T]} \mathbb{E}_{\nu_\rho} \left[ \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} \phi'(\rho) \nabla \cdot \mathcal{A} \nabla G(x) \left( \eta_s(x) - \rho \right) \right]^2 < \infty.$$

$$\sup_N \sup_{t \in [0,T]} \mathbb{E}_{\nu_\rho} \left[ \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} \phi(\rho) \sum_{y \in \mathbb{T}_N^d} p_N(x,y) \left[ G_N^\lambda(y) - G_N^\lambda(x) \right]^2 \right]^2 < \infty.$$

b) For every $G \in \mathcal{S}(U^d)$ there exists $\delta(t, G, N)$ such that $\lim_{N \to \infty} \delta(t, G, N) = 0$ and

$$\lim_N \mathbb{P}_{\nu_\rho} \left( \sup_{0 \le s \le t} \left| \mathcal{Y}_s^{N,\lambda}(G) - \mathcal{Y}_{s-}^{N,\lambda}(G) \right| > \delta(t, G, N) \right) = 0.$$

The first expectation in a) is bounded by $||G_N^\lambda||_\infty \chi(\rho)$, which in turn is bounded by $C||R^\lambda G||_\infty$. The second expectation in a) is bounded by $C||\nabla \cdot \mathcal{A} \nabla G||_2^2$ and the last one bounded by $C||R^\lambda G||_2^4$.

To prove b) we only have to remark that by definition of the process it holds that $\sup_{0 \le s \le t} |\mathcal{Y}_s^{N,\lambda}(G) - \mathcal{Y}_{s-}^{N,\lambda}(G)| \le \frac{||G_N^\lambda||_\infty}{N^{d/2}}$.

By the results proved $\{\mathcal{Q}_N^\lambda\}_{n \ge 1}$ is tight and we have identified above a unique limit point $\mathcal{Q}$ that corresponds to the Ornstein-Uhlenbeck process; consequently the whole sequence converges to $\mathcal{Q}$.

## 5. Boltzmann-Gibbs Principle

This section is devoted to the proof of Theorem (4.5). Let $f : \chi \times \Omega_N^d \to +\infty$ be a local, uniformly Lipschitz function and take $f(x, \eta) = f(\theta_{Nx}\omega, \tau_x \eta)$.

Fix a function $G \in \mathcal{S}(U^d)$ and an integer $K$ that shall increase to $\infty$ after $N$. For each $N$, we subdivide $\mathbb{T}_N^d$ in non overlapping cubes of linear size $K$. Denote them by $\{I_j, 1 \le j \le M^d\}$, where $M = [\frac{2N}{K}]$. Let $I_0$ be the set of points that are not included in any $I_j$ which implies that $|I_j| \le dKN^{d-1}$. If we restrict the sum in the expression that appears inside the integral in (4.4) to the set $I_0$, then its $L^2(\mathbb{P}_{\nu_\rho})$ norm clearly vanishes as $N \to +\infty$.

Let $\Lambda_{s_f}$ be the smallest cube centered at the origin that contains the support of $f$ and define $s_f$ as the radius of $\Lambda_{s_f}$. Denote by $I_j^0$ the interior of the interval $I_j$, namely the sites $x$ in $I_j$ that are at a distance at least $s_f$ from the boundary:

$$I_j^0 = \{x \in I_j, d(x, \mathbb{T}_N^d \setminus I_j) > s_f\}.$$

Denote also by $I^c$ the set of points that are not included in any $I_j^0$. By construction it is easy to see that $|I^c| \le dN^d(\frac{c(g)}{K} + \frac{K}{N})$. Using the notation just settled, we have that

$$\frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} H(x) V_f(x, \eta_t) = \frac{1}{N^{d/2}} \sum_{x \in I^c} H(x) V_f(x, \eta_t) +$$

$$+ \frac{1}{N^{d/2}} \sum_{j=1}^{M^d} \sum_{x \in I_j^0} \left[ H(x) - H(y_j) \right] V_f(x, \eta_t) + \frac{1}{N^{d/2}} \sum_{j=1}^{M^d} H(y_j) \sum_{x \in I_j^0} V_f(x, \eta_t),$$

where $y_j$ is a point in $I_j$. We assume that the points $y_j$ have the same relative position on each of the cubes. The first step is to prove that

$$\lim_{K \to \infty} \lim_{N \to \infty} \mathbb{E}_{\nu_\rho} \left[ \int_o^t \frac{1}{N^{d/2}} \sum_{x \in I^c} H(x) V_f(x, \eta_t) ds \right]^2 = 0.$$

