CONTACT NORMAL SEMI-INVARIANT SPACELIKE SUBMANIFOLDS IN AN INDEFINITE SASAKIAN MANIFOLD

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Abstract. In this paper we study the contact normal, semi-invariant, spacelike submanifolds M^s isometrically immersed in a Sasakian manifold $\widetilde{M}_q^{2m+1}$ with pseudo-Riemannian metric of index $q = (2m+1) - s$ (so with timelike characteristic vector field). We study the integrability of the distributions D and $D^\perp$ and the geometry of its leaves. We show that a contact normal spacelike semi-invariant hypersurface is invariant, that is $TM = D$, and totally geodesic. We also obtain a classification result for submanifolds totally umbilic.

1. Intruduction

Let $\widetilde{M}$ be a $(2m+1)$-dimensional manifold and $\Gamma(T\widetilde{M})$ the Lie algebra of vector fields on $\widetilde{M}$. Recall that an *almost contact structure* on $\widetilde{M}$ is defined by a (1,1)-tensor φ, a vector field ξ and a 1-form η on $\widetilde{M}$ such that for any $p \in \widetilde{M}$, we have

$$\varphi_p^2 = -I + \eta_p \otimes \xi_p, \quad \eta_p(\xi_p) = 1, \tag{1}$$
$$\eta(\varphi(\widetilde{X})) = 0 \quad \widetilde{X} \in \Gamma(T\widetilde{M}),$$

where I denote the identity transformation of the tangent space $T_p\widetilde{M}$ at p. Manifolds equipped whit an almost contact structure are called *almost contact manifolds.*

If a manifold $\widetilde{M}^{2m+1}$ with a (φ, ξ, η)-structure admits a Riemannian metric $\langle\ ,\ \rangle$ such that

$$\langle \varphi\widetilde{X}, \varphi\widetilde{Y}\rangle = \langle \widetilde{X}, \widetilde{Y}\rangle - \eta(\widetilde{Y})\eta(\widetilde{Y})$$

for any $\widetilde{X}, \widetilde{Y} \in \Gamma(T\widetilde{M})$, then $\widetilde{M}^{2m+1}$ is said to have a $(\varphi, \xi, \eta, \langle\ ,\ \rangle)$-structure or an *almost contact metric manifolds.*

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A manifold $\widetilde{M}$ with pseudo-Riemannian metric tensor $\langle\ ,\ \rangle$ and an almost contact structure $(\varphi, \xi, \eta, \varepsilon)$ such that

$$\langle \xi, \xi \rangle = \varepsilon, \quad \varepsilon = \pm 1, \quad \eta \circ \varphi = 0,$$

$$\eta(\widetilde{X}) = \varepsilon \langle \xi, \widetilde{X} \rangle, \quad \widetilde{X} \in \Gamma(T\widetilde{M}), \tag{2}$$

$$\langle \varphi \widetilde{X}, \varphi \widetilde{Y} \rangle = \langle \widetilde{X}, \widetilde{Y} \rangle - \varepsilon \eta(\widetilde{X}) \eta(\widetilde{Y}), \quad \widetilde{X}, \widetilde{Y} \in \Gamma(T\widetilde{M})$$

is an *almost contact metric manifold*.

The fundamental 2-form Ψ of an almost contact metric manifold $\left(\widetilde{M}, \varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon\right)$ is defined by

$$\Psi(\widetilde{X}, \widetilde{Y}) = \langle \varphi \widetilde{X}, \widetilde{Y} \rangle,$$

for all $\widetilde{X}, \widetilde{Y} \in \Gamma(T\widetilde{M})$. When $d\eta = \Psi$, the associated structure is a *contact metric structure* and $\widetilde{M}$ is an *almost Sasakian manifold*.

In 1969, T. Takahashi [11] introduced almost contact manifolds equipped with associated pseudo-Riemannian matric. An almost Sasakian manifold $\left(\widetilde{M}, \varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon\right)$ is called a *Sasakian manifold* if

$$[\varphi \widetilde{X}, \varphi \widetilde{Y}] + \varphi^2 [\widetilde{X}, \widetilde{Y}] - \varphi [\widetilde{X}, \varphi \widetilde{Y}] - \varphi [\varphi \widetilde{X}, \widetilde{Y}] = -2 d\eta(\widetilde{X}, \widetilde{Y}) \xi \tag{3}$$

for all $\widetilde{X}, \widetilde{Y} \in \Gamma(T\widetilde{M})$. A necessary and sufficient condition for an almost contact metric manifold to be a Sasakian manifold is (see [11])

$$\left(\widetilde{\nabla}_{\widetilde{X}} \varphi\right) \widetilde{Y} = \varepsilon \eta(\widetilde{Y}) \widetilde{X} - \langle \widetilde{X}, \widetilde{Y} \rangle \xi, \tag{4}$$

for all $\widetilde{X}, \widetilde{Y} \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian metric $\langle\ ,\ \rangle$.

