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Lectures on sectional-Anosov flows

– Monograph –

Preface

This book arose out from lecture notes of two graduate courses given by the second author in 2009 at the Federal University of Rio de Janeiro in Brazil. The first one entitled *"Anosov flows"* dealth with the theory of Anosov flows. The second *"sectional-Anosov flows"* followed the first author thesis *"Sobre conjuntos hiperbólicos-singulares"*, awarded Honorable Mention in the competition for best thesis of the CAPES/Brazil in 2006.

The *Anosov flows* are by definition vector fields exhibiting in the whole ambient manifold a tangent bundle decomposition formed by a contracting subbundle, an expanding subbundle and the flow-direction. The study of these flows belongs to one of the most influencial mathematical areas encompasing Algebra, Analysis and PDE, ODE, Geometry and Riemannian geometry, number theory etc.. This name comes from Professor D. V. Anosov who studied them with the name *U-systems*. The huge importance of Anosov flows was surely the motivation behind their current extensions. Among these we can mention the Anosov group actions and foliations, the pseudo-Anosov and projectively Anosov systems and so on.

This work focuses on *sectional-Anosov flows*, a further extension corresponding to vector fields exhibiting, in the maximal invariant set, a dominated splitting formed by a contracting subbundle and a subbundle where the flow's derivative *expands the area of parallelograms*. Indeed, we expose several aspects of the theory of sectional-Anosov flows developed by the authors during the first decate of the twenty one century. The exposition is done in a way to put together both Anosov and sectional-Anosov flows in the same context. Aspects of the theory of Anosov flows are then included for the sake of completeness.

We organized this book in the following way.

In Chapter 1 we give some preliminaries for the study of Anosov and sectional-Anosov flows. In Chapter 2 we introduce the definition of hyperbolic and sectional-hyperbolic sets. Properties including shadowing lemma, Axiom A and spectral decomposition, the hyperbolic lemma, classification of singularities and existence of singular partition under sectional-hyperbolicity will be discussed. In Chapter 3 we define the Anosov and sectional-Anosov flows. Examples including suspended Anosov flows, geodesic flows on negatively curved manifolds, BL-flows, Anomalous flows, those obtained by Dehn surgery, the geometric Lorenz attractor, the annular attractors, the venice masks and pathological sectional-Anosov flows will be presented.

In Chapter 4 we present some basic properties of the Anosov and sectional-Anosov flows.

In Chapter 5 we study codimension one Anosov flows and prove the Plante-Thurston-Margulis Theorem and the Verjovsky Theorem. We also analyze Anosov flows on closed 3-manifolds. In particular, we introduce the product Anosov flows, prove the Armendariz Theorem on solvable 3-manifolds and classify Anosov flows on closed 3-manifolds whose fundamental group exhibits a non-cyclic abelian normal subgroup.

In Chapter 6 we study sectional-Anosov flows on compact 3-manifolds. First we present dynamical properties including existence of periodic orbits, the sectional-Anosov closing and connecting lemmas, and the dynamics of venice masks. Afteward we study the perturbation theory of sectional-Anosov flows on compact 3-manifolds. Indeed, we obtain a bound for the number of attractors that may appear after a perturbation and also the perturbation of venice maks. An study of the omega-limit sets for small perturbation of transitive sectional-Anosov flows on compact 3-manifolds is given. Finally, we present some topological properties as the topology of ambient manifolds, transverse geometry and existence of Lorenz-like singularities when the ambient manifold is the 3-ball.

We finish with a list of problems in Chapter 7.

Knowledge of basic dynamical systems, geometric theory of foliations and topology are prerequisites for the full comprension of the topics of this book.

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Contents

Chapter 1
Preliminaires

In this chapter we introduce some preliminaries to study Anosov flows. The idea is to explore the relationship between dynamics, topology and algebra on manifolds supporting Anosov flows. Some background on differentiable manifolds, basic algebraic topology and group theory is needed.

1.1 Algebraic preliminaires

We start with some definitions. A good reference is the Rotman's book [133]. The *cardinality* of a set B will be denoted by $\#B$ (the alternative notation $|B|$ is also used).

A *group* is a pair $(G,*)$ where G is a nonempty set and $*: G\times G\to G$ is a binary operation in G satisfying the following properties:

1. There is an element $e\in G$ (called unity or identity element) such that:
   a. $g*e=e*g=g$ for every $g\in G$.
   b. For every $g\in G$ there is $g^{-1}\in G$ such that $g*g^{-1}=g^{-1}*g=e$.
2. The operation $*$ is associative, i.e. for every $g,f,h\in G$ one has

$$(g*f)*h=g*(f*h).$$

It is convenient to write G instead of $(G,*)$ when refering to a group. It is also convenient to write gf instead of $g*f$ when refering to the binary operation $*$ associated to the group G. A group G is *abelian* (or *commutative*) if $gf=fg$ for every $g,f\in G$. If G is an abelian group it is customary to denote the operation of G by $+$, the identity element of G by 0 and the power g^n of $g\in G$ by ng.

Given a group G we say that $H\subset G$ is a *subgroup* of G if $gf\in H$ for every $g,f\in H$. We write $H\leq G$ to indicate that H is a subgroup of G. A necessary and sufficient condition for $H \leq G$ is that $e \in H$ and $gf^{-1} \in H$ for every $g, f \in H$. If G is a group and $A, B \subset G$ we use the notation

$$AB = \{ab : (a,b) \in A \times B\}.$$

If $A = \{a\}$ consists of a single element a we denote $aB = AB$. Analogous notation holds when $B = \{b\}$ consists of a single element b. If $H \leq G$ we say that H is *normal* if $gHg^{-1} = H$ for all $g \in G$. We use the notation $H \triangleleft G$ to indicate that H is a normal subgroup of G. If $H \leq G$ we define the quotient

$$G/H = \{gH : g \in G\}.$$

In the case when $H \triangleleft G$ we endow G/H with the operation $(gH)(fH) = (gf)H$. We can easily check that if $H \triangleleft G$ then G/H is a group if equipped with such an operation. If $H \leq G$ we define the index $[G;H]$ as the cardinality $\#(G/H)$ of G/H.