Applying Schwarz inequality, since $\nu_\rho$ is an invariant product measure and since $V_f$ has mean zero with respect to the measure $\nu_\rho$, the last expectation is bounded above by

$$\frac{t^2}{N^d} \sum_{\substack{x,y \in I^c \\ |x-y| \leq 2s_f}} H(x)H(y)E_{\nu_\rho}\big[V_f(x,\eta)V_f(y,\eta)\big].$$

Since $V_f$ belongs to $\mathcal{L}^2(\nu_\rho)$ and $|I^c| \leq dN^d(\frac{c(f)}{K} + \frac{K}{N})$, the last expression vanishes by taking first $N \to +\infty$ and then $K \to +\infty$.

Applying the same arguments, it is not hard to show that

$$\lim_{N \to \infty} \mathbb{E}_{\nu_\rho}\Big[\int_0^t \frac{1}{N^{d/2}} \sum_{j=1}^{M^d} \sum_{x \in I_j^0} \big[H(x) - H(y_j)\big]V_f(x,\eta_t)ds\Big]^2 = 0.$$

In order to finish the proof it remains to show that

$$\lim_{K \to \infty} \lim_{N \to \infty} \mathbb{E}_{\nu_\rho}\Big[\int_0^t \frac{1}{N^{d/2}} \sum_{j=1}^{M^d} H(y_j) \sum_{x \in I_j^0} V_f(x,\eta_t)ds\Big]^2 = 0.$$

Let $\mathbb{L}_N$ be the generator of the zero-range process without the random environment (that is, taking $a(\omega) \equiv 1$ in (2.3)), and without the diffusive scaling $N^2$. For each $j = 0,..,M^d$ denote by $\zeta_j$ the configuration $\{\eta(x), x \in I_j\}$ and by $\mathbb{L}_{I_j}$ the restriction of the generator $\mathbb{L}_N$ to the interval $I_j$, namely:

$$\mathbb{L}_{I_j}h(\eta) = \sum_{\substack{x,y \in I_j \\ |x-y|=1/N}} g\big(\eta(x)\big)\big[h(\eta^{x,y}) - h(\eta)\big].$$

We point out here that we are introducing a slightly different generator than the one that generates the dynamics, namely $\mathcal{L}_N$. The reason for doing this stands on the fact that the dynamics generated by this operator is translation invariant. The generator that we choose to introduce here is not random, but due to the ellipticity assumption on the environment, it is mutually bounded with the one that we have started with.

Now we introduce some notation. Fix a local function $h : \chi \times \Omega_N^d \to \mathbb{R}$, measurable with respect to $\sigma(\eta(x), x \in I_1)$, such that $E[\int h(\omega,\eta)^2 d\nu_\rho] < \infty$ and let $h_j$ be the translation of $h$ by $y_j - y_0$: $h_j(x,\eta) = h(\theta_{(y_j-y_0)N}\omega, \tau_{y_j-y_0}\eta)$. Denote by $\mathcal{L}^2(\nu_\rho \times P)$ the set of such functions. Consider

$$V_{H,h}^N(\eta) = \frac{1}{N^{d/2}} \sum_{j=1}^{M^d} H(y_j)\mathbb{L}_{I_j}h_j(\zeta_j).$$

By proposition A 1.6.1 of [9] and the ellipticity assumption, it is not hard to show that

$$\mathbb{E}_{\nu_\rho}\Big[\int_0^t \frac{1}{N^{d/2}} \sum_{j=1}^{M^d} H(y_j)\mathbb{L}_{I_j}h_j(\zeta_j(s))ds\Big]^2 \leq 20\epsilon_0^{-1}t|||V_{H,h}^N|||_{-1}^2,$$

where the norm $||| \cdot |||_{-1}$ is given by the variational formula

$$|||V_{H,h}^N|||_{-1}^2 = \sup_{F \in \mathcal{L}^2(\nu_\rho)}\Big\{2\int V_{H,h}^N(\eta)F(\eta)d\nu_\rho - N^2\langle F, -\mathbb{L}_N F\rangle_\rho\Big\}, \tag{5.1}$$

where $\langle \cdot, \cdot \rangle_\rho$ denotes the inner product in $\mathcal{L}^2(\nu_\rho)$.