Example 1.1 ([11]). Let $\mathbb{R}^{2m+2}_{2s}$ be the pseudo-Euclidian space with the indefinite standard Kaehler structure. The pseudo-sphere

$$S^{2m+1}_{2s}(1) = \{p \in \mathbb{R}^{2m+2}_{2s}\ ;\ \langle p, p \rangle = 1\}$$

and the pseudo-hyperbolic space

$$H^{2m+1}_{2s-1} = \{p \in \mathbb{R}^{2m+2}_{2s}\ ;\ \langle p, p \rangle = -1\}$$

are hyperquadrics of $\mathbb{R}^{2m+2}_{2s}$, both of dimension $2m+1$ of index $2s$ an $2s-1$ and of constant sectional curvature 1 and -1 respectively. They have a canonical structure of Sasakian indefinite manifolds, with characteristic vector field ξ spacelike and timelike respectively.

Suppose that $\left(\widetilde{M}^{2m+1}_q, \varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon\right)$ is a Sasakian manifold. Let

$$\Omega_p = \{\widetilde{X} \in T_p\widetilde{M}\ ;\ \eta(\widetilde{X}) = 0\}.$$

For a non-null vector $\widetilde{X}$ in Ω_p, $\widetilde{X}$ and $\varphi\widetilde{X}$ span a non-degenerate 2-plane and hence, we can consider a sectional curvature $K(\widetilde{X}) = K(\widetilde{X}, \varphi\widetilde{X})$. If $K(\widetilde{X})$ is constant for all non-null vectors $\widetilde{X} \in \Omega_p$, we call M_q^{2m+1} to be of constant φ-sectional curvature at p. If $K(\widetilde{X})$ is constant φ-sectional curvature at every point, $K(\widetilde{X})$ is a function of $p \in \widetilde{M}_q^{2m+1}$, say $c(p)$. In this case, if $c(p) = c$ is constant on M_q^{2m+1}, we call $\widetilde{M}_q^{2m+1}$ to be a *Sasakian space form* and is denoted by $\widetilde{M}_q^{2m+1}(c)$.

The curvature tensor of a Saskian space form $\widetilde{M}_q^{2m+1}(\mathbf{c})$ is given by [11]

$$
\begin{aligned}
\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \ & \tfrac{1}{4}(c+3\varepsilon)\{\langle \widetilde{Y},\widetilde{Z}\rangle \widetilde{X} - \langle \widetilde{X},\widetilde{Z}\rangle \widetilde{Y}\} \\
& + \tfrac{1}{4}(\varepsilon c - 1)\{\eta(\widetilde{X})\eta(\widetilde{Z})\widetilde{Y} - \eta(\widetilde{Y})\eta(\widetilde{Z})\widetilde{X}\} \\
& + \tfrac{1}{4}(c-\varepsilon)\{\langle \widetilde{X},\widetilde{Z}\rangle \eta(\widetilde{Y})\xi - \langle \widetilde{Y},\widetilde{Z}\rangle \eta(\widetilde{X})\xi \\
& + \langle \varphi\widetilde{Y},\widetilde{Z}\rangle \varphi\widetilde{X} + \langle \varphi\widetilde{Z},\widetilde{X}\rangle \varphi\widetilde{Y} - 2\langle \varphi\widetilde{X},\widetilde{Y}\rangle \varphi\widetilde{Z}\},
\end{aligned} \tag{5}
$$

for any $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma\left(T\widetilde{M}_q^{2m+1}(c)\right)$.

An interest topic in differential geometry is the study of submanifold in space endowed with an additional structure. For a contact Riemannian manifold $\widetilde{M}^{2m+1}$ with a $(\varphi, \xi, \eta, \langle\ ,\ \rangle)$-structure, A. Bejancu and N. Papaghiuc [4], considere a submanifold M tangent to ξ, they called it semi-invariant if $TM = D \oplus D^{\perp} \oplus < \xi >$, where the first distribution is invariante ($\varphi D = D$) and the second anti-invariant ($\varphi D^{\perp} \subset TM^{\perp}$).

In [1] P. Alegre obtained some results for submanifolds in a Sasakian indefinite manifolds, considering only submanifolds tangent to characteristic vector field ξ.