An *homomorphism* between groups G, G' is a map $\varphi : G \to G'$ respecting the operations in G and G', namely $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in G$. The kernel $Ker(\varphi)$ of φ is the preimage of the identity element of G'. The image $Im(\varphi)$ of φ is the image set of φ. It is immediate to see that $Ker(\varphi) \triangleleft G$ and $Im(\varphi) \leq G'$. A homomorphism φ is an epimorphisms or a monomorphism depending on whether it is onto or injective (as a map). It is immediate to see that φ is an epimorphism (resp. monomorphism) if and only if $Im(\varphi) = G'$ (resp. $Ker(\varphi) = \{e\}$).

An *isomorphism* between two groups G, G' is a homomorphism $\varphi : G \to G'$ which is both an epimorphism and a monomorphism (if such a φ does exist then we say that G and G' are isomorphic groups). We often say that two groups are equal if they are isomorphic groups. Given a group G we denote by $Aut(G)$ the set of all isomorphism $\varphi : G \to G$. The set $Aut(G)$ equipped with the composition operation is a group with the identity map as identity element. A *fixed point* of $\varphi \in Aut(G)$ is an element $g \in G$ such that $\varphi(g) = g$.

The following results are part of the so-called *isomorphism theorems*.

Theorem 1.1 (First Isomorphism Theorem)**.** *If $\varphi : G \to G'$ is a homomorphism between groups G, G' then*

$$G/Ker(\varphi) = Im(\varphi).$$

Proof. If $h \in Ker(\varphi)$ and $g_1 = g_2h$ we have $\varphi(g_1) = \varphi(g_2h) = \varphi(g_2)$. So we have that if $g_1Ker(\varphi) = g_2Ker(\varphi)$ then $\varphi(g_1) = \varphi(g_2)$. This says that the following application $\psi : G/Ker(\varphi) \to Im(\varphi)$ given by $\psi(gKer(\varphi)) = \varphi(g)$ is a well-defined surjective homomorphism. Now observe that $ker(\psi)$ is the set of $g(Ker(\varphi))$ such that $\varphi(g) = e_{G'}$. This means that $Ker(\psi) = \{gKer(\varphi); g \in Ker(\varphi)\} = \{e_{G/Ker(\varphi)}\}$, then ψ is injective, and we have that it is an isomorphism. □

Theorem 1.2 (Third Isomorphism Theorem)**.** *Let G a group and K,H be normal subgroups of G with $K \leq H$. Then, $H/K \lhd G/K$ and*

$$G/H = (G/K)/(H/K).$$

Proof. If $g_1 = g_2k$ with $k \in K \leq H$ then we have that $g_1H = g_2kH = g_2H$, so the following application $\varphi : G/k \to G/H$ defined by $\varphi(gK) = gH$ is a well-defined surjective homomorphism. Now, if $lK \in H/K$ then $\varphi(lK) = lH = e_{G/H}$. So $Ker(\varphi) = H/K$. Then, by the previous theorem we have $(G/K)/(H/K) = G/H$. □

By a *category of groups* we mean a set whose elemens are groups. Let B a set and $\mathscr{V}$ be category of groups. A *free group with free basis B* in the category $\mathscr{V}$ is a group $F \in \mathscr{V}$ satisfying the following "universal" property: There is a map $\varphi : B \to F$ such that if $F' \in \mathscr{V}$ is another group and $\varphi' : B \to F'$ is another map then there is a unique homomorphism $\Phi : F \to F'$ such that $\Phi \circ \varphi = \varphi'$. When $\mathscr{V}$ consists of all groups we say that F is *free* and when $\mathscr{V}$ consists of all abelian groups we say that F is *free abelian*. Given a nonempty set B it exists a unique group F which is free (free abelian) with basis B. The rank $Rank(F)$ of a free abelian group F is the cardinality $\#B$ of a free basis B of F. It turns out that the rank of a free abelian group F does not depend on the free basis B.

If G is a group and $g \in G$ we say that g has *finite index in G* if there is $n \in \mathbb{N} \setminus \{0\}$ such that $g^n = e$. The set $T(G)$ of all elements with finite index of G is easily seem to be a subgroup of G called *torsion group* of G. A group G is either *torsion free* of a *torsion group* depending on whether $T(G) = \{e\}$ or $T(G) = G$.