By the Cauchy-Schwarz inequality,

$$\int \mathbb{L}_{I_j}h_j(\zeta_j)F(\eta)d\nu_\rho \leq \frac{1}{2\gamma_j}\langle -\mathbb{L}_{I_j}h_j, h_j\rangle_\rho + \frac{\gamma_j}{2}\langle F, -\mathbb{L}_{I_j}F\rangle_\rho$$

for each $j$, where $\gamma_j$ is a positive constant. Therefore,

$$2\int V_{H,h}^N(\eta)F(\eta)d\nu_\rho \leq \frac{2}{N^{d/2}} \sum_{j=1}^{M^d} H(y_j)\Big\{\frac{1}{2\gamma_j}\langle -\mathbb{L}_{I_j}h_j, h_j\rangle_\rho + \frac{\gamma_j}{2}\langle F, -\mathbb{L}_{I_j}F\rangle_\rho\Big\}.$$

Taking for each $j$, $\gamma_j = N^{2+\frac{d}{2}}|H(y_j)|^{-1}$ we have that

$$\frac{2}{N^{d/2}} \sum_{j=1}^{M^d} \left|H(y_j)\right| \frac{\gamma_j}{2} \langle F, -\mathbb{L}_{I_j} F \rangle_\rho \leq N^2 \langle F, -\mathbb{L}_N F \rangle_\rho,$$

and the expectation becomes bounded by

$$\frac{20\epsilon_0^{-1}t}{N^{d/2}} \sum_{j=1}^{M^d} \frac{|H(y_j)|}{\gamma_j} \langle -\mathbb{L}_{I_j} h_j, h_j \rangle_\rho \leq \frac{20 t M^d ||H||_\infty}{\epsilon_0^2 N^{2+d}} \sum_{j=1}^{M^d} \frac{1}{M^d} \langle -\mathbb{L}_j h_j, h_j \rangle_\rho.$$

By the ergodic theorem, the sum in the previous expression converges as $N \to \infty$ to a finite value and therefore this last expression vanishes as $N \to \infty$. To conclude the proof of the theorem we need to show that

$$\lim_{K \to \infty} \inf_{h \in \mathcal{L}^2(\nu_\rho \times P)}$$

$$\lim_{N \to \infty} \mathbb{E}_{\nu_\rho} \left[ \int_0^t \frac{1}{N^{d/2}} \sum_{j=1}^{M^d} H(y_j) \left\{ \sum_{x \in I_j^0} V_f(x, \eta_s) - \mathbb{L}_{I_j} h_j(\zeta_j(s)) \right\} \right]^2 = 0.$$

By Schwarz inequality the expectation in the previous expression is bounded by

$$\frac{t^2}{N^d} \sum_{j=1}^{M^d} ||H||_\infty^2 E_{\nu_\rho} \left( \sum_{x \in I_j^0} V_f(x, \eta) - \mathbb{L}_{I_j} h_j(\zeta_j) \right)^2$$

because the measure $\nu_\rho$ is invariant under the dynamics and also translation invariant and the supports of $V_f(x, \eta) - \mathbb{L}_{I_i} h_i(\zeta_i)$ and $V_f(y, \eta) - \mathbb{L}_{I_j} h_j(\zeta_j)$ are disjoint for $x \in I_i^0$ and $y \in I_j^0$, with $i \neq j$.

By the ergodic theorem, as $N \to \infty$ this expression converges to

$$\frac{t^2}{K^d} ||H||_\infty^2 E \left[ \int \left( \sum_{x \in I_1^0} V_f(x, \eta) - \mathbb{L}_{I_1} h(\omega, \eta) \right)^2 d\nu_\rho \right]. \tag{5.2}$$

So it remains to show that

$$\lim_{K \to \infty} \frac{t^2}{K^d} ||H||_\infty^2 \inf_{h \in \mathcal{L}^2(\nu_\rho \times P)} E \left[ \int \left( \sum_{x \in I_1^0} V_f(x, \eta) - \mathbb{L}_{I_1} h(\omega, \eta) \right)^2 d\nu_\rho \right] = 0.$$

Denote by $\mathcal{R}(\mathbb{L}_{I_1})$ the range of the generator $\mathbb{L}_{I_1}$ in $\mathcal{L}^2(\nu_\rho \times P)$ and by $\mathcal{R}(\mathbb{L}_{I_1})^\perp$ the space orthogonal to $\mathcal{R}(\mathbb{L}_{I_1})$. The infimum of (5.2) over all $h \in \mathcal{L}^2(\nu_\rho \times P)$ is equal to the projection of $\sum_{x \in I_1^0} V_f(x, \eta)$ into $\mathcal{R}(\mathbb{L}_{I_1})^\perp$.