On the other hand, in the last 40 years, there has been an increasing interest in studying the structure of spacelike submanifolds. This goes back to 1976, when S.Y. Chen and S.T. Yau proved [7] the Calabi-Bernstein conjecture concerning complete maximal spacelike hypersurface of $\mathbb{R}_1^{n+1}$, namely, that the only ones are the spacelike hyperplanes. In [9], S. Nashikawa proved that a complete maximal spacelike hypersurface in $N_q^{n+q}(1)$ is totally geodesic. In [3], J.O. Baek, Q.M. Cheng and Y.J. Suh, obtained an optimal estimate of the squared norm of the second fundamental form for complete spacelike hypersurfaces with constant mean curvature in a locally symmetric Lorentz space satisfying some curvature conditions and characterized the totally umbilical hypersurfaces. In particular, semi-Riemannian space forms $N_p^{n+p}(c)$ are examples of locally symmetric semi-Riemannian spaces. Recently, A. Brasil, R.M. Chaves and M. Mariano [5] extended the result in [3] for higher codimensional spacelike submanifolds with parallel mean curvature vector in a semi-Riemannian space form $N_p^{n+p}(c)$ and extended also to spacelike submanifolds a gap theorem obtained by A. Brasil, A.G. Colares and O. Palmas in [6] for hypersurfaces.

In the context of submanifolds, it is well known T. Ishihara's result (see [8]) that, for an n-dimensional complete maximal spacelike submanifold M^n immersed in $N_p^{n+p}(c)$, if

$c \geq 0$, then M^n is totally geodesic and if $c < 0$, then the square norm S of the second fundamental form satisfies $0 \leq S \leq -npc$.

Motivated by above, in this paper we study the contact normal, semi-invariant, spacelike submanifolds M^s isometrically immersed in a Sasakian manifold $\widetilde{M}_q^{2m+1}$ with pseudo-Riemannian metric of index $q = (2m+1) - s$ whose characteristic vector field is timelike.

The paper organized as follows. Section 2 is devoted to the some basic definitions and proprieties of contact normal spacelike submanifolds. In section 2, we prove that a contact normal spacelike semi-invariant hypersurface is invariant and totally geodesic. In section 3 we study the integrability of both of the distributions D and $D^{\perp}$, the geometry of its leaves, and we obtain a classification result for submanifolds totally umbilic (see theorem 4.10).

2. Preliminaries

In this paper, manifolds and tensor fields are supposed to be class C^{∞}.

Let $\widetilde{M}_q^{2m+1} = \left(\widetilde{M}_q^{2m+1}, \varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon\right)$ be a Sasakian manifold and let M^s be an s-dimensional (nondegenerate) submanifold isometrically immersed in M_q^{2m+1}. M^n is said to be a *contact normal submanifold* if the characteristic vector field ξ is normal to M^s

A contact normal submanifold M^s of M_q^{2m+1} is called a semi-invariant submanifold if there exist on M^s two differentiable orthogonal distributions D and $D^{\perp}$ such that the following conditions are fulfilled:

(1) $TM = D \oplus D^{\perp}$.
(2) The distribution D is invariant by φ, that is, $\varphi(D_p) = D_p$ for each $p \in M^s$.
(3) The distribution $D^{\perp}$ is anti-invariant by φ, that is, $\varphi(D_p^{\perp}) \subset D_p^{\perp}$ for each $p \in M^s$.

The distribution D and $D^{\perp}$ are called the invariant distribution and the anti-invariant distribution respectively, of M^s. A contact normal, semi-invariant submanifold M^s is called invariant (resp. anti-invariant) if $D^{\perp} = \{0\}$ (resp. $D = \{0\}$).

Example 2.1. Let (M^{2n}, h, J) be a Kaehlerain manifold. Let $\widetilde{M}^{2n+1}$ be the manifold

$$\widetilde{M} = \{M^{2n} \times \mathbb{R}, \langle\ ,\ \rangle = h - dt^2\}$$

Denote a vector field on $\widetilde{M}$ by $\widetilde{X} = (X, \eta(\widetilde{X})\frac{d}{dt})$, where X is tangent to M^{2n}, t is the coordenate of $\mathbb{R}$ and $\eta(\widetilde{X})$ is a smooth function on $\widetilde{M}$. Set $\eta = dt$ so that $\xi = (0, \frac{d}{dt})$ is a timelike global vector field. Then with

$$\varphi(X, \eta(\widetilde{X})\tfrac{d}{dt}) = (JX, 0)$$

$$\left\langle (X, \eta(\widetilde{X})\tfrac{d}{dt}), (Y, \eta(\widetilde{Y})\tfrac{d}{dt}) \right\rangle = h(X,Y) - \eta(\widetilde{X})\eta(\widetilde{Y}),$$

we recover a Sasakian manifold $\left(\widetilde{M}, \varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon\right)$ with $\varepsilon = -1$.