Proposition 1.3. *If A is an abelian torsion free group of rank 1, then $Aut(A)$ is abelian.*

Proof. Let A be as in the statement. First we claim that if $\phi \in Aut(A)$ has a fixed point $\gamma_0 \neq 1$ then $\phi = Id$. Indeed, if $\gamma \in A$ then there are $m,n \in \mathbb{Z}^*$ such that $\gamma^n = \gamma_0^m$ since $Rank(A) = 1$ and A is torsion free. But $\phi(\gamma^n) = \phi(\gamma_0^m) = (\phi(\gamma_0))^m = \gamma_0^m = \gamma^n$. Hence $(\phi(\gamma)\gamma^{-1})^n = 1$ and so $\phi(\gamma) = \gamma$ since $n \neq 0$ and A is torsion free. As γ is arbitrary the claim follows. Now fix $\phi, \xi \in Aut(A)$ and $\gamma \in A - 1$. Hence $\phi(\gamma), \xi(\gamma) \neq 1$. As A is abelian and $Rank(A) = 1$ we have as above that there are $m,n,r,s \in \mathbb{Z}^*$ such that $\xi(\gamma^n) = \gamma^m$ and $\phi(\gamma^r) = \gamma^s$. If $\gamma_0 = \gamma^{msnr}$ we have that $\gamma_0 \neq 1$ since A is torsion free. Moreover, $\phi\xi(\gamma_0) = \phi(\gamma^{m^2sr}) = \gamma^{m^2s^2}$ and $\xi\phi(\gamma_0) = \xi(\gamma^{ms^2n}) = \gamma^{m^2s^2}$. We conclude that $\phi\xi\phi^{-1}\xi^{-1}$ has a fixed point $\gamma_0 \neq 1$ and so $\phi\xi = \xi\phi$ proving the result. □

Given a subset A of a group G we define $< A >$ as the smallest subgroup of G containing A. One can describe $< A >$ using words. More precisely, a *word* on A is an element $x \in G$ of the form

$$x = a_1^{e_1} a_2^{e_2} \cdots a_n^{e_n},$$

where each $a_i \in A$, each $e_i = \pm 1$ and $n \geq 1$. It follows that $< A >= \{e\}$ (if $A = \emptyset$) or the set of all the words on A (otherwise). We say that A is a *generating set* of G if $G =< A >$. A generating set A is *symmetric* if $a^{-1} \in A$ for all $a \in A$. G is said to be *finitely generated* if it has a finite generating set A. A group is *cyclic* if it has a generating set consisting of a single element. It is an exercise to prove that if G is cyclic then G is either $\mathbb{Z}$ or $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for some $p \in \mathbb{N}$.

If G is a finitely generated group then it has a finite symmetric generating set A. In such a case we have that every element $g \in G$ equals to some some word $a_1^{p_1} a_2^{p_2} \cdots a_k^{p_k}$ where $p_1, \cdots, p_k \in \mathbb{Z}$. The number $|p_1| + |p_2| + \cdots + |p_k|$ is called the *length of the word*. We define the number $||g||$ as the minimal length of those words which are equal to g. A finitely generated group G with finite symmetric generating set A has *polynomial* or *exponential growth* depending on whether there are two positive numbers d and C such that the ball $B(r) = \{g \in G : ||g|| \leq r\}$ satisfies

$$\#B(r) \leq Cr^d, \quad \forall r \geq 1$$

or

$$\#B(r) \geq C \cdot exp(dr), \quad \forall r \geq 1.$$

It is easy to see that the definition of polynomial and exponential growths are independent of the generating set A.

Example 1.1. For free (non-Abelian) groups with a prefered set of n generators, we have that $\#B(r) = \frac{n(2n-1)^k - 1}{n-1}$ if $r \in [k, k+1)$. Indeed, if we have a word with length $s-1$ we can obtain a word with length s attaching a letter on the left of the word only if the letter is not the inverse of the first letter of the original word. So we have that

$$\#B(r) = 1 + \sum_{s=1}^{k} 2n(2n-1)^{s-1} = \frac{n(2n-1)^k - 1}{n-1}$$

In particular for the free group with 2 generators we have $\#B(r) = 2.3^k - 1$ for $r \in [k, k+1)$. Compare with Lemma 3 in [89] and [57] p. 54.

Example 1.2. The free Abelian groups with n generators has polynomial growth, since every word can be reorganized as $a_1^{p_1} \ldots a_n^{p_n}$ where $p_i \in \mathbb{Z}$ and $\{a_i\}$ is the set of generators. So the number of words with length k is the number of combinations of $(p_1, \ldots, p_n) \in \mathbb{Z}^n$ such that $p_1 + \cdots + p_n \leq k$ and this number is bounded by Ck^n. In fact if the group has two generators one has $\#B(r) = 2k^2 + 2k + 1$ for $r \in [k, k+1)$.

A group G is *solvable* if there is a normal series

$$G_0 = 1 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$$

such that G_{i+1}/G_i is abelian for all $0 \leq i \leq n-1$. The group G is *virtually solvable* (or *solvable-by-finite*) if it contains a finite index solvable subgroup.

If G is a group we define the *center of G* as the set

$$Z(G) = \{x \in G : xg = gx \forall g \in G\}.$$

Clearly the center $Z(G)$ of a group G is a normal subgroup of G. A group G is *nilpotent* if there is a central series

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$$

such that $G_{i+1}/G_i \subset Z(G/G_i)$. We say that G is *almost nilpotent* if it exhibits a nilpotent normal subgroup of finite index. The following result gives a link between nilpotency and polynomial growth [57].

Theorem 1.4. *A finitely generated almost nilpotent group has polynomial growth.*

We also use the following lemma.