It is not hard to show that $\mathcal{R}(\mathbb{L}_{I_1})^\perp$ is the space of functions that depends on $\eta$ only through the total number of particles on the box $I_1$. So, the previous expression is equal to

$$\lim_{K \to \infty} \frac{t^2 ||H||_\infty^2}{K^d} E \left[ \int \left( E_{\nu_\rho} \left[ \sum_{x \in I_1^0} V_f(x, \eta) \Big| \eta^{I_1} \right] \right)^2 d\nu_\rho \right] \tag{5.3}$$

where $\eta^{I_1} = K^{-d} \sum_{x \in I_1} \eta(x)$.

Let us call this last expression $\mathcal{I}_0$. Define $\psi(x, \rho) = E_{\nu_\rho}[f(\theta_x \omega)]$. Notice that $V_f(x, \eta) = f(x, \eta) - \psi(x, \rho) - E[\psi'(x, \rho)](\eta(x) - \rho)$, since in the last term the derivative commutes with the expectation with respect to the random environment. In order to estimate the expression (5.3) using the elementary inequality $(x + y)^2 \leq 2x^2 + 2y^2$, we split it into three pieces: $\mathcal{I}_0 \leq 4(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)$, where

$$\mathcal{I}_1 = \frac{1}{K^d} E \left[ \int \left( \sum_{x \in I_1^0} E_{\nu_\rho}[f(x, \eta)|\eta^{I_1}] - \psi(x, \eta^{I_1}) \right)^2 d\nu_\rho \right],$$

$$\mathcal{I}_2 = \frac{1}{K^d} E\Big[ \int \Big( \sum_{x \in I_1^0} \psi(x, \eta^{I_1}) - \psi(x, \rho) - \psi'(x, \rho)[\eta^{I_1} - \rho] \Big)^2 d\nu_\rho \Big],$$

$$\mathcal{I}_3 = \frac{1}{K^d} E\Big[ E_{\nu_\rho}\Big[ \Big( \sum_{x \in I_1^0} (\psi'(x, \rho) - E[\psi'(x, \rho)]) [\eta^{I_1} - \rho] \Big)^2 \Big] \Big].$$

We will make use of the following lemma, known as the equivalence of ensembles.

**Lemma 5.1.** *Let $h : \Omega_N^d \to \mathbb{R}$ a local, uniformly Lipschitz function. Then, for each $\beta \geq 0$ there exists a constant $C$ that depends on $h$ only through its support and its Lipschitz constant, such that*

$$\Big| E_{\nu_\rho}[h(\eta)|\eta^N] - E_{\nu_{\eta^N}}[h(\eta)] \Big| \leq \frac{C}{N^d}$$

*whenever $\rho, \eta^N \leq \beta$, where*

$$\eta^N = \sum_{|x| \leq N} \eta(x).$$

In order to estimate $\mathcal{I}_1$ and $\mathcal{I}_2$, we introduce the indicator functions $\mathbf{1}(\eta^{I_1} \leq \beta)$. By a large deviations estimate, $\nu_\rho(\eta^{I_1} \geq \beta) \leq \exp(-C(\beta)K^d)$. Since $f$ is Lipschitz, it has bounded exponential moments of any order and a simple Schwarz estimate shows that we can introduce the indicator function $\mathbf{1}(\eta^{I_1} \leq \beta)$ into the integrals in $\mathcal{I}_1$ and $\mathcal{I}_2$. By Lemma 5.1,

$$\frac{1}{K^d} E\Big[ \int \Big( \sum_{x \in I_1^0} E_{\nu_\rho}[f(x, \eta)|\eta^{I_1}] - \psi(x, \eta^{I_1}) \Big)^2 \mathbf{1}(\eta^{I_1} \leq \beta) d\nu_\rho \Big] \leq \frac{C}{K^d},$$

which vanishes as $K \to \infty$.