In this case, M^{2n} is a contact normal invariant submanifold of $\widetilde{M}_1^{2n+1}$ and if N^n is a totally real submanifold of M^{2n} then N^n is a contact normal anti-invariant submanifold of $\widetilde{M}_1^{2n+1}$.

For a $(2m+1)$-dimensional Sasakian manifold $\widetilde{M}_q^{2m+1}$ we have the following result:

Lemma 2.2. *If* $\left(\widetilde{M}_q^{2m+1}, \varphi, \xi, \eta, \langle\ ,\ \rangle, \varepsilon\right)$ *is a Sasakian manifold, then*

$$\widetilde{\nabla}_{\widetilde{X}}\xi = \varepsilon\varphi(\widetilde{X})$$

for any $\widetilde{X} \in \Gamma(T\widetilde{M})$, *where* $\widetilde{\nabla}$ *is the Levi-Civita connection of the pseudo-Riemannian metric* $\langle\ ,\ \rangle$.

Proof. From $\eta(\varphi(\widetilde{Y})) = \langle \xi, \varphi\widetilde{Y}\rangle = 0$ we get

$$\langle \widetilde{\nabla}_{\widetilde{X}}\xi, \varphi\widetilde{Y}\rangle + \langle \xi, (\widetilde{\nabla}_{\widetilde{X}}\varphi)\widetilde{Y}\rangle = 0, \tag{6}$$

and using (4) in (6) we obtain

$$\langle \widetilde{\nabla}_{\widetilde{X}}\xi, \varphi\widetilde{Y}\rangle = -\varepsilon\{\varepsilon\eta(\widetilde{X})\eta(\widetilde{Y}) - \langle \widetilde{X}, \widetilde{Y}\rangle\}. \tag{7}$$

Hence, from (2) we get

$$\langle \widetilde{\nabla}_{\widetilde{X}}\xi, \varphi\widetilde{Y}\rangle = \varepsilon\langle \varphi\widetilde{X}, \varphi\widetilde{Y}\rangle.$$

□

Now, we assume that M^s is a contact normal semi-invariant spacelike submanifold in a Sasakian manifold M_q^{2m+1}, of codimension q. Note that the codimension is equal to the index. Hence $\varepsilon = -1$ and the characteristic vector field ξ is timelike.

As usual, $\widetilde{\nabla}$ (resp. ∇) be Levi-Civita connection with respect to $\langle\ ,\ \rangle$ (resp. $\langle\ ,\ \rangle|_M$) and $\nabla^\perp$ the connection in the normal bundle on M^{2n}. The Gauss and the Weingarten formulas are given respectively by

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y), \\ \widetilde{\nabla}_X N &= -A_N X + \nabla^\perp_X N, \end{aligned} \tag{8}$$

for any X, Y vectors tangent to M^{2n} and any N vector normal to M, where A_N is the shape operator in direction N and σ is the second fundamental form of M. The shape operator and the second fundamental form are related by

$$\langle \sigma(X, Y), N\rangle = \langle A_N X, Y\rangle. \tag{9}$$

Let R and $\widetilde{R}$ the curvature tensors of ∇ and $\widetilde{\nabla}$, respectively. Then, the Gauss equation is given by

$$\begin{aligned} \langle R(X, Y)Z, W\rangle = &\ \langle \widetilde{R}(X, Y)Z, W\rangle + \langle \sigma(X, Z), \sigma(Y, W)\rangle \\ &-\langle \sigma(X, W), \sigma(Y, Z)\rangle, \end{aligned} \tag{10}$$

From (4) we have,

$$\left(\widetilde{\nabla}_{\widetilde{X}}\varphi\right)\widetilde{Y} = -\eta(\widetilde{Y})\widetilde{X} - \langle \widetilde{X}, \widetilde{Y}\rangle \xi, \tag{11}$$

and from lemma 2.2,

$$\widetilde{\nabla}_{\widetilde{X}}\xi = -\varphi(\widetilde{X}) \tag{12}$$

If $X, Y \in \Gamma(TM)$, from (11)

$$(\widetilde{\nabla}_X\varphi)Y = -\langle X, Y\rangle\xi. \tag{13}$$

The projection morphisms of TM to D and $D^{\perp}$ are denoted respectively by P and Q. Then, we have

$$X = PX + QX \tag{14}$$

for all $X \in \Gamma(TM)$.

If N is a vector field in the normal bundle $TM^{\perp}$, we put

$$\varphi N = tN + fN, \tag{15}$$

where tN and fN are the tangential and the normal components of φN, respectively.