Lemma 1.1. *A group exhibiting an infinite cyclic normal subgroup with abelian quotient is almost nilpotent.*

Proof. Let G be a group exhibiting an infinite cyclic normal subgroup H with abelian quotient $K = G/H$. Let x be the generator of H. Then, for all $g \in G$ there is a unique $k(g) \in \mathbb{Z}$ such that $gxg^{-1} = x^{k(g)}$. Clearly $k(gg') = k(g) \cdot k(g')$ and $k(1) = 1$. Hence $k(g^{-1}) = k(g)^{-1}$ and then $k(g) = \pm 1$ for all $g \in G$. Thus $gxg^{-1} = x^{\pm 1}$ for all $g \in G$. Define $G_+ = \{g \in G : gxg^{-1} = x\}$. Clearly G_+ is a subgroup with index two of G. Hence G_+ is normal and has finite index in G. Let us prove that G_+ is nilpotent. Indeed, note that $H \subset G_+$ by the definition of G_+. Next, consider the serie $G_0 = 1 \lhd G_1 = H \lhd G_2 = G_+$. Note that $H \subset Z(G_+)$ since if $y \in H$ and $g \in G_+$ then $y = x^h$ for some $h \in \mathbb{Z}$ and so $gyg^{-1} = gx^h g^{-1} = (gxg^{-1})^h = x^h = y \therefore y$ commutes with all the elements of $G_+ \Rightarrow y \in Z(G_+)$. It follows that $G_1/G_0 \subset Z(G_+/G_0)$. On the other hand, $G_2/G_1 = G_+/H \subset G/H = K$ and since K is abelian we have that $G_2/G_1 \subset Z(G_+/G_1)$. These remarks prove that G_+ is nilpotent as claimed. Since G_+ has finite index in G we conclude that G is almost nilpotent as desired. □

One can easily see that a torsion free group with non-trivial center has an infinite cyclic normal subgroup (e.g. the cyclic group generated by a central element). A sort of converse holds for by the following lemma.

Lemma 1.2. *A group exhibiting an infinite cyclic normal subgroup also exhibits an index ≤ 2 normal subgroup with non-trivial center.*

Proof. Let A be an infinite cyclic normal subgroup of a group G. Then the set $Aut(A)$ of automorphisms of A is isomorphic to $\mathbb{Z}_2$. In particular $Aut(A)$ has order 2. Let $\phi : G \to Aut(A)$ be the action of G in A by conjugation, namely $\phi(g)(a) = gag^{-1}$, $\forall a \in A$. ϕ is well defined since A is normal. Let H be the kernel of ϕ. Hence H is normal with index $\mid M/H \mid = \mid Im(\phi) \mid \leq \mid Aut(A) \mid \leq 2$. Clearly $A \subset H$ since A is abelian. Moreover, $ga = ag$ for all $g \in H$ and $a \in A$ since $H = Ker(\phi)$. Hence $A \leq Z(H)$. As A is not trivial we have that $Z(H)$ is non-trivial as well. The proof follows. □

Let V and G be a manifold and a group respectively. An action (on the left) of G on V is a differentiable map

$$\varphi : G \times V \to V$$

with the following properties:

1. $\varphi(e, v) = v$ for every $v \in V$.
2. $\varphi(g, \varphi(f, v)) = \varphi(gf, v)$ for every $g, f \in G$ and $v \in V$.

Analogously one defines actions on the right $V \times G \to V$. It is usual to denote $\varphi(g, v)$ by gv and $G \times V \to V$ for the corresponding action. Under this notation we can see that every element $g \in G$ represents a unique diffeomorphism given by $v \in V \to gv$. This diffeomorphism is still denoted by g. An element $v \in V$ is a fixed point of $g \in G$ if $g(v) = v$. The set of fixed points of $g \in G$ is denoted by $Fix(g)$. The *isotropy group* (stabilizer) of $v \in V$ is the subset $Stab(v) \subset G$ formed by those $g \in G$ such that $g(v) = v$. It is easy to see that $Stab(v)$ is a subgroup of G.

Lemma 1.3. *Let $A \times V \to V$ be an action of a non-trivial abelian group A on a connected simply connected 1-manifold V. Suppose that every fixed point of every element of A is hyperbolic (either attracting or repelling). Suppose in addition that $Stab(f)$ is either trivial or cyclic for all $f \in V$. Then, $\cup_{\beta \in A \setminus \{0\}} Fix(\beta)$ is discrete.*

Proof. By contradiction suppose that $\cup_{\beta \in A \setminus \{0\}} Fix(\beta)$ is not discrete. Then there is $\beta_1 \in A \setminus \{0\}$ and $f_1 \in Fix(\beta_1)$ such that $\cup_{\beta \in A \setminus \{0\}} Fix(\beta) \setminus \{f_1\}$ is not closed. We can assume that β_1 is orientation-preserving and that f_1 is attracting for β_1. Let $I_1 \subset V$ be the corresponding local basin of attraction. It turns out that I_1 is an open interval containing f_1 and $\forall f \in I_1$ one has $\beta_1^n(f) \in I_1$ $(\forall n \in \mathbb{N})$ and $\lim_{n\to\infty} \beta_1^n(f) = f_1$. Since $\cup_{\beta \in A-1} Fix(\beta) \setminus \{f_1\}$ is not closed we can find $\beta_2 \in A - 1$ and $f_2 \in Fix(\beta_2) \cap I_1 \setminus \{f_1\}$. Clearly we can assume that β_2 is orientation-preserving. Since I_1 is an interval we have that f_2 and $\beta_1(f_2)$ bounds an interval $[f_2, \beta_2(f_2)]$ in I_1. Define

$$J = \cup_{n\in\mathbb{Z}} \beta_1^n([f_2, \beta_1(f_2)]).$$

Hence J is an open interval. Moreover, f_1 is a boundary point of J because of $\lim_{n\to\infty} \beta_1^n(f_2) = f_1$. Since A is abelian, and $\beta_1, \beta_2 \in A$, we have that β_1 and β_2 commute. Hence

$$\beta_2(J) = \cup_{n\in\mathbb{Z}} \beta_1([\beta_2(f_2), \beta_1(\beta_2(f_2))] = J$$

because $\beta_2(f_2) = f_2$. This implies that J is β_2-invariant. Since β_2 is orientation-preserving and f_1 is a boundary point of J we have that $\beta_2(f_1) = f_1$. In other words $\beta_2 \in Stab(f_1)$. This proves that $\beta_1, \beta_2 \in Stab(f_1)$. Since the last set is infinite cyclic we have that $\beta_1^{n_1} = \beta_2^{n_2}$ for some $n_1, n_2 \in \mathbb{Z}^*$. Hence $\beta_1^{n_1}(f_2) = \beta_2^{n_2}(f_2) = f_2$, i.e. f_2 is fixed by a non-trivial power of β_1 contradicting $f_2 \in I_1 \setminus \{f_1\}$. The lemma is proved. □

Now we prove the classical Holder's Theorem [65].