Using a Taylor expansion for $\psi(x, \rho)$, we see that

$$\frac{1}{K^d} E\Big[ \int \Big( \sum_{x \in I_1^0} \psi(x, \eta^{I_1}) - \psi(x, \rho) - \psi'(x, \rho)[\eta^{I_1} - \rho] \Big)^2 d\nu_\rho \Big] \leq \frac{C}{K^d}$$

and also goes to 0 as $K \to \infty$.

Finally, we see that

$$\mathcal{I}_3 = E_{\nu_\rho}\big[ (\eta(0) - \rho)^2 \big] \cdot E\Big[ \Big( \frac{1}{K^d} \sum_x (\psi'(x, \rho) - E[\psi'(x, \rho)]) \Big)^2 \Big]$$

and it goes to 0 as $K \to \infty$ by the $\mathcal{L}^2$-ergodic theorem.

5.1. **Some applications of the Boltzmann-Gibbs principle.** In the proof of Theorem 2.8, we need to use the Boltzmann-Gibbs Principle 4.5 for the function $g(\eta(0))$, that does not depend on the random environment. In particular, the results of the previous section are not needed in the proof of Theorem 2.8, since the proof for the non-random case applies directly for functions that do not depend on the random environment. We point out here two applications for the Boltzmann-Gibbs principle as stated in Theorem 4.5.

**First application:** Consider, for simplicity, some local, bounded and uniformly Lipschitz function $f(\omega, \eta)$ that does not depend on the value of $\eta(0)$. For each $\eta \in \Omega_N$, define

$$\Theta_x^+ \eta(z) = \begin{cases} \eta(x) + 1, z = x \\ \eta(z), z \neq x, \end{cases}$$

$$\Theta_x^- \eta(z) = \begin{cases} \eta(x) - 1, z = x \\ \eta(z), z \neq x. \end{cases}$$

Notice that $\Theta_x^- \eta$ is well defined only if $\eta(x) \geq 1$. We can define a reaction-diffusion model adding to the zero-range dynamics a Glauber dynamics as follows:

$$\mathcal{L}_N^{rd} F(\eta) =: N^2 \mathcal{L}_N + \sum_{x \in \mathbb{T}_N^d} f(x, \eta) \big[ F(\Theta_x^+ \eta) - F(\eta) \big]$$
$$+ \sum_{x \in \mathbb{T}_N^d} \alpha(\rho) \frac{f(x, \Theta_x^- \eta)}{g(\eta(x))} \big[ F(\Theta_x^- \eta) - F(\eta) \big],$$

where we define $f(x, \Theta_x^- \eta)/g(\eta(x)) = 0$ if $\eta(x) = 0$. We have chosen the annihilation rate in such a way that the measure $\nu_\rho$ is invariant for this process. Therefore, we can obtain the equilibrium fluctuations for this model as in Section 4.

**Second application:** This one has to do with the convergence of additive functionals of Markov processes. For each $f$ satisfying the conditions of Theorem 4.5, define the density fluctuation field for $f$ acting on functions $G \in \mathcal{S}(U^d)$ as

$$\mathcal{Z}_t^{N,f}(G) = \frac{1}{N^{d/2}} \sum_{x \in \mathbb{T}_N^d} G(x)) \{ f(x, \eta_s) - E_{\nu_\rho}[f(x, \eta)] \}.$$

Note that for $f(x, \eta) = \eta(x) - \rho$, the density fluctuation field for $f$ is the density fluctuation field introduced above and denoted by $\mathcal{Y}_s^N(G)$.

For fixed $f$ as above, define the additive functional

$$\mathcal{I}_f^N(t) = \int_0^t \mathcal{Z}_t^N(G) ds.$$

Then, by Theorems 2.8 and 4.5,

$$\lim_{N \to \infty} \mathcal{I}_f^N(t) = \partial_\rho E \Big[ \int f(\omega, \eta) d\nu_\rho \Big] \int_0^t \mathcal{Y}_s(G) ds \text{ in distribution.}$$

