Density of Axiom A for area contracting surface embeddings

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Abstract

We prove that Axiom A is open and dense in the space of C^1 area contracting orientation-preserving embeddings on compact orientable surfaces with boundary. This settles the area contracting version of the *Smale's conjecture* [**23**].

1. *Introduction*

In 1967 S. Smale raised the question whether the set of Axiom A diffeomorphisms is open and dense in the space of C^r diffeomorphisms of a compact manifold, $\forall r \geq 1$ (c.f. p. 779 in [**23**]). The answer is known to be negative in dimension greater than 2 ($\forall r \geq 1$) and in dimension 2 ($\forall r \geq 2$) by classical results [**1**], [**18**], [**24**]. The remainder part is the nowadays called

Smale's conjecture. *The set of Axiom A diffeomorphisms is open and dense in the space of C^1 diffeomorphisms of a compact surface.*

Partial solutions have been given elsewhere in the literature. For instance, [**8**], [**9**], [**21**] and [**13**] proved respectively that Axiom A is dense in the interval, the circle, in the complement of the closure of the sets of surface diffeomorphisms exhibiting homoclinic tangencies and in the Benedicks-Carleson toy models. See also the recent result [**16**] where it is proved that a C^1 generic surface diffeomorphism satisfies that there are no tangencies between the leaves of the stable and unstable foliations of any pair of compact, locally maximal, hyperbolic invariant sets of saddle type.

This paper proves a result which, at the same time, settles the area contracting version of the conjecture:

Theorem 1.1. *The set of Axiom A embeddings is open and dense in the space of C^1 area contracting orientation-preserving embeddings of a compact orientable surface with boundary.*

By a *surface* we mean a two-dimensional Riemannian manifold M. It turns out that the space of C^1 embeddings of M is a Baire metric space if endowed with the C^1 topology. A *surface with boundary* is one whose boundary ∂M is nonempty. An embedding f is *area contracting*, if $|\det(Df(x))| < 1$ for all $x \in M$ where $\det(\cdot)$ denotes the jacobian operation, and *Axiom A* if its nonwandering set $\Omega(f)$ is both hyperbolic and the closure of the periodic points. Recall that $\Omega(f)$ consists of those points $p \in M$ such that $U \cap (\cup_{n\in\mathbb{N}^+} f^n(U)) \neq \emptyset$ for all neighborhood U of p while a point $p \in M$ is *periodic* if there is an integer $n \in \mathbb{N}$ such that $f^n(p) = p$. A compact invariant set Λ is *hyperbolic* if there is a continuous tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^u_\Lambda$ over Λ and positive constants K, λ such that

$$\|Df^n(x)/E^s_x\| \leq Ke^{-\lambda n} \quad \text{and} \quad m(Df^n(x)/E^u_x) \geq K^{-1}e^{\lambda n}, \quad \forall x \in \Lambda, n \in \mathbb{N},$$

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where $m(\cdot)$ denotes the co-norm operation.

Theorem 1.1 will be obtained from a powerful dichotomy for surface diffeomorphisms stated as follows.

Recall that if f is Axiom A, then the Spectral Decomposition Theorem asserts that $\Omega(f)$ is the disjoint union of finitely many isolated homoclinic classes $\Lambda_1, \cdots, \Lambda_k$. We then say that f *has no cycles* if there is no $\{i_1, \cdots, i_{r+1}\} \subset \{1, \cdots, k\}$ with $i_1 = i_{r+1}$ such that

$$(W^u(\Lambda_{i_j}) \setminus \Lambda_{i_j}) \cap (W^s(\Lambda_{i_{j+1}}) \setminus \Lambda_{i_{j+1}}) \neq \emptyset, \qquad \forall j \ (\bmod r).$$

Here $W^s(\Lambda)$ and $W^u(\Lambda)$ denotes the stable and unstable sets of a hyperbolic set Λ defined by

$$W^s(\Lambda) = \left\{ x \in M : \lim_{n\to\infty} d(f^n(x), \Lambda) = 0 \right\}$$

and

$$W^u(\Lambda) = \left\{ x \in M : \lim_{n\to-\infty} d(f^n(x), \Lambda) = 0 \right\}$$

respectively [**7**].

We denote by $\mathrm{Per}(f)$ the set of periodic points of f. If $p \in \mathrm{Per}(f)$ we denote by n_p (or $n_{p,f}$ to emphasize f) the minimum of the set of positive integers n satisfying $f^n(p) = p$ and call it the *period* of p. By the eigenvalues of $p \in \mathrm{Per}(f)$ we mean those of the linear map $Df^{n_p}(p) : T_pM \to T_pM$. We say that p is *hyperbolic* if none of its eigenvalues has modulus 1; a *sink* if its eigenvalues have moduli less than 1; and a *source* if it is a sink of f^{-1}. Denote by $\mathrm{Sink}(f)$, $\mathrm{Source}(f)$ and $\mathrm{Spir}(f)$ the set of sinks, sources and periodic points with nonreal eigenvalues of f respectively. Define

$$\mathrm{Sink}_{\mathbb{C}}(f) = \mathrm{Spir}(f) \cap \mathrm{Sink}(f) \quad \text{and} \quad \mathrm{Source}_{\mathbb{C}}(f) = \mathrm{Spir}(f) \cap \mathrm{Source}(f).$$

Let $\mathrm{Diff}^1(M)$ denote the space of C^1 diffeomorphisms of M endowed with the usual C^1-topology. A subset R of $\mathrm{Diff}^1(M)$ is *residual* if it contains the intersection of a countable family of open and dense subsets. We say that *a C^1 generic diffeomorphism satisfies a property (P)* if there is $R \subset \mathrm{Diff}^1(M)$ residual such that (P) holds for every $f \in R$. By a *closed surface* we mean a compact boundaryless surface. The closure operation is denoted by $\mathrm{Cl}(\cdot)$.