Theorem 1.5. *If G is a group for which there is a fixed point free action $G \times \mathbb{R} \to \mathbb{R}$, then G is abelian.*

Proof. There is an order $<$ in G by setting $\alpha < \gamma \Leftrightarrow \alpha(f) < \gamma(f)$ for some $f \in \mathbb{R}$. This order is well defined since $G \times \mathbb{R} \to \mathbb{R}$ is fixed point free. The following arquinedean property holds: $\forall \alpha \in G$ with $\alpha > Id$ and $\gamma \in G$ there is $n \in \mathbb{N}$ such that $\alpha^n > \gamma$. Hereafter we fix an element $\gamma_0 > Id$ of G. We define for all $\gamma \in G$ and $n \in bn$ the integer

$$\overline{X}(\gamma, n) = \inf\{k \in \mathbb{N} : \gamma^n < \gamma_0\}.$$

Clearly $\overline{X}(\gamma, n)$ is well defined by the arquimedean property. One can easily prove that $\overline{X}(\gamma, \cdot)$ is subadditive, namely $\overline{X}(\gamma, m+n) \leq \overline{X}(\gamma, n) + \overline{X}(\gamma, n)$ for all $m, n \in \mathbb{N}$. From this it follows that the limit

$$\overline{X}(\gamma) = \lim_{n\to\infty} \frac{\overline{X}(\gamma, n)}{n}$$

exists for all $\gamma \in G$ and satisfies $\overline{X}(Id) = 0$ (this is a well known trick in Ergodic Theory, [31]). This defines a map

$$\overline{X} : G \to \mathbb{R}$$

which satisfies: $\overline{X}(\gamma^n) = n\overline{X}(\gamma)$, $\overline{X}(Id) = 0$ and if $\alpha < \gamma$ then $\overline{X}(\alpha) < \overline{X}(\gamma)$. Let us prove the following requality:

$$\lim_{n\to\infty} \frac{\overline{X}(\alpha^n \gamma^n)}{n} = \overline{X}(\alpha) + \overline{X}(\gamma), \tag{1.1}$$

$\forall \alpha, \gamma \in G$. In fact by the definition of $\overline{X}(\gamma, n)$ one has

$$\gamma_0^{\overline{X}(\gamma,n)-1} < \gamma^n < \gamma_0^{\overline{X}(\gamma,n)},$$

and

$$\gamma_0^{\overline{X}(\alpha,n)-1} < \alpha^n < \gamma_0^{\overline{X}(\alpha,n)}.$$

Hence

$$\gamma_0^{\overline{X}(\alpha,n)+\overline{X}(\gamma,n)-2} < \alpha^n \gamma^n < \gamma_0^{\overline{X}(\alpha,n)+\overline{X}(\gamma,n)}.$$

Appliying $\overline{X}$ to the above inequality and using the properties one has

$$\overline{X}(\alpha, n) + \overline{X}(\gamma, n) - 2 < \overline{X}(\alpha^n \gamma^n) < \overline{X}(\alpha, n) + \overline{X}(\gamma, n).$$

Dividing the above inequality by n and taking the limit we obtain the result.

Next we prove the following inequality

$$\alpha^n \gamma^n < (\alpha\gamma)^n < \gamma^n \alpha^n$$

for all $\alpha,\gamma \in G$ and $n \in \mathbb{N}$. In fact, it is clear that the above inequality holds if α and γ commutes. So we assume that $\alpha\gamma \neq \gamma\alpha$. Since the action is fixed point free we can assume that $\alpha\gamma < \gamma\alpha$ or $\alpha\gamma > \gamma\alpha$. We shall assume $\alpha\gamma < \gamma\alpha$ since the proof for the another case is similar. Clearly $\alpha\gamma < \gamma\alpha$ implies $\alpha^n\gamma^n < (\alpha\gamma)^n$ for all n. To prove the another inequality we only observe that $\alpha\gamma < \gamma\alpha$ also implies $\alpha^i\gamma\alpha^{-i} < \gamma$ for all $i \in \mathbb{N}$. And since

$$(\alpha\gamma)^n = (\alpha\gamma\alpha^{-1})(\alpha^2\gamma\alpha^{-2})\cdots(\alpha^n\gamma\alpha^{-n})\alpha^n$$

we have that $(\alpha\gamma)^n < \gamma^n\alpha^n$ and the proof follows.