## APPENDIX A. SOME ESTIMATES FOR $\eta_t$

A.1. **Entropy production.** Denote by $\mu_N(t) = S_t^N \mu_N$ the distribution of $\eta_t$ in $\Omega_N^d$ under $\mathbb{P}_{\mu_N}$ and define $f_t^N = \frac{d\mu_N(t)}{d\nu_{\bar{\rho}}}$. The density $f_t^N$ satisfies the Kolmogorov equation

$$\frac{d}{dt} f_t^N(\eta) = \mathcal{L}_N f_t^N(\eta).$$

For each density $f : \Omega_N^d \to \mathbb{R}_+$, define the Dirichlet form $\mathcal{D}_N(f)$ by

$$\mathcal{D}_N(f) = \sum_{\substack{x, y \in \mathbb{T}_N \\ |x-y|=1/N}} \int g(\eta(x)) \big[ \sqrt{f(\eta^{xy})} - \sqrt{f(\eta)} \big]^2 d\nu_{\bar{\rho}},$$

and the entropy $H_N(f) = \int f \log f d\nu_{\bar{\rho}}$. By the ellipticity assumption in $p_N(x, y)$, the entropy production is bounded by the Dirichlet form of $f_t^N$ [9]:

$$\frac{d}{dt} H_N(f_t^N) \leq -2\epsilon_0 N^2 \mathcal{D}_N(f_t^N).$$

Assume that $H_N(\mu_N | \nu_{\bar{\rho}}) \leq K_0 N^d$, or in other words that $H_N(f_0^N) \leq K_0 N^d$. Since the Dirichlet form and the entropy are convex functions of $f$, integrating the previous inequality we obtain the bounds

$$H_N(\bar{f}_T^N) \leq \frac{K_0}{T} N^d, \quad \mathcal{D}_N(\bar{f}_T^N) \leq \frac{K_0}{2\epsilon_0 T} N^{d-2},$$

where

$$\bar{f}_T^N(\eta) = \frac{1}{T} \int_0^T f_t^N(\eta) dt.$$

A.2. **Attractiveness of $\eta_t$.** Take two probability measures $\mu$, $\nu$ in $\Omega_N$ such that $\mu \preceq \nu$. When the jump rate $g(\cdot)$ is non-decreasing, it is possible to construct a process $(\eta_t, \eta_t')$ in $\Omega_N \times \Omega_N$, starting from a coupling $\bar{\mu}$ of $\mu$ and $\nu$, such that for every $t \in [0, T]$

   i) The distribution of $\eta_t$ in $\mathcal{D}([0, T], \Omega_N)$ is equal to $\mathbb{P}_\mu$.
   ii) The distribution of $\eta_t'$ in $\mathcal{D}([0, T], \Omega_N)$ is equal to $\mathbb{P}_\nu$.
   iii) The distribution of $(\eta_t, \eta_t')$ in $\mathcal{D}([0, T], \Omega_N \times \Omega_N)$ is concentrated on the set $\{(\eta, \eta') \in \Omega_N \times \Omega_N; \eta \preceq \eta'\}$.

In this case the process $\eta_t$ is said to be attractive. We say that a function $h : \Omega_N^d \to \mathbb{R}$ is non-decreasing if for $\eta \preceq \eta'$ then $h(\eta) \leq h(\eta')$. The following proposition is an immediate consequence of the existence of the process $(\eta_t, \eta_t')$.

**Proposition A.1.** *Let $\mu$, $\nu$ be two probability measures in $\Omega_N^d$ such that $\mu \preceq \nu$. Let $h : \Omega_N^d \to \mathbb{R}$ be a non-decreasing function. Then,*

$$\mathbb{E}_\mu\big[h(\eta_t)\big] \leq \mathbb{E}_\nu\big[h(\eta_t)\big]$$

*for all $t \in [0, T]$.*

A.3. **An $\mathcal{L}^2$ estimate for $\eta_t$.** Consider the process $\eta_t$ starting from the equilibrium measure $\nu_\rho$. Define the $\mathcal{L}_N^2$-norm of $\eta_t$ by

$$||\eta_t||_{0,N}^2 = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N} \eta_t(x)^2.$$

By Dynkin's formula,

$$\mathcal{M}_t^N = ||\eta_t||_{0,N}^2 - ||\eta_0||_{0,N}^2 - \int_0^t \mathcal{L}_N ||\eta_s||_{0,N}^2 ds \tag{A.1}$$

is a martingale of quadratic variation

$$\langle \mathcal{M}_t^N \rangle = \int_0^t ||\eta_s||_{0,N}^2 (-\mathcal{L}_N) ||\eta_s||_{0,N}^2 ds.$$

Explicit computations show that $\mathbb{E}_{\nu_\rho}\big[\langle \mathcal{M}_t^N \rangle\big] \leq C/N^{d-2}$. Therefore, by Doob's inequality,

$$\mathbb{E}_{\nu_\rho}\Big[ \sup_{t \in [0,T]} |\mathcal{M}_t^N|^2 \Big] \leq C/N^{d-2}.$$