A result announced by Sambarino in 1997 (and which was a precursor of his joint paper with Pujals [**21**]) asserted that every diffeomorphism f of a closed surface for which $\mathrm{Cl}(\mathrm{Sink}(f)) \cap \mathrm{Cl}(\mathrm{Source}(f)) = \emptyset$ can be C^1 approximated either by one which exhibits a homoclinic bifurcation or by one which is essentially hyperbolic (a property that is weaker than Axiom A). The result below strengths it as follows.

Theorem 1.2. *A C^1 generic orientation-preserving diffeomorphisms f of a closed orientable surface either is Axiom A without cycles or satisfies*

$$\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \cap \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f)) \neq \emptyset.$$

Proof of Theorem 1.1. Let M be a compact orientable manifold with boundary ∂M. Denote by $\mathrm{Int}(\cdot)$ the interior operation in M. If an embedding $f : M \to M$ satisfies $f(M) \subset \mathrm{Int}(M)$ we can cap each component of ∂M with a 2-dimensional disk in order to obtain a closed orientable surface $\hat{M}$ and a C^1 orientation-preserving diffeomorphism $\hat{f}$ of $\hat{M}$ such that M is a compact 2-dimensional submanifold with boundary of $\hat{M}$ satisfying $\hat{f}/M = f$. Furthermore, we can do the capping in a way that each of the above disks belongs to the unstable set of a source of $\hat{f}$. Because of this, the fact that f is area contracting and $\hat{f}/M = f$ we have that there is a neighborhood $\hat{\mathcal{U}}$ of $\hat{f}$ in $\mathrm{Diff}^1(\hat{M})$ such that $\mathrm{Cl}(\mathrm{Sink}(\hat{g})) \cap \mathrm{Cl}(\mathrm{Source}(\hat{g})) = \emptyset$ (and so $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(\hat{g})) \cap \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(\hat{g})) \neq \emptyset$) for all $\hat{g} \in \hat{\mathcal{U}}$. Then, by Theorem 1.2, there is an open subset of diffeomorphisms $\hat{g}$ which are C^1 close to $\hat{f}$ all of which are Axiom A Taking $g = \hat{g}/M$

we obtain an open subset of Axiom A embeddings which is C^1 close to f. Then, the result follows since the set of embeddings $f : M \to M$ with $f(M) \subset Int(M)$ is open and dense in the space of C^1 embeddings of M. □

Let us present more corollaries of the above dichotomy. The first one improves the Mañé dichotomy (c.f. Corollary II p. 506 in [**12**]) in the orientation-preserving case as well as [**15**] (see also [**14**]).

Corollary 1.3. *A C^1 generic orientation-preserving diffeomorphism on a closed orientable surface either has infinitely many sinks* and *sources* with nonreal eigenvalues *or is Axiom A without cycles.*

For the next corollary we recall that a compact invariant set Λ of f is *transitive* if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$ where $\omega(x)$ is the omega-limit set of x (see [**6**] for details). We say that Λ is an *attractor* if it is transitive and exhibits a neighborhood U such that

$$\Lambda = \bigcap_{n \in \mathbb{N}} f^n(U)$$

A *repeller* is an attractor for f^{-1}. An attractor or repeller is hyperbolic if it does as a compact invariant set. Hyperbolic attractors (resp. repellers) include the sinks (resp. sources) but not conversely. The following implies the abundance of diffeomorphisms on closed orientable surfaces exhibiting both hyperbolic attractors and hyperbolic repellers.

Corollary 1.4. *The set of C^1 diffeomorphisms exhibiting both hyperbolic attractors* and *hyperbolic repellers is open and dense in the set of all C^1 orientation-preserving diffeomorphisms of a closed orientable surface.*

Our next corollary is related to the nowadays classical Araujo's thesis [**2**] which claims that a C^1 generic diffeomorphism on a closed surface exhibits either infinitely many sinks or finitely many hyperbolic attractors to which every positive orbit in a full Lebesgue measure set converge. Indeed, we improve this result in the orientation-preserving case as follows.

Corollary 1.5. *A C^1 generic orientation-preserving diffeomorphisms of a closed orientable surface either has infinitely many sinks or is Axiom A without cycles and the union of the stable sets of the attractors form a full Lebesgue measure set.*

Proof. Let f be a C^1 generic orientation-preserving diffeomorphism with finitely many sinks of a closed orientable surface M. Since $\mathrm{Sink}(f)$ is finite we have $\mathrm{Sink}_{\mathbb{C}}(f)$ also is and then $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \cap \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f)) = \emptyset$. Consequently, Theorem 1.2 and the genericity of f imply that f is Axiom A without cycles. It remains to prove that the union of the stable sets of the attractors has full Lebesgue measure in M. For this we appeal to an argument due to Araujo [**2**].

Define the set-valued map $\Phi : \mathrm{Diff}^1(M) \to 2^M_c$ by $\Phi(h) = m(B(h))$, where m is the Lebesgue measure in M and $B(h)$ denotes the union of the stable sets of the attractors of h. It is not difficult to prove that Φ is lower semicontinuous, so, there is a residual subset $\mathcal{R}^1 \subset \mathrm{Diff}^1(M)$ such that every $h \in \mathcal{R}^1$ is a semicontinuity point of Φ ([**10**],[**11**]). Clearly we can assume that $f \in \mathcal{R}^1$, i.e., f is a continuity point of Φ. Now, let g be a diffeomorphism of class C^2 which is C^1 close to f. By the Ω-stability Theorem [**17**] we have that g is Axiom A. In this case we have that $\Phi(g) = m(M)$ since g is an Axiom A diffeomorphism of class C^2 (see [**4**]). As g is arbitrarily close to f which is a continuity point of Φ we conclude that $m(B(f)) = m(M)$. This completes the proof. $\square$

To state our final corollary we need some short definitions. We say that a nonempty compact invariant set Λ of $f \in \mathrm{Diff}^1(M)$ has a *dominated splitting* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ over Λ with $E_x \neq 0$ and $F_x \neq 0$ for all $x \in \Lambda$ and positive numbers K, λ such that

$$\frac{\|Df^n(x)/E_x\|}{m(Df^n(x)/F_x)} \leq Ke^{-\lambda n}, \qquad \forall x \in \Lambda, \forall n \in \mathbb{N}.$$

In case that $\|Df^n(x)/E_x\| \leq Ke^{-\lambda n}$ for all $x \in \Lambda$ and $n \in \mathbb{N}$ we say that the dominated splitting has a *contracting direction*.