Next we claim that $\overline{X}$ is an homomorphism. In fact, by appliying $\overline{X}$ to $\alpha^n\gamma^n < (\alpha\gamma)^n < \gamma^n\alpha^n$ one has

$$(1/n)\overline{X}(\alpha^n\gamma^n) < \overline{X}(\alpha\gamma) < (1/n)\overline{X}(\gamma^n\alpha^n).$$

By taking the limit and applying Eq.(1.1) one has

$$\overline{X}(\alpha) + \overline{X}(\gamma) \leq \overline{X}(\alpha\gamma) \leq \overline{X}(\gamma) + \overline{X}(\alpha).$$

This proves that $\overline{X} : G \to (\mathbb{R},+)$ is an homomorphism. To finish let us consider an element $\gamma \in Ker(\overline{X}) - 1$ (i.e. $\overline{X}(\gamma) = 0$). We can assume that $\gamma > Id$ for otherwise we consider γ^{-1}. As above one observes

$$\gamma_0^{\overline{X}(\gamma,n)-1} < \gamma^n < \gamma_0^{\overline{X}(\gamma,n)}, \ \forall n.$$

By applying $\overline{X}$ to the above inequality one has

$$\overline{X}(\gamma,n) - 1 < n\overline{X}(\gamma) < \overline{X}(\gamma,n).$$

Because $\overline{X}(\gamma) = 0$ we conclude from the above inequality that $\overline{X}(\gamma,n) = 1$ for all n. This implies $\gamma^n < Id$ for all n which is absurd. We conclude that $Ker(\overline{X}) = 0$ and so G is isomorphic to the image $Im(\overline{X})$. Since $(\mathbb{R},+)$ is abelian and $Im(\overline{X}) \subset \mathbb{R}$ we obtain that G is abelian. The proof follows. □

1.2 Group cohomology

In this section we give a very brief description of group cohomology to be used in the next section. The exposition we give here is the one of the Suzuki's book [141].

A *ring* is a triple a triple $R = R,+,\cdot)$ consisting of an abelian group $(R,+)$ and a bynary operation $\cdot$ on R with the following properties:

1. $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3)$ for all $r_1, r_2, r_3 \in R$.

2. $r_1 \cdot (r_2 + r_3) = (r_1 \cdot r_2) + (r_1 \cdot r_3)$ for every $r_1, r_2, r_3 \in R$.

If additionally there is an element $1_R \in R$ such that $r \cdot 1_R = 1_R \cdot r = r$ for all $r \in R$ one says that R is *a ring with unity* 1_R. One says that R is a *commutative ring* if $r_1 \cdot r_2 = r_2 \cdot r_1$ for every $r_1, r_2 \in R$. As usual the product $g \cdot f$ of a pair of elements g, f in a ring R is denoted by gf.

Given a group G and a non-trivial commutative ring with unity R we can define the *group ring RG of G over R* as the set RG all *finite* sums $\Sigma_{g\in G} r_g g$, that is $r_g \in R$ is zero except by finitely many g's in G, equipped with the operations

$$(\Sigma_{g\in G} r_g g) + (\Sigma_{g\in G} s_g g) = \Sigma_{g\in G} (r_g + s_g) g$$

and

$$(\Sigma_{g\in G} r_g g) \cdot (\Sigma_{g\in G} s_g g) = \Sigma_{g\in G} \left(\Sigma_{h\in G} r_h s_{h^{-1}g} \right) g.$$

It turns out that RG is a ring.

Recall that a module over R (or R-*module* for short) is an abelian group A equipped with an action $A \times R \to A$, $(a, r) \to ar$, such that $a(r + s) = ar + as$, $a(rs) = (ar)s$ and $(a + b)r = ar + br$ for all $(a, b, r, s) \in A \times A \times R \times R$.

A *homomorphism of R-modules* is a map $f : A \to B$ between R-modules A and B such that $f(a + b) = f(a) + f(b)$ and $f(ar) = f(a)r$ for all $(a, b, r) \in A \times A \times R$.

An R-module A is *freely generated* by a subset $X \subset A$ whenever for all R-module B and for all map $g : X \to B$ there is a unique homomorphism of R-modules $f : A \to B$ such that $g = f \circ i$. We say that A is a *free R-module* if it is freely generated by some $X \subset A$. In such a case the cardinality $\#X$ of X will be refered to as the *rank* of A.

Observe that R itself is a RG-module if equipped with the operation

$$(\Sigma_{g\in G} r_g g)\, r = \Sigma_{g\in G} r_g r.$$

A *free G-resolution of R* is a sequence of free RG-modules $X = \{X_i : i = 0, 1, \cdots\}$ and a sequence of homomorphisms of RG-modules $d = \{d_i : X_i \to X_{i-1}\}$ such that the sequence below

$$\cdots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} R \longrightarrow 0$$

is exact.

(As customary we shall say *free resolution* instead of free G-resolution.)

If A is any RG-module, and X is a free resolution of R, then we have a homomorphism of RG-modules sequence $d_i^* : Hom_{RG}(X_{i-1}, A) \to Hom_{RG}(X_i, A)$ in the standard way,

$$d_i^*(f_{i-1}) = f_{i-1} \circ d_i.$$

The quotient group

$$H^n(G,A) = \frac{Ker(d^*_{n+1})}{Im(d^*_n)}$$

is the *n-th cohomology group of A with respect to RG* (or G).

By a standard diagram chasing argument involving two free resolutions of R we can see that $H^n(G,A)$ does not depend on free resolutions.

The *cohomology dimension of G over R* is the maximun integer $n = cd_R G$ such that $H^n(G,A)$ is non-trivial for some RG-module A. We denote $cd_{\mathbb{Z}}G = cdG$ when $R = \mathbb{Z}$ the set of integers.

Next we state without proof a result due to R. Bieri [26].

Theorem 1.6. *The center of a non-abelian group with cohomological dimension ≤ 2 is either trivial or infinite cyclic.*

We shall prove the following result which is a weaker version of a result by Hillman [67]. We define $\mathscr{G}$ as the set of groups G with finite cdG such that $cdN < cdG - 1$ for all subgroup N of infinite index in G.