For the integral term in A.1, we have the following estimate:

$$\mathbb{E}_{\nu_\rho}\Big[ \big( \sup_{t \in [0,T]} \int_0^t \mathcal{L}_N ||\eta_s||_{0,N}^2 ds \big)^2 \Big] \leq Ct \mathbb{E}_{\nu_\rho}\Big[ ||\eta||_{0,N}^2 (-\mathcal{L}_N) ||\eta||_{0,N}^2 \Big].$$

Therefore, for dimension $d \geq 2$, we conclude that $\mathbb{E}_{\nu_\rho}\big[ \sup_t ||\eta_t||_{0,N}^2 \big]$ is uniformly bounded in $N$. Since $||\eta||_{0,N}^2$ is an increasing function, we have proved the following result:

**Lemma A.2.** *Fix $\rho > 0$. Let $\{\mu_N\}_{N \geq 1}$ be a sequence of measures such that $\mu_N \preceq \nu_\rho$ for all $N$. Then,*

$$\sup_{N \in \mathbb{N}} \mathbb{E}_{\mu_N}\Big[ \sup_{t \in [0,T]} ||\eta_t||_{0,N}^2 \Big] < +\infty.$$

## Appendix B. Functional analysis in the spaces $\mathcal{L}_N^2$, $\mathcal{H}_{1,N}$

B.1. **Convergence in $\mathcal{L}_N^2$, $\mathcal{H}_{1,N}$.**

Fix $f \in \mathcal{H}_{1,N}$. We define the linear interpolation $\mathcal{T}_N^1 f$ of $f$ as follows. To fix ideas, take $d = 3$. We divide each of the cubes of size $1/N$ in $\mathbb{T}_N^d$ into six tetrahedrons with vertices in $\mathbb{T}_N^d$. The way we do this is not important, but we do it in the same way for every cube in $\mathbb{T}_N^d$.

For a point $u$ in one of such tetrahedrons, we define $\mathcal{T}_N^1 f(u)$ as the linear interpolation of the values of $f$ on the vertices of the tetrahedron. In this way we have defined a function $\mathcal{T}_N^1 f$ in $\mathcal{H}_1(U^d)$.

We say that $f_N \in \mathcal{H}_{1,N}$ converges strongly (resp. weakly) in $\mathcal{H}_{1,N}$ to $f \in \mathcal{H}_1(U^d)$ if

$$\lim_{N\to\infty} \mathcal{T}_N^1 f_N = f \text{ strongly (resp. weakly) in } \mathcal{H}_1(U^d).$$

In an analogous way, for each $u \in U^d$ we define $\mathcal{T}_N^0 f(u) = f(x)$ if $|u - x| \leq 1/2N$. We say that $f_N$ converges strongly (resp. weakly) in $\mathcal{L}_N^2$ to $f \in \mathcal{L}^2(U^d)$ if $\mathcal{T}_N^0 f_N$ converges strongly (resp. weakly) to $f$ in $\mathcal{L}^2(U^d)$.

A sequence $f_N \in \mathcal{H}_{-1,N}$ converges to $f \in \mathcal{H}_{-1}(U^d)$ strongly (resp. weakly) if for any sequence $g_N \in \mathcal{H}_{1,N}$ and $g \in \mathcal{H}_1(U^d)$ such that $g_N \to g$ weakly in (resp. strongly) $\mathcal{H}_{1,N}$ we have

$$\lim_{N\to\infty} \langle f_N, g_N \rangle_N = \langle f, g \rangle.$$

## B.2. Resolvent estimates.

Let $f$ be a regular function and let $u_N$ be the solution of the resolvent equation

$$\lambda u_N(x) - L_N u_N(x) = f(x). \tag{B.1}$$

**Lemma B.1.** *There exists a constant $c = c(\lambda)$ such that*

$$\max\{||u_N||_{0,N}, ||u_N||_{1,N}\} \leq c||f||_1.$$

*Proof.* By Lax-Milgram's lemma, this equation has a unique solution in $\mathcal{H}_{1,N}$. Taking the inner product of equation (B.1) with respect to $u_N$, we see that

$$\lambda ||u_N||_{0,N}^2 + \langle u_N, -L_N u_N \rangle_N \leq \langle f, u_N \rangle_N.$$