Dominated splittings play fundamental role in the study of hyperbolic dynamical systems as shown the breakthrough results [**12**], [**21**]. In 2000 Asaoka [**3**] proved that for C^2 diffeomorphisms f with infinitely many sinks of M there is no dominated splitting with a contracting direction on $\mathrm{Cl}(\mathrm{Sink}(f)) \setminus \mathrm{Sink}(f)$. Afterward [**15**] considered the possibility of proving such a property among the C^1 generic surface diffeomorphisms instead. As a partial answer it was proved in Corollary 1.2 of [**15**] that a C^1 generic surface orientation-preserving diffeomorphism of a closed orientable surface satisfies that there is no dominated splitting either for $\mathrm{Cl}(\mathrm{Sink}(f)) \setminus \mathrm{Sink}(f)$ or $\mathrm{Cl}(\mathrm{Source}(f)) \setminus \mathrm{Source}(f)$. The following corollary (extending the aforementioned corollary in [**15**]) gives positive answer for the above question in the orientation-preserving case:

Corollary 1.6. *For a C^1 generic orientation-preserving diffeomorphism f of a closed orientable surface there is no dominated splitting for* both $\mathrm{Cl}(\mathrm{Sink}(f)) \setminus \mathrm{Sink}(f)$ *and* $\mathrm{Cl}(\mathrm{Source}(f)) \setminus \mathrm{Source}(f)$.

Proof. Suppose by contradiction that there is a dominated splitting on $\mathrm{Cl}(\mathrm{Sink}(f)) \setminus \mathrm{Sink}(f)$ (say). Since $\mathrm{Sink}_{\mathbb{C}}(f) \subset \mathrm{Sink}(f)$ if $\mathrm{Sink}_{\mathbb{C}}(f)$ were infinite we would also have a dominated splitting on the infinite compact invariant set $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \setminus \mathrm{Sink}_{\mathbb{C}}(f)$. As this is clearly impossible (for such a property forces the eigenvalues of nearby periodic orbits to be real) we conclude that $\mathrm{Sink}_{\mathbb{C}}(f)$ is finite and then $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \cap \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f)) = \emptyset$. Since f is C^1 generic we would obtain that f is Axiom A by Corollary 1.3. In particular, $\mathrm{Sink}(f)$ is finite and so $\mathrm{Cl}(\mathrm{Sink}(f)) \setminus \mathrm{Sink}(f) = \emptyset$ which is absurd. $\square$

Theorem 1.2 is motivated by a result of Pliss [**19**] which says that a *star diffeomorphism* (i.e. a C^1 diffeomorphisms far from ones exhibiting nonhyperbolic periodic orbits) have finitely many sinks and sources. Indeed, we would like to prove the same but for C^1 generic surface diffeomorphisms f satisfying $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \cap \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f)) = \emptyset$. What we shall prove instead is that $\mathrm{Spir}(f)$ is finite for all such diffeomorphisms. This assertion will be obtained by studying families of complex periodic sequences of linear isomorphisms which are *complex uniformly bounded* (i.e. there is an uniform bound for all nearby complex families). The key estimative is given in Lemma 2.2 from which the finitude of $\mathrm{Spir}(f)$ follows from classical arguments. Theorem 1.2 then follows from [**15**], [**14**].

2. *Proof of Theorem 1.1*

Denote by $GL(2)$ the group of linear isomorphisms of $\mathbb{R}^2$ equipped with the Euclidean norm $\|\cdot\|$. We denote by $I \in GL(2)$ the identity. The product of $A, B \in GL(2)$ is denoted AB or by $\prod_{i=1}^n A_i$ for finite sequences $A_1, \cdots, A_n \in GL(2)$. Given $A \in GL(2)$ we denote by $tr(A)$ and $det(A)$ the trace and determinant of A respectively. We also define the *discriminant* $\Delta(A) = tr^2(A) - 4det(A)$. Denote by $GL^+(2)$ the subgroup of all $A \in GL(2)$ which are orientation-preserving, i.e., $det(A) > 0$. Special elements of $GL^+(2)$ are the rotations,

$$P_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

Clearly $\Delta(p_\theta A) \leq 0$ for all $A \in GL^+(2)$ with $\Delta(A) \leq 0$. Therefore, since $tr(AB) = tr(BA)$ and $det(AB) = det(BA)$ for all $A, B \in GL(2)$ we see that if $A_1 \cdots, A_n \in GL^+(2)$ and the product $\prod_{i=1}^n A_i$ satisfy $\Delta(\prod_1^n A_i) \leq 0$, then

$$\Delta\left(\prod_{i=1}^{n+1} B_i\right) = \Delta\left(P_\theta \prod_{i=1}^{n} A_i\right) \leq 0 \tag{2.1}$$

where $B_i = A_i$ for $1 \leq i < j$, $B_j = P_\theta$, $B_i = A_{i-1}$ for $j < i \leq n$. This remark will be useful later in the proof of Lemma 2.2.

Meanwhile we state a result about rotations which will play the role of Lemma II.6 in [**12**].