Lemma 2.3. *Let M^s be a contact normal semi-invariant spacelike submanifold in a Sasakian manifold $\widetilde{M}_q^{2m+1}$. Then,*

(1) $P(\nabla_X\varphi PY) - P(A_{\varphi QY}X) = \varphi(P\nabla_X Y)$;
(2) $Q(\nabla_X\varphi PY) - Q(A_{\varphi QY}X) = t\sigma(X, Y)$;
(3) $\sigma(X, \varphi PY) + \nabla^{\perp}_X\varphi QY = \varphi(Q\nabla_X Y) + f\sigma(X, Y) - \langle X, Y\rangle\xi$,

for any $X, Y \in \Gamma(TM)$.

Proof. From the Gauss and the Weingarten formulas (8), we have

$$\begin{aligned} \widetilde{\nabla}_X\varphi PY = &\ \nabla_X\varphi PY + \sigma(X, \varphi PY), \\ \widetilde{\nabla}_X\varphi QY = &\ -A_{\varphi QY}X + \nabla^{\perp}_X\varphi QY. \end{aligned} \tag{16}$$

By using (13) we obtain

$$\widetilde{\nabla}_X\varphi PY = \varphi(\widetilde{\nabla}_X PY) - \langle X, PY\rangle\xi$$

From (16), we get

$$\widetilde{\nabla}_X\varphi PY = \varphi(\nabla_X PY) + \varphi\sigma(X, PY) - \langle X, PY\rangle\xi. \tag{17}$$

Similarly, we obtain

$$\widetilde{\nabla}\varphi QY = \varphi(\nabla_X QY) + \varphi\sigma(X, QY) - \langle X, QY\rangle\xi. \tag{18}$$

Therefore,

$$\widetilde{\nabla}\varphi PY + \widetilde{\nabla}\varphi QY = \varphi(\nabla_X Y) + \varphi\sigma(X, Y) - \langle X, Y\rangle\xi,$$

which implies

$$\nabla \varphi PY + \sigma(X, PY) - A_{\varphi QY}X + \nabla^{\perp}_{X}\varphi QY = \varphi(\nabla_X Y) + \varphi\sigma(X,Y) - \langle X, Y\rangle\xi. \tag{19}$$

Then, (1)-(3) follows from (19) by taking the components of D and $D^{\perp}$. □

From the Gauss and Weingarten formulas for M^s, we have

LEMMA 2.4. *Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}^{2m+1}_q$. Then,*

(1) *$A_\xi PX = \varphi PX$ and $\nabla^{\perp}_{PX}\xi = 0$,*
(2) *$A_\xi QX = 0$ and $\nabla^{\perp}_{QX}\xi = -\varphi QX$,*

for all $X \in \Gamma(TM)$.

3. CONTACT NORMAL SPACELIKE HYPERSURFACES

LEMMA 3.1. *Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}^{2m+1}_q$. Then, for any $X, Y \in \Gamma(TM)$, $\sigma(X,Y) \in <\xi>^{\perp}$, where $<\xi>$ is the 1-dimensional distribution spanned by ξ on $\widetilde{M}^{2m+1}_q$.*

Proof. Note that, if $X \in D^{\perp}$ (or $Y \in D^{\perp}$), from lemma 2.4, $A_\xi X = 0$ (or $A_\xi Y = 0$). Hence, in this case we have

$$\langle \sigma(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle = 0.$$

If $X, Y \notin D^{\perp}$, then

$$\begin{aligned} \langle \xi, \sigma(X,Y)\rangle = &\ \langle A_\xi X, Y\rangle = \langle \varphi X, Y\rangle \\ = &\ -\langle X, \varphi Y\rangle = \langle X, \varphi Y\rangle = -\langle A_\xi Y, X\rangle \\ = &\ -\langle \xi, \sigma(X,Y)\rangle. \end{aligned}$$

Thus, $\langle \xi, \sigma(X,Y)\rangle = 0$. □

Remember that a submanifold M^s is totally geodesic if the second fundamental form vanishes identically, that is, $\sigma = 0$.

THEOREM 3.2. *Let M^{2n} be a contact normal spacelike hypersurface in a Sasakian manifold $\widetilde{M}^{2n+1}_1$. Then,*

(1) *M^{2n} is an invariant and totally geodesic submanifold,*
(2) *$(M^{2n}, \varphi, \langle\ ,\ \rangle)$ is a Kaehlerian manifold. In particular, if $M^{2n+1}_1(c)$ is of constant φ-sectional curvature c, then $(M^{2n}, \varphi, \langle\ ,\ \rangle)$ is a space of constant holomorphic sectional curvature c.*

Proof. The assertion (1) follow from lemma 3.1. Now, note that restricted to M^{2n}, φ satisfies $\varphi^2 = -I$ that is, φ is an almost complex structure on M^{2n}. Moreover we have $\eta(X) = 0$ for any $X \in \Gamma(TM^{2n})$. Therefore, from (3), we get

$$[\varphi X, \varphi Y] - [X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] = 0. \tag{20}$$

Hence, φ is a complex structure in M^{2n}.