Theorem 1.7. *(Hillman) Let G be a finitely generated group in $\mathscr{G}$. Suppose that G containing an abelian normal subgroup A which either is $\mathbb{Z}^2$ or has rank* 1. *If $G/Z_G(A)$ is infinite, then G is virtually solvable.*

Proof. First we assume that $A = \mathbb{Z}^2$. We claim that G/A has an element of infinite order. Indeed, suppose by contradiction that every element of G/A has finite order. Then G/A is a torsion group which is finitely generated since G is. Let $\Phi : G/A \to Aut(A)$ be the homomorphism by conjugation, namely $\Phi(\gamma A) = \phi_{\gamma A}$ with $\phi_{\gamma A}(\alpha) = \gamma\alpha\gamma^{-1}$ for all $\alpha \in A$. Φ is well defined because A is abelian. Moreover, $ker(\Phi) = Z_G(A)/A$ and

$$Im(\Phi) = (G/A)/(Z_G(A)/A).$$

$Im(\Phi)$ is a finitely generated torsion group since G/A is. Hence $Im(\Phi)$ is a finitely generated torsion group of $Aut(A) = GL(2,\mathbb{Z})$. According to a classical result [74] we know that $Im(\Phi)$ is locally finite. Since $Im(\Phi)$ is finitely generated we conclude that $Im(\Phi)$ is finite. Thus $G/Z_G(A) = (G/A)/(Z_G(A)/A)$ is finite, a contradiction. This proves that G/A has an element of infinite order αA. Note that $< \alpha, A > /A =< \alpha A >$ because A is abelian. Hence $< \alpha, A > /A = \mathbb{Z}$ because αA has infinite order. Thus $< \alpha, A >$ is a Poly-$\mathbb{Z}$ group with Hirsch number 3. As $G \in \mathscr{G}$ we conclude that $N =< \alpha, A >$ has finite order in G proving the result. Now we assume that $Rank(A) = 1$. It follows from Proposition 1.3 that $Aut(A)$ is abelian since A is torsion free. Let Φ be the representation $G \to Aut(A)$ given by conjugation, namely $\Phi(\gamma) = \phi_\gamma$ with $\phi_\gamma(\alpha) = \gamma\alpha\gamma^{-1}$. Note that the kernel of Φ is precisely the centralizer $Z_G(A)$ of A in G. As $Aut(A)$ is abelian we conclude that $G/Z_G(A)$ is abelian. But $G/Z_G(A)$

is infinite by assumption. Then, $cdZ_G(A) \leq 2$ since $G \in \mathscr{G}$. On the other hand, it is clear from the definition of center that $A \subset Z(Z_G(A))$. Thus $Z_G(A)$ has nontrivial center. As A is not cyclic we have that $Z_G(A)$ is abelian by the Bieri's Theorem. Hence both $G/Z_G(A)$ and $Z_G(A)$ are abelian proving that $N = G$ is solvable. The proof follows. □

1.3 Topological preliminars

We give some background in 3-manifold topology. Basic reference here is the Hatcher's book [63] (see also the classical book by Hempel [66]).

In the sequel we state one of the most important results in 3-manifold topology.

Theorem 1.8 (Loop Theorem)**.** *Let M be a 3-manifold with boundary ∂M. If thre is a map $f\colon (D^2, \partial D^2) \to (M, \partial M)$ with $f(\partial D^2)$ not null-homotopic in ∂M, then there is an embedding with the same property.*

We shall not prove this theorem here (see [63]). Instead we state some corollaries. A submanifold S of a manifold M is *two-sided* if there is an embedding $S \times I \to U$ where U is a neighborhood of S in M.

Corollary 1.9. *Let T be a two-sided torus on a closed 3-manifold M. If the homomorphism $\pi_1(T) \to \pi_1(M)$ induced by the inclusion $T \to M$ has nontrivial kernel, then there is a disk D in M such that $D \cap T = \partial D$ and ∂D is* not *null homotopic in T.*

Proof. If $\text{Ker}(\pi_1(T) \to \pi_1(M)) \neq 0$ then exists map $f\colon D^2 \to M$ s.t. $f(\partial D^2)$ is not null-homotopic in T. By transversality theory we can assume that T and $f(D^2)$ are in general position. Hence $f^{-1}(T)$ is a finite collection of circles one of which is ∂D^2. If the f-image of one of such circles is null-homotopic in T we can redefine f so as to eliminate it as below:

Hence we can assume that the f-image of these circles are not null-homotopic in T. Afterward we choose one of such circles so that it is minimal by inclusion (this choice is similar to one in the Haefliger Theorem's proof). In his way we can further assume the following

$$f(D^2) \cap T = f(\partial D^2) \qquad (*)$$

is not null-homotopic in T. Since T is two-sided we can cut M open along T to obtain a 3-manifold with boundary N (possibly disconnected). As M is closed the connected components of ∂N are copies of T. By (*) $f(D^2)$ intersect T only in $f(\partial D^2)$. Hence f induces a map $g\colon (D^2, \partial D^2) \to (N, \partial N)$ so that $g(\partial D^2) = f(\partial D^2)$ *is not null-homotopic in* ∂N. By the Loop Theorem we can assume that g is an *embedding*. Thus $D = g(D^2)$ is a disk in N satisfying