LEMMA 2.1. *For every $A \in GL(2)$, $v \neq 0$ and $0 < \delta < 1$ there is a rotation P_θ such that $\|P_\theta - I\| \leq 2\delta$ and $\|AP_\theta v\| \geq \frac{\delta}{2}\|A\| \cdot \|v\|$.*

Proof. We can assume as in the proof of Lemma II.6 in [**12**] that $A = \mathrm{diag}(\lambda_1, \lambda_2)$ with $|\lambda_1| = \max\{|\lambda_1|, |\lambda_2|\}$ thus $\|A\| = |\lambda_1|$. Setting $v = (v_1, v_2)$ we obtain

$$\|AP_\theta v\| \geq |\lambda_1| \cdot |v_1 \cos\theta + v_2 \sin\theta|.$$

We have two cases, namely, either $|v_1| \geq \frac{1}{2}\|v\|$ or $|v_2| \geq \frac{1}{2}\|v\|$.

First suppose that $|v_1| \geq \frac{1}{2}\|v\|$. Take $\epsilon_0 > 0$ such that

$$\|P_\theta - I\| \leq 2\delta \quad \text{and} \quad \delta < \cos\theta \quad \text{for} \quad |\theta| \leq \epsilon_0.$$

We need θ with $|\theta| \leq \epsilon_0$ such that

$$|v_1 \cos\theta + v_2 \sin\theta| \geq \delta \cdot |v_1| \tag{2.2}$$

which is equivalent to

$$v_1^2(\cos^2\theta - \delta^2) + v_2(v_2 \sin^2\theta + 2v_1 \cos\theta \sin\theta) \geq 0.$$

Since $\delta < \cos\theta$ (and so $\delta^2 < \cos^2\theta$) for $|\theta| \leq \epsilon_0$ we only need to find θ with $|\theta| \leq \epsilon_0$ such that

$$v_2(v_2 \sin^2\theta + 2v_1 \cos\theta \sin\theta) \geq 0. \tag{2.3}$$

For this we proceed as follows:

If $v_2 > 0$ (thus $v_2 \sin\theta > 0$) and $v_1 > 0$ (thus $2v_1 \cos\theta > 0$) we take $0 < \theta < \epsilon_0$ yielding (2.3).

If $v_2 > 0$ (thus $v_2 \sin\theta > 0$) and $v_1 < 0$ (thus $2v_1 \cos\theta \leq 0$) we take $-\epsilon_0 < \theta < 0$ yielding (2.3).

If $v_2 < 0$ (thus $v_2 \sin\theta < 0$) and $v_1 > 0$(thus $2v_1 \cos\theta > 0$) we take $-\epsilon_0 < \theta < 0$ yielding (2.3).

If $v_2 < 0$ (thus $v_2 \sin\theta < 0$) and $v_1 < 0$ (thus $2v_1 \cos\theta < 0$) we take $0 < \theta < \epsilon_0$ yielding (2.3).

It follows that we can always find θ with $|\theta| \leq \epsilon_0$ such that (2.3) holds. Then, such a θ satisfies not only $\|P_\theta - I\| \leq 2\delta$ but also (2.2) so

$$\|AP_\theta v\| \geq |\lambda_1| \cdot \delta \cdot |v_1| \geq \frac{\delta}{2}\|A\| \cdot \|v\|.$$

Now we suppose $|v_2| \geq \frac{1}{2}\|v\|$. As before we search θ satisfying (2.2) and for this we need to satisfy

$$|v_1 \cos\theta + v_2 \sin\theta| \geq \delta |v_2| \tag{2.4}$$

Take $\epsilon_1 > 0$ such that

$$\left|\frac{\cos\theta - 1}{\theta}\right| < \frac{1}{2} \quad \text{and} \quad \frac{1}{2} < \left|\frac{\sin\theta}{\theta}\right| < 2 \quad \text{for} \quad 0 < |\theta| \leq \epsilon_1.$$

Thus, $\|P_\theta - I\| \leq 2|\theta|$ and so

$$\|P_\theta - I\| \leq 2\delta \quad \text{for} \quad |\theta| \leq \delta.$$

On the other hand, we will obtain (2.4) as soon as $|v_1 \cos\theta + v_2 \sin\theta| \geq |\sin\theta| \cdot |v_2|$ which in turns is equivalent to

$$v_1(v_1 \cos^2\theta + 2v_2 \cos\theta \sin\theta) \geq 0.$$

Proceeding as before we obtain this inequality for suitable θ with $0 < |\theta| \leq \min\{\delta, \epsilon_1\}$. This ends the proof. □

We shall need some formalism about families of periodic sequences of linear isomorphisms [**12**].

A *sequence of linear orientation-preserving isomorphism of* $\mathbb{R}^2$ is a map $L : \mathbb{Z} \to GL^+(2)$, $i \in \mathbb{Z} \mapsto L_i \in GL^+(2)$ (for simplicity we call them *sequence of linear isomorphisms*). We say that L is *periodic* if there is a minimal integer $n_0 \geq 1$ (called the *period* of L) such that $L_{i+n_0} = L_i$ for all $i \in \mathbb{Z}$. In such a case we can define

$$M_L = \prod_{i=0}^{n_0-1} L_i \in GL^+(2).$$

A *family of periodic sequences of linear isomorphisms* is a collection $\xi = \{\xi^\alpha : \alpha \in \mathcal{A}\}$ such that each ξ^α is a periodic sequence of linear isomorphisms which is bounded, i.e.,

$$\sup\{\|\xi_i^\alpha\| : (\alpha, i) \in \mathcal{A} \times \mathbb{Z}\} < \infty.$$

We shall use the notation $M_\xi(\alpha) = M_{\xi^\alpha}$.

Given two families of periodic sequences of linear isomorphism $\xi = \{\xi^\alpha : \alpha \in \mathcal{A}\}$ and $\eta = \{\eta^\alpha : \alpha \in \mathcal{A}\}$ we define

$$d(\xi, \eta) = \sup\{\|\xi_i^\alpha - \eta_i^\alpha\| : (\alpha, i) \in \mathcal{A} \times \mathbb{Z}\}.$$

and say that they are *periodically equivalent* if the periods of ξ^α and η^α coincide for all $\alpha \in \mathcal{A}$.