Using the fact that $\langle \varphi X, \varphi Y\rangle = \langle X, Y\rangle$ for any X, Y on M^{2n}, we conclude that $(M^{2n}, \varphi, \langle\ ,\ \rangle)$ is a Kaehlerian manifold.

From Gauss equation (10), we have

$$\widetilde{R}(X, \varphi X, X, \varphi X) = R(X, \varphi X, X, \varphi X),$$

Therefore, if $M_1^{2n+1}(c)$ is of constant φ-sectional curvature c then, taking account (5) we obtain

$$R(X, \varphi X, X, \varphi X) = c.$$

□

4. Integrability of Distribution D and $D^\perp$ and Geometry of Leaves

Lemma 4.1. *Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. Then, for any $X, Y \in \Gamma(D^\perp)$*

$$A_{\varphi X} Y = A_{\varphi Y} X. \tag{21}$$

Proof. If $Z \in \Gamma(TM)$, from (13)

$$\varphi \widetilde{\nabla}_Z Y = \langle Z, Y\rangle \xi + \widetilde{\nabla}_Z \varphi Y \tag{22}$$

Hence, if $X, Y \in \Gamma(D^\perp)$, from Gauss equation, by using (22), for any $Z \in \Gamma(TM)$ we obtain

$$\begin{aligned} \langle A_{\varphi X} Y, Z\rangle = &\ \langle \sigma(Y, Z), \varphi X\rangle = \langle \widetilde{\nabla}_Z Y, \varphi X\rangle \\ = &\ -\langle \varphi \widetilde{\nabla}_Z Y, Z\rangle = \langle A_{\varphi Y} Z, X\rangle \\ = &\ \langle A_{\varphi Y} X, Z\rangle. \end{aligned}$$

□

Theorem 4.2. *Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. Then, the anti-invariant distribution $D^\perp$ is integrable.*

Proof. For any $X, Y \in \Gamma(D^\perp)$, from assertion (1) of lemma 2.3 we obtain

$$\varphi P(\nabla_X Y) = -P(A_{\varphi Y} X). \tag{23}$$

Applying φ to (23) we get

$$-\varphi P(A_{\varphi Y} X) = -P(\nabla_X Y). \tag{24}$$

Hence, from lemma (4.1),

$$P([X,Y]) = P(\nabla_X Y - \nabla_Y X) = \varphi P(A_{\varphi Y} X) - \varphi P(A_{\varphi X} Y) = 0,$$

that is $[X,Y] \in \Gamma(D^{\perp})$. □

THEOREM 4.3. *Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. Then, the invariant distribution D is integrable if and only if,*

$$\sigma(X, \varphi Y) = \sigma(\varphi X, Y), \tag{25}$$

for any $X, Y \in \Gamma(D)$.

Proof. From assertion (3) of lemma 2.3, for any $X, Y \in \Gamma(D)$, we have

$$\sigma(X, \varphi Y) = \varphi Q \nabla_X Y + f\sigma(X,Y) - \langle X, Y\rangle \xi, \quad \text{and}$$

$$\sigma(\varphi X, Y) = \varphi Q \nabla_Y X + f\sigma(X,Y) - \langle X, Y\rangle \xi.$$

Hence,

$$\sigma(X, \varphi Y) - \sigma(\varphi X, Y) = \varphi Q([X,Y]). \tag{26}$$

So, D is integrable if only if $Q([X,Y]) = 0$. □

LEMMA 4.4. *The condition (25) is satisfied if only if*

$$\langle \sigma(X, \varphi Y) - \sigma(Y, \varphi X), \varphi Z\rangle = 0 \tag{27}$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Proof. If (25) is satisfied then (27) is satisfied. Suppose that (27) is satisfied. Then, from (26) for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$ we have

$$\langle \varphi Q([X,Y]), \varphi z\rangle = \langle Q([X,Y]), Z\rangle = 0. \tag{28}$$

If we take in (28) $Z = Q([X,Y])$, we obtain $\|Q([X,Y])\|^2 = 0$. Then (25) follows from (26). □

Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. We say that M^s is *D-geodesic submanifold* if its second fundamental form satisfies

$$\sigma(X,Y) = 0 \quad \text{for any} \quad X, Y \in \Gamma(D).$$

THEOREM 4.5. *Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. Then,*

(1) *the distribution D is integrable and its leaves are totally geodesic in M^s if only if,*

$$\langle \sigma(X,Y), \varphi Z\rangle = 0$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$;