By the Cauchy-Schwarz inequality, $|\langle f, u_N \rangle_N \leq ||u_N||_{0,N} ||f||_{0,N}$. Using the ellipticity assumption, we obtain the estimates

$$||u_N||_{0,N} \leq \lambda^{-1} ||f||_{0,N}$$
$$||u_N||_{1,N}^2 \leq \left[ (\lambda \epsilon_0)^{-1} + \lambda^{-2} \right] ||f||_{0,N}^2.$$

By the finite elements theory [1], there exists a constant $\gamma$ independent of $N$ such that for every $f \in \mathcal{H}_1(U^d)$, $||f||_{0,N} \leq \gamma ||f||_1$. Therefore, it is enough to take $c = \gamma \max\{\lambda^{-1}, (\lambda \epsilon_0)^{-1} + \lambda^{-2}\}$. $\square$

Since the operator $L_N$ is the generator of a random walk in $\mathbb{T}_N^d$, the solutions of (B.1) satisfy the maximum principle:

$$\inf_{x\in\mathbb{T}_N^d} \lambda^{-1} f(x) \leq \inf_{x\in\mathbb{T}_N^d} u_N(x) \leq \inf_{x\in\mathbb{T}_N^d} u_N(x) \leq \sup_{x\in\mathbb{T}_N^d} \lambda^{-1} f(x).$$

In particular, for $f$ continuous, $||u_N||_\infty \leq \lambda^{-1} ||f||_\infty$.

## REFERENCES

[1] Braess, D.: Finite elements. Theory, fast solvers, and applications in solid mechanics, Second edition. Cambridge University Press, Cambridge, 2001.

[2] Chang, C. C.: Equilibrium fluctuations of gradient reversible particle systems. Probab. Theory Related Fields 100 (1994), no. 3, 269–283.

[3] Faggionato, A.: Bulk diffusion of 1D exclusion process with bond disorder. Preprint.

[4] Faggionato, A. and Martinelli, F.: Hydrodynamic limit of a disordered lattice gas. Probab. Theory Related Fields 127 (2003), no. 4, 535–608.

[5] Ferrari, P.A., Presutti, E. and Vares, M. E.: Nonequilibrium fluctuations for a zero range process. Ann. Inst. H. Poincaré Probab. Statist. 24 (1988), no. 2, 237–268.

[6] Guo, M., Papanicolaou, G. and Varadhan, S.R.S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. Comm. Math. Phys. 118 (1988), no. 1, 31–59.

[7] Jara, M.: Hydrodynamic limit for the simple exclusion process on non-homogeneous graphs. Preprint.

[8] Jara, M. and Landim, C.: Quenched nonequilibrium central limit theorem for a tagged particle in the exclusion process with bond disorder. To appear in Ann. Inst. H. Poincaré Probab. Statist.

[9] Kipnis, C. and Landim, C. (1999): *Scaling Limits of Interacting Particle Systems*, Springer-Verlag, New York.

[10] Mitoma, I.: Tightness of probabilities on $C([0,1]; \mathcal{S}')$ and $D([0,1]; \mathcal{S}')$. Ann. Probab. 11 (1983), no. 4, 989–999.

[11] Olla, S. and Siri, P.: Homogenization of a bond diffusion in a locally ergodic random environment. Stochastic Process. Appl. 109 (2004), no. 2, 317–326.

[12] Piatnitski, A. and Remy, E.: Homogenization of elliptic difference operators. SIAM J. Math. Anal. 33 (2001), no. 1, 53–83 .

[13] Quastel, J.: Diffusion of color in the simple exclusion process. Comm. Pure Appl. Math. 45 (1992), no. 6, 623–679.

[14] Quastel, J.: Bulk Diffusion in a System with Site Disorder. Ann. Probab. 34 (2006), no. 5.

[15] Stroock, D. and Zheng, W. (1997): Markov chain approximations to symmetric diffusions. Ann. Inst. H. Poincaré Probab. Statist. 33 (1997), no. 5, 619–649. 619-649.

[16] Varadhan, S.R.S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. II. Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), 75–128, Pitman Res. Notes Math. Ser., 283, Longman Sci. Tech., Harlow, 1993

IMPA, Estrada Dona Castorina 110, CEP 22460-320, Rio de Janeiro, Brazil
*E-mail address*: patg@impa.br, monets@impa.br