A periodic sequence of linear isomorphism L is $\mathbb{C}$*-periodic* if $\Delta(M_L) \leq 0$. A *family of* $\mathbb{C}$*-periodic sequences of linear isomorphisms* is a family of periodic sequences of linear isomorphisms $\xi = \{\xi^\alpha : \alpha \in \mathcal{A}\}$ such that ξ^α is $\mathbb{C}$-periodic, $\forall \alpha \in \mathcal{A}$.

A family of periodic sequences of linear isomorphisms ξ is *uniformly* $\mathbb{C}$*-bounded* if there are positive numbers ϵ and H such that every periodically equivalent family of $\mathbb{C}$-periodic sequences of linear isomorphism η with $d(\xi, \eta) \leq \epsilon$ satisfies

$$\|M_\eta(\alpha)\| \leq H, \qquad \forall \alpha \in \mathcal{A}.$$

Denote by $[r]$ the integer part of a real number r.

Lemma 2.2. *For every uniformly* $\mathbb{C}$*-bounded family of* $\mathbb{C}$*-periodic sequences of linear isomorphisms* $\xi = \{\xi^\alpha : \alpha \in \mathcal{A}\}$ *there exist* $K_0 > 0$, $0 < \lambda < 1$ *and* $m_0 \in \mathbb{N}^+$ *such that if* $\alpha \in \mathcal{A}$

and ξ^α has period $n \geq m_0$, then

$$\prod_{j=0}^{\left[\frac{n}{m_0}\right]-1} \left\| \prod_{i=0}^{m_0-1} \xi^\alpha_{i+m_0 j} \right\| \leq K_0 \lambda^{\left[\frac{n}{m_0}\right]}.$$

Proof. By hypothesis there are ϵ and H positive such that every periodically equivalent family of $\mathbb{C}$-periodic sequences of linear isomorphism η with $d(\xi,\eta) \leq \epsilon$ satisfies

$$\|M_\eta(\alpha)\| \leq H, \qquad \forall \alpha \in \mathcal{A}. \tag{2.5}$$

(this property corresponds to Lemma II.4 p. 527 in [**12**].)

Now, we follow closely the proof of Lemma II.5 p. 528 in [**12**]. Define

$$C = \inf\{\|Tv\| : \|v\| = 1, \|T - \xi^\alpha_i\| \leq \epsilon \text{ for some } (\alpha, i) \in \mathcal{A} \times \mathbb{Z}\},$$

$$C_0 = \sup\{\|T\| : \|T - \xi^\alpha_i\| \leq \epsilon \text{ for some } (\alpha, i) \in \mathcal{A} \times \mathbb{Z}\}$$

and $\epsilon_0 = \frac{\epsilon}{2C_0}$. Shrinking ϵ if necessary we can assume that $0 < \epsilon_0 < 2$. Take $m_0 \in \mathbb{N}^+$ such that $(1+\epsilon_0)^{m_0} \frac{\epsilon_0}{4} > 1$. Define $\lambda = \left((1+\epsilon_0)^{m_0}\frac{\epsilon_0}{4}\right)^{-1}$ and $K_0 = \frac{H}{C^{m_0}}$.

Fix $\alpha \in \mathcal{A}$ such that ξ^α has period $n \geq m_0$ and also $v \in \mathbb{R}^2$. By Lemma 2.1, since $0 < \frac{\epsilon_0}{2} < 1$, we can take finitely many rotations $P_0, \cdots, P_{\left[\frac{n}{m_0}\right]-1}$ such that $\|P_j - I\| \leq \epsilon_0$ for $0 \leq j \leq \left[\frac{n}{m_0}\right] - 1$ and

$$\left\| \left(\prod_{i=0}^{m_0-1} \xi^\alpha_{i+m_0 j} \right) P_j v_j \right\| \geq \frac{\epsilon_0}{4} \left\| \prod_{i=0}^{m_0-1} \xi^\alpha_{i+m_0 j} \right\| \|v_j\|,$$

where $v_0 = v$ and

$$v_j = \left(\prod_{i=0}^{m_0-1} \xi^\alpha_{i+m_0(j-1)} \right) P_{j-1} v_{j-1} \quad \text{for } j \geq 1.$$

Define the periodic sequence of linear isomorphisms ν by the condition of having period n and

$$\nu_i = \begin{cases} (1+\epsilon_0)\xi^\alpha_i, & \text{if} \quad i \text{ is not a multiple of } m_0 \\ (1+\epsilon_0)\xi^\alpha_{m_0 j} P_j, & \text{if} \quad i = m_0 j \text{ is a multiple of } m_0 \end{cases}$$

By (2.1) since $\Delta(M_\xi(\alpha)) \leq 0$ we have $\Delta(M_\nu) \leq 0$.

On the other hand, $\|\nu_i - \xi^\alpha_i\|$ is either

$$\|(1+\epsilon_0)\xi^\alpha_i - \xi^\alpha_i\| \leq C_0 \epsilon < \epsilon$$

or

$$\|(1+\epsilon_0)\xi^\alpha_{m_0 j} P_j - \xi^\alpha_{m_0 j}\| \leq \|(1+\epsilon_0)\xi^\alpha_{m_0 j} P_j - \xi^\alpha_{m_0 j} P_j\| + \|\xi^\alpha_{m_0 j} - \xi^\alpha_{m_0 j}\| \leq 2C_0\epsilon_0 = \epsilon.$$