(2) *the distribution D is integrable and its leaves are totally geodesic in $\widetilde{M}_q^{2m+1}$ if only if, M^s is D-geodesic.*

Proof. Suppose D is integrable and each leaf of D is totally geodesic in M^s. Then $\nabla_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Hence, if $Z \in \Gamma(D^\perp)$ we have

$$\begin{aligned}\langle \sigma(X,Y), \varphi Z\rangle = & -\langle \varphi\sigma(X,Y), Z\rangle = -\langle \varphi(\widetilde{\nabla}_X Y), Z\rangle \\ = & -\langle \widetilde{\nabla}_X \varphi Y), Z\rangle = -\langle \nabla_X \varphi Y), Z\rangle = 0.\end{aligned}$$

Conversely, if $\langle \sigma(X,Y), \varphi Z\rangle = 0$ for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$, then

$$\langle \sigma(X, \varphi Y) - \sigma(\varphi X, Y), \varphi Z\rangle = 0.$$

Thus, from lemma 4.1, D is integrable.

Now, for $X, Y \in \Gamma(TM)$, using (13) and Gauss formula, we obtain

$$\varphi(\nabla_X Y) = \sigma(X, \varphi Y) - \varphi(\sigma(X,Y)) + \langle X, Y\rangle\xi + \nabla_X \varphi Y. \tag{29}$$

Therefore, for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$, from (29) we obtain

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = & \ \langle \varphi(\nabla_X Y), \varphi Z\rangle \\ = & \ \langle \sigma(X, \varphi Y), \varphi Z\rangle = 0,\end{aligned}$$

that is, $\nabla_X Y \in \Gamma(D)$ and each leaf of D is totally geodesic in M^s. Thus we get the assertion (1).

Suppose that D is integrable and its leaves are totally geodesic in $\widetilde{M}_q^{2m+1}$. Then $\widetilde{\nabla}_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$.

Hence, for any $X, Y \in \Gamma(D)$ and $N \in \Gamma(TM^\perp)$, we have,

$$\langle \sigma(X,Y), N\rangle = \langle \widetilde{\nabla}_X Y, N\rangle = 0,$$

that is, M^s is D-geodesic.

Now, if M^s is D-geodesic, then for any $X, Y \in \Gamma(D)$, $\sigma(X, \varphi Y) = \sigma(\varphi X, Y) = 0$. Hence, from lemma 4.3, D is integrable and for $X, Y \in \Gamma(D)$ and $N \in \Gamma(TM^\perp)$,

$$\langle \widetilde{\nabla}_X Y, N\rangle = \langle \sigma(X,Y), N\rangle = 0.$$

Thus, $\widetilde{\nabla}_X Y \in \Gamma(TM)$.

Hence, if $Z \in \Gamma(D^\perp)$ then, for all $X.Y \in \Gamma(D)$, using (13) and Gauss formula we have

$$\begin{aligned}\langle \widetilde{\nabla}_X Y, Z\rangle = & \ \langle \varphi\widetilde{\nabla}_X Y, \varphi Z\rangle \\ = & \ \langle \widetilde{\nabla}_X \varphi Y, \varphi Z\rangle \\ = & \ \langle \nabla_X Y + \sigma(X, \varphi Y), \varphi Z\rangle = 0.\end{aligned}$$

So, $\widetilde{\nabla}_X Y \in \Gamma(D)$ therefore each leaf of D is totally geodesic in $\widetilde{M}^{2m+1}$. $\square$

Let ϑ be the orthogonal complementary subbundle to $\varphi D^\perp$ in $TM^\perp$ and let $M_\perp$ be a leaf of $D^\perp$.

THEOREM 4.6. *The submanifold $M_\perp$ is totally geodesic in M^s if only if $\sigma(X,Z) \in \Gamma(\vartheta)$ for all $X \in \Gamma(D^\perp)$ and $Z \in \Gamma(D)$.*

Proof. From assertion (1) of lemma 2.3, for any $X, Y \in \Gamma(D^\perp)$ we have

$$\varphi(P\nabla_X Y) = -P\left(A_{\varphi Y}X\right).$$

Thus, for any $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D)$ we obtain

$$\begin{aligned}\langle \sigma(X,Z), \varphi Y\rangle &= \langle P\left(A_{\varphi Y}X\right), Z\rangle \\ &= -\langle \varphi(P\nabla_X Y), Z\rangle \\ &= \langle P\nabla_X Y, \varphi Z\rangle\end{aligned}$$

Note that $M_\perp$ is totally geodesic in M^s if only if for any $X, Y \in \Gamma(D^\perp)$, $\nabla_X Y \in \Gamma(D^\perp)$. Hence, $\langle \sigma(X,Z), \varphi Y\rangle = 0$ if only if $M_\perp$ is totally geodesic in M^s. □

A contact normal semi-invariant spacelike submanifold M^s in $\widetilde{M}_q^{2m+1}$ is called a *mixed geodesic submanifold* if its second fundamental form satisfies $\sigma(X,Z) = 0$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$.