Therefore, the family $\eta = \{\eta^\beta : \beta \in \mathcal{A}\}$ defined by $\eta^\beta = \xi^\beta$ for $\beta \neq \alpha$ and $\eta^\alpha = \nu$ is a periodically equivalent family of $\mathbb{C}$-periodic sequences of linear isomorphism satisfying $d(\xi,\eta) \leq \epsilon$. As $M_\eta(\alpha) = \prod_{j=0}^{n-1} \nu_j$ we can apply (2.5) to obtain

$$\left\| \left(\prod_{j=0}^{n-1} \nu_j \right) v \right\| \leq H \|v\|.$$

Then, the definition of ν_j and the property of the P_j's yield

$$
\begin{aligned}
H\|v\| &\geq \left\| \left(\prod_{j=0}^{n-1} \nu_j \right) v \right\| \\
&\geq C^{m_0} \left\| \left(\prod_{j=0}^{\left[\frac{n}{m_0}\right] m_0 - 1} \nu_j \right) v \right\| \\
&\geq C^{m_0} (1+\epsilon_0)^{\left[\frac{n}{m_0}\right] m_0} \left\| \left(\prod_{j=0}^{\left[\frac{n}{m_0}\right]-1} \left(\prod_{i=0}^{m_0-1} \xi_{i+m_0 j} \right) P_j \right) v \right\| \\
&\geq C^{m_0} \left((1+\epsilon)^{m_0} \frac{\epsilon_0}{4} \right)^{\left[\frac{n}{m_0}\right]} \left(\prod_{j=0}^{\left[\frac{n}{m_0}\right]-1} \left(\left\| \prod_{i=0}^{m_0-1} \xi^{\alpha}_{i+m_0 j} \right\| \right) \right) \|v\|.
\end{aligned}
$$

Dividing this expression by $C^{m_0} \left((1+\epsilon)^{m_0} \frac{\epsilon_0}{4} \right)^{\left[\frac{n}{m_0}\right]} \|v\|$ and using the definition of K_0, λ we obtain the result. □

In the sequel we state the *Franks's Lemma* [**5**] (c.f. Lemma 2.1.1 in [**21**]).

LEMMA 2.3. *Let $f \in \mathrm{Diff}^1(M)$ and $\mathcal{W}(f) \subset \mathrm{Diff}^1(M)$ be a neighborhood of f. Then, there are $\epsilon > 0$ and a neighborhood $\mathcal{W}_0(f) \subset \mathcal{W}(f)$ of f such that if $g' \in \mathcal{W}_0(f)$, $\{x_1, \cdots, x_n\} \subset M$ is a finite set, $U \subset M$ is a neighborhood of $\{x_1, \cdots, x_n\}$ and $L_i : T_{x_i}M \to T_{g'(x_i)}M$ are linear maps satisfying $\|L_i - Dg'(x_i)\| < \epsilon$ ($\forall i = 1, \cdots, n$), then there is $g \in \mathcal{W}(f)$ such that $g(x) = g'(x)$ in $\{x_1, \cdots, x_n\} \cup (M \setminus U)$, and $Dg(x_i) = L_i$ for every $i = i, \cdots, n$.*

To any $f \in \mathrm{Diff}^1(M)$ it corresponds the family of periodic sequences of linear isomorphisms $\xi(f) = \{\xi(f)^p : p \in \mathrm{Sink}_{\mathbb{C}}(f)\}$ where $\xi(f)^p_i$ stands for the matrix of $Df(f^i(p))$ written with respect to orthonormal basis of $T_{f^i(p)}M$ and $T_{f^{i+1}(p)}M$. The Franks's lemma is used to prove the following key property of this family.

LEMMA 2.4. *Let M a closed orientable surface and $\mathrm{Diff}^1_+(M)$ be the set of orientation-preserving diffeomorphisms of M. There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_+(M)$ such that if $f \in \mathcal{R}$ satisfies $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \cap \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f)) = \emptyset$, then $\xi(f)$ is an uniformly $\mathbb{C}$-bounded family of $\mathbb{C}$-periodic sequences of linear isomorphisms.*

Proof. Denote by 2^M_c the set of all compact subsets of M endowed with the Hausdorff metric. Define the maps $\Phi_1, \Phi_2 : \mathrm{Diff}^1_+(M) \to 2^M_c$ by

$$\Phi_1(f) = \mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \quad \text{and} \quad \Phi_2(f) = \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f)).$$

It is clear that these maps are lower semicontinuous, so, they are semicontinuous in a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_+(M)$ (c.f. [**10**], [**11**]). We shall prove that every $f \in \mathcal{R}$ satisfies the conclusion of the lemma.

Take $f \in \mathcal{R}$ satisfying

$$\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \cap \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f)) = \emptyset.$$

Since Φ_1 and Φ_2 are semicontinuous at f there are neighborhoods U and V of $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f))$ and $\mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f))$ respectively with

$$U \cap V = \emptyset \tag{2.6}$$

and a neighborhood $\mathcal{W}(f)$ of f, such that $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(g)) \subset U$ and $\mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(g)) \subset V$ for all $g \in \mathcal{W}(f)$. Let $\mathcal{W}_0(f)$ and $\epsilon > 0$ be as in the Franks's lemma for such a neighborhood $\mathcal{W}(f)$.

It follows from the definition of $\mathrm{Spir}(f)$ that $\xi(f)$ is a family of $\mathbb{C}$-periodic sequences of linear isomorphisms. It remains to prove that $\xi(f)$ is uniformly $\mathbb{C}$-bounded.

For this we shall need two claims.

The first one is that if η is a periodically equivalent family of $\mathbb{C}$-periodic sequences of linear isomorphism with $d(\xi(f), \eta) \leq \epsilon$ and $p \in \mathrm{Spir}(f)$, then both eigenvalues of $M_\eta(p)$ have modulus less than 1. Otherwise, since $\Delta(M_\eta(p)) \leq 0$ one has that both eigenvalues have modulus bigger than 1. Take $g' = f$, $x_i = f^i(p)$ and observe that $\|\eta_i^p - Dg'(x_i)\| \leq \epsilon$ ($\forall i = 1, \cdots, n$). Then, by the Franks's lemma, there is $g \in \mathcal{W}(f)$ such that $g(x) = f(x)$ in $\{x_1, \cdots, x_n\}$, and $Dg(x_i) = \eta_i^p$ for every $i = i, \cdots, n$. It follows that $p \in \mathrm{Per}(g)$, $n_{p,g} = n_{p,f}$ and $Dg^{n_{p,g}}(p_\alpha) = M_\eta(p)$. Then, the two eigenvalues of p as a periodic point of g have common moduli bigger than 1. It is then easy to find h close to g (and then $h \in \mathcal{W}(f)$) such that $p \in \mathrm{Source}_{\mathbb{C}}(h)$. Therefore, $p \in V$. However, $p \in \mathrm{Sink}_{\mathbb{C}}(f)$ and evidently $f \in \mathcal{W}(f)$ so $p \in U$. We conclude that $p \in U \cap V$ which contradicts (2.6). This contradiction proves our first claim.

The second claim is that there is $K > 0$ such that if η is a periodically equivalent family of $\mathbb{C}$-periodic sequences of linear isomorphism with $d(\xi(f), \eta) \leq \epsilon$, $p \in \mathrm{Spir}(f)$ and

$$M_\eta(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then} \quad |b - c| \leq K.$$

Suppose by contradiction that this is not true. Then, for arbitrarily large $K > 0$ we can select η and p as above with $|b - c| \geq K$. Notice that $tr(P_\theta M_\eta(p)) = (trA)\cos\theta + (c - b)\sin\theta$ thus

$$|tr(P_\theta M_\eta(p))| \geq |b - c| \cdot |\sin\theta| - |tr(M_\eta(p))| \cdot |\cos\theta|, \qquad \forall\theta.$$

By the previous claim we have $|tr(M_\eta(p))| \leq 2$ whence

$$|tr(P_\theta M_\eta(p))| \geq K|\sin\theta| - 2.$$

Take $\theta = \theta_K \to 0$ as $K \to \infty$ in a way that $|\sin\theta| = \frac{5}{K}$. Then, $|tr(P_\theta M_\eta(p))| \geq 3$ and so $P_\theta M_\eta(p)$ has at least one eigenvalue of modulus bigger than 1. Now consider the periodically equivalent family of periodic sequences of linear isomorphism $\tilde{\eta}$ given by $\tilde{\eta}^q = \eta^q$ (for $q \neq p$ in $\mathrm{Spir}(f)$), $\tilde{\eta}^p$ has period n_p, $\tilde{\eta}_i^p = \eta_i^p$ (for $0 \leq i \leq n_p - 2$) and $\tilde{\eta}_{n_p-1}^p = P_\theta \eta_{n_p-1}^p$. Since η is $\mathbb{C}$-periodic we also have that $\tilde{\eta}$ is $\mathbb{C}$-periodic by (2.1). Moreover, taking K large we have $d(\xi(f), \tilde{\eta}) \leq \epsilon$ and since $M_{\tilde{\eta}}(p) = P_\theta M_\eta(p)$ we see that $M_{\tilde{\eta}}(p)$ has at least one eigenvalue of modulus bigger than 1. Since this contradicts the first claim we obtain the second claim.

Now we use these claims to bound the entries a, b, c and d of $M_\eta(p)$:

To start with we notice that since $0 \leq (b + c)^2 = b^2 + c^2 + 2bc = b^2 + c^2 - 2bc + 4bc$ we get $-4bc \leq (b - c)^2$ so

$$-4bc \leq K^2.$$

In addition, η is $\mathbb{C}$-periodic so

$$0 \geq \Delta(M_\eta(p)) = (a - d)^2 + 4bc \tag{2.7}$$

thus $(a - d)^2 \leq -4bc$ yielding

$$|a - d| \leq K.$$

But by the first claim we have $|a + d| = |tr(M_\eta(p))| < 2$ whence

$$|a| = \frac{1}{2}|2a| = \frac{1}{2}|a + a| = \frac{1}{2}|a + d + a - d| \leq \frac{1}{2}(|a + d| + |a - d|)$$

therefore

$$|a| \leq \frac{1}{2}(2+K). \tag{2.8}$$

Then, since $|d| = |d+a-a| \leq |a+d| + |a|$ we obtain

$$|d| \leq 2 + \frac{1}{2}(2+K). \tag{2.9}$$

Next we observe that since $(b+c)^2 = (b-c)^2 + 4bc$ and $4bc \leq 0$ (because of (2.7)) we obtain $|b+c| \leq |b-c|$ so

$$|b+c| \leq K.$$

Finally, we see that

$$|b| = \frac{1}{2}|2b| = \frac{1}{2}|b+b| = \frac{1}{2}|b+c+b-c| \leq \frac{1}{2}(|b+c| + |b-c|)$$

thus

$$|b| \leq K \tag{2.10}$$

so

$$|c| = |c-b+b| \leq 2K. \tag{2.11}$$

Putting together (2.8), (2.9), (2.10) and (2.11) we obtain that every periodically equivalent family of $\mathbb{C}$-periodic sequences of linear isomorphism η with $d(\xi(f), \eta) \leq \epsilon$ satisfies

$$\|M_\eta(p)\| \leq H, \qquad \forall p \in \mathrm{Spir}(p),$$

where

$$H = \max\left\{2 + \frac{1}{2}(2+K), 2K\right\}.$$

Then, $\xi(f)$ is uniformly $\mathbb{C}$-bounded and the proof follows. □

The next ingredient is the following version of the *Pliss's lemma* (c.f. Lemma 3.0.2 in [**21**]).

Lemma 2.5. *For every $g \in \mathrm{Diff}^1(M)$ and $0 < \gamma_1 < \gamma_2$ there are $N \in \mathbb{N}^+$ and $c > 0$ such that if $x \in M$ and $n \geq N$ is an integer satisfying*

$$\prod_{i=1}^{n} \|Dg(g^i(x))\| \leq \gamma_1^n,$$

then there are $0 \leq n_1 < n_1 < \cdots < n_l \leq n$ with $l \geq cn$ such that

$$\prod_{i=n_r}^{j} \|Dg(g^i(x))\| \leq \gamma_2^{j-n_r}, \qquad \textit{for every } r \in \{1, \cdots, l\} \textit{ and } j \in \{n_r, \cdots, n\}.$$

Next we state and prove a lemma which seems to be well-known (see p. 213 of [**19**] or p. 1978 of [**20**]).