As a consequence of the theorem 4.6, we have the following result:

COROLLARY 4.7. *Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. If M^s is a mixed geodesic submanifold then, each leaf of $D^\perp$ is totally geodesic.*

Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. We say that M^s is $D^\perp$*-geodesic submanifold* if its second fundamental form satisfies

$$\sigma(X,Y) = 0 \quad \text{for any} \quad X, Y \in \Gamma(D^\perp).$$

THEOREM 4.8. *The submanifold $M_\perp$ is totally geodesic in $\widetilde{M}_q^{2m+1}$ if only if M^s is $D^\perp$-geodesic and*

$$\langle \sigma(X,Z), \varphi Y\rangle = 0$$

for all $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D)$.

Proof. Note that, $M_\perp$ is totally geodesic in $\widetilde{M}_q^{2m+1}$ if only if $\widetilde{\nabla}_X Y \in \Gamma(D^\perp)$ for any $X, Y \in \Gamma(D^\perp)$. Hence, from Gauss formula, $M_\perp$ is totally geodesic in $\widetilde{M}_q^{2m+1}$ if only if $\nabla_X Y \in \Gamma(D^\perp)$ and $\sigma(X,Y) = 0$ for any $X, Y \in \Gamma(D^\perp)$.

Therefore, $M_\perp$ is totally geodesic in $\widetilde{M}_q^{2m+1}$ if only if $M_\perp$ is totally geodesic in M^s and $D^\perp$-geodesic.

From theorem 4.6, $M_\perp$ is totally geodesic in $\widetilde{M}_q^{2m+1}$ if only if $\sigma(X,Z) \in \Gamma(\vartheta)$ for all $X \in \Gamma(D^\perp)$ and $Z \in \Gamma(D)$, that is, $\langle \sigma(X,Z), \varphi Y\rangle = 0$ for all $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D)$. □

Let H be the mean curvature vector field. We recall that M^s is a maximal submanifold if $\|H\| = 0$ and M^s is totally umbilical submanifold if $\sigma(X,Y) = \langle X,Y\rangle H$.

Theorem 4.9. *Let M^s be a contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. If M^s is invariant then, it is maximal.*

Proof. In this case M^s is even dimensional. So we can choose an orthonormal basis on M^s $\{e_1, ..e_r, e_{r+1} = \varphi e_1, ..., e_{2r} = \varphi e_r\}$.

Because M^s is invariant, from lemma 4.3, $\sigma(X, \varphi Y) = \sigma(\varphi X, Y)$ for any $X, Y \in \Gamma(TM)$. Hence

$$\sigma(\varphi X, \varphi Y) = \sigma(Y, \varphi^2 Y) = -\sigma(X,Y).$$

Therefore, $\sigma(\varphi e_i, \varphi e_i) = \sigma(e_i, e_i)$ for $i = 1, ..., r$. So $\sum_{i=1}^{2r} \sigma(e_i, e_i) = 0$. □

We have the following classification theorem for totally umbilical contact normal semi-invariant spacelike submanifold M^s of a Sasakian manifold M_q^{2m+1}.

Theorem 4.10. *Let M^s be a totally umbilical contact normal, semi-invariant spacelike submanifold of a Sasakian manifold $\widetilde{M}_q^{2m+1}$. Then*

(1) *M^s is totally geodesic or*
(2) *M^s is anti-invariant.*

Proof. Let M^s be a totally umbilical. If M^s is invariant, from theorem 4.9, M^s is totally geodesic. So, if M^s is not totally geodesic then, $\dim D^\perp \neq 0$.

Suppose M^s is totally umbilical but not totally geodesic and take $X \in \Gamma(D)$. Hence,

$$\sigma(X, \varphi X) = \langle X, \varphi X\rangle H = 0, \tag{30}$$

and

$$\sigma(X,X) = \langle X, X\rangle H = \|X\|^2 H. \tag{31}$$

Taking account (30) and (31), from assertion (3) of lemma 2.3, we obtain

$$\langle f\sigma(X,X), \xi\rangle + \|X\|^2 = 0.$$

Thus, $\|X\|^2 \langle fH, \xi\rangle + \|X\|^2 = 0$. Note that $\langle fH, \xi\rangle = 0$, therefore $X = 0$, that is, $\dim D = 0$ and M^s is anti-invariant. □

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