Lemma 2.6. *Let $f \in \mathrm{Diff}^1(M)$ be such that every periodic point of f is hyperbolic. Then, a nonempty subset $S \subset \mathrm{Sink}(f)$ is finite if and only if there exist K_0, $0 < \lambda < 1$ and $m_0 \in \mathbb{N}$ such that*

$$\prod_{j=0}^{\left[\frac{n_p}{m_0}\right]-1} \|Df^{m_0}(f^{m_0 j}(p))\| \leq K_0 \lambda^{\left[\frac{n_p}{m_0}\right]}, \quad \forall p \in S \textit{ with period } n_p \geq m_0. \tag{2.12}$$

Proof. Since the necessity is clear we only need to prove the sufficiency. Suppose by contradiction that (2.12) holds but S is infinite. Then, there is an infinite sequence $p_k \in S$. Since every periodic point is hyperbolic we obtain $n_{p_k} \to \infty$. Choose $g = f^{m_0}$, $0 < \lambda < \gamma_1 < \gamma_2 < 1$ and let $N \in \mathbb{N}^+$ and $c > 0$ be as in the Pliss's lemma for these choices. As $n_{p_k} \to \infty$ we have $\left[\frac{n_{p_k}}{m_0}\right] \to \infty$ too. Therefore, can assume that $\left[\frac{n_{p_k}}{m_0}\right] \geq N$ and, since $\lambda < \gamma_1$, we can also assume that $K_0 \lambda^{\left[\frac{n_{p_k}}{m_0}\right]} \leq \gamma_1^{\left[\frac{n_{p_k}}{m_0}\right]}$ for all k. Then, (2.12) and the Pliss's lemma imply that for all $k \in \mathbb{N}$ there are $0 \leq n_1^k < n_2^k < \cdots < n_{l_k}^k \leq \left[\frac{n_{p_k}}{m_0}\right] - 1$ with $l_k \geq c\left(\left[\frac{n_{p_k}}{m_0}\right] - 1\right)$ such that

$$\prod_{j=n_r^k}^{s} \|Dg(g^j(p_k))\| \leq \gamma_2^{s-n_r^k}, \quad \forall r \in \{1, \cdots, l_k\} \text{ and } s \in \left\{n_r^k, \cdots, \left[\frac{n_{p_k}}{m_0}\right] - 1\right\} \tag{2.13}$$

Now consider the sequences $x_k = g^{n_1^k}(p_k)$ which, by compactness, can be assumed to be convergent to some $x_\infty \in M$. Since $l_k \geq c\left(\left[\frac{n_{p_k}}{m_0}\right] - 1\right)$ we get $\left(\left[\frac{n_{p_k}}{m_0}\right] - 1\right) - n_1^k \geq l_k \geq c\left(\left[\frac{n_{p_k}}{m_0}\right] - 1\right)$ so $\left(\left[\frac{n_{p_k}}{m_0}\right] - 1\right) - n_1^k \to \infty$ as $k \to \infty$. This together with (2.13) implies

$$\|Dg^l(x_\infty)\| \leq \gamma_2^l, \qquad \forall l \in \mathbb{N}.$$

From this it follows that the omega-limit set $\omega(x_\infty) = \{y \in M : y = \lim_{r\to\infty} g^{s_r}(x_\infty)$ for some sequence $s_r \to \infty\}$ is a hyperbolic set with zero-dimensional expanding subbundle. Therefore, $\omega(x_\infty)$ is a sink of g. As $x_k = g^{n_1^k}(p_k) \to x_\infty$ we conclude $g^{n_1^k}(p_k)$ belongs to the basin of attraction of this sink for all k large. From this we easily conclude that the sequence p_k is in fact finite, a contradiction. This ends the proof. □

Proof of Theorem 1.2. By the main result in [**15**] there is a residual subset $\mathcal{R}_0 \subset \mathrm{Diff}^1_+(M)$ such that if $f \in \mathcal{R}_0$, then f is Axiom A without cycles if and only if $\mathrm{Spir}(f)$ is a finite set. We can further assume by the Kupka-Smale theorem [**6**] that every periodic point of every $f \in \mathcal{R}_0$ is hyperbolic.

Let $\mathcal{R}$ the residual subset in Lemma 2.4 and $\mathcal{R}_+ = \mathcal{R} \cap \mathcal{R}_0$ which is clearly residual in $\mathrm{Diff}^1_+(M)$. Let us prove that every $f \in \mathcal{R}_+$ satisfies the conclusion of the theorem.

Take $f \in \mathcal{R}_+$ such that $\mathrm{Cl}(\mathrm{Sink}_{\mathbb{C}}(f)) \cap \mathrm{Cl}(\mathrm{Source}_{\mathbb{C}}(f)) = \emptyset$. Since $f \in \mathcal{R}^+ \subset \mathcal{R}$ we have by Lemma 2.4 that $\xi(f)$ is a uniformly $\mathbb{C}$-bounded family of $\mathbb{C}$-periodic sequences of linear isomorphisms. Then, by Lemma 2.2, there exist $K_0 > 0$, $0 < \lambda < 1$ and $m_0 \in \mathbb{N}^+$ for which the set $S = \mathrm{Sink}_{\mathbb{C}}(f) \subset \mathrm{Sink}(f)$ satisfies (2.12). Therefore, $\mathrm{Sink}_{\mathbb{C}}(f)$ is finite by Lemma 2.6.

Applying the same argument to f^{-1} we can prove up to passing to another residual subset if necessary that $\mathrm{Source}_{\mathbb{C}}(f)$ is finite. Since every periodic point is hyperbolic we have $\mathrm{Spir}(f) = \mathrm{Sink}_{\mathbb{C}}(f) \cup \mathrm{Source}_{\mathbb{C}}(f)$, and so, $\mathrm{Spir}(f)$ is finite too. Since $f \in \mathcal{R}_0$ we conclude that f is Axiom A without cycles. □

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