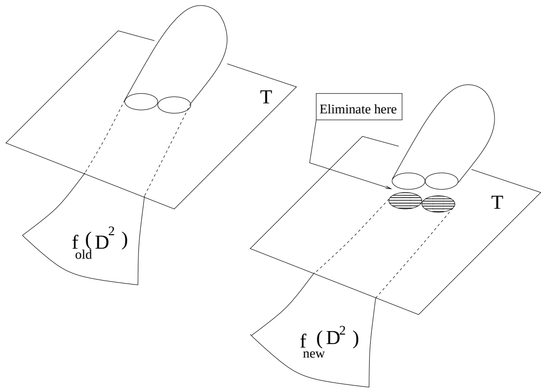


Fig. 1.1

$$D \cap \partial N = \partial D \quad \text{is not null-homotopic in } T.$$

Reversing the proceeding yielding N we can observe that D is a disk in M satisfying

$$D \cap T = \partial D \quad \text{not null-homotopic in } T.$$

The proof follows. □

Definition 1.10. *An n-manifold M is called* irreducible *if every two-sided embedded* $(n-1)$*-sphere in M bounds a n-ball in M (c.f. [84]).*

Remark 1.11. *The Sphere Theorem implies that if M orientable irreducible 3-manifold, then* $\Rightarrow \pi_2(M) = 0$.

It is in general *difficult* to prove (or disprove) that a manifold is irreducible.

Definition 1.12. *An embedded surface S on a* 3*-manifold M is called* incompressible *if the homomorphism* $\pi_1(S) \to \pi_1(M)$ *induced by the inclusion is injective.*

Corollary 1.13. *A 2-sided torus T on a closed irreducible 3-manifold M either is incompressible or bounds a solid torus or belongs to a 3-ball.*

Proof. Assume that T is not incompressible, i.e. $\mathrm{Ker}(\pi_1(T) \to \pi_1(M)) \neq 0$. Then, by Corollary 1, there is a disk D in M such that $D \cap T = \partial D$ and $c = \partial D$ is not null-homotopic in T. Let $\mathscr{C}$ be a tubular neighborhood of D, which is a cylinder-like 3-ball, such that $\partial\mathscr{C} \cap T = A$ is an annulus neighborhod of C in T.

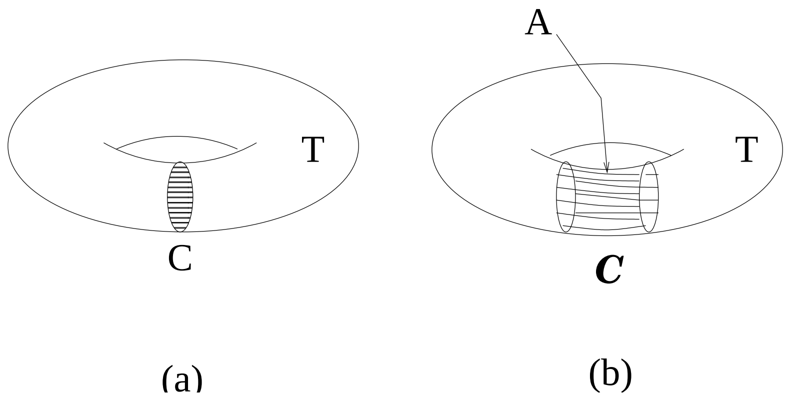


Fig. 1.2

Note that $\partial\mathscr{C}$ is formed by two 2-disks D_+ , D_- and the annulus A

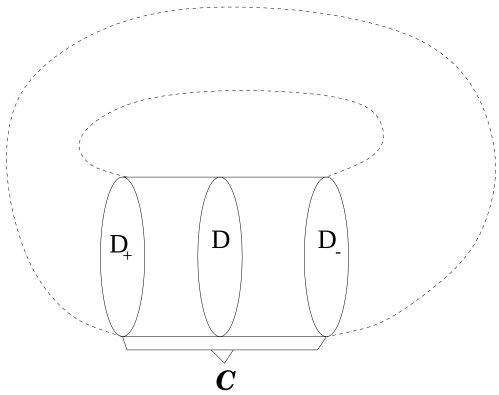


Fig. 1.3

Consider the manifold

$$S = (T - A) \cup D_+ \cup D_-$$

Clearly S is a (tamely embedded) 2-sphere in M. Since M is irreducible we have that S bounds a 3-ball B. We have two possibilities:

$$\begin{cases} \bullet\, B \cap D = \emptyset \\ \bullet\, B \cap D \neq \emptyset \end{cases}$$

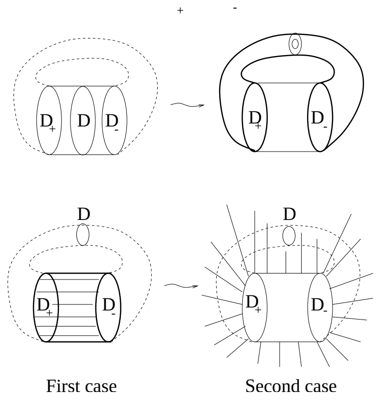


Fig. 1.4

In the first case we see that $ST \overset{\text{def}}{=} B \cup \mathscr{C}$ is a solid torus bounded by T. In the second case we have that $T \subseteq B$. □

Remark 1.14. *In the third conclusion of this corollary we can assume that T belongs to the interior of the 3-ball.*

The following two results are well known facts in 3-manifold topology. The first one is the famous Stalling's Fibration Theorem [139]. The second one is due to Evans and Moser [43].

Theorem 1.15. *Let M be a closed irreducible 3-manifold and N be a finitely generated normal subgroup of $\pi_1(M)$ with infinite cyclic quotient group $\pi_1(M)/N$. If $N \neq \mathbb{Z}_2$ then N is the fundamental group of a closed surface F and M fibers over S^1 with fibre F.*

Theorem 1.16. *If M is a closed 3-manifold with solvable fundamental group, then every subgroup of $\pi_1(M)$ is finitely generated.*

1.4 Group homology and manifolds

Definition 1.17. *A space X is aspherical (or* Eilenberg-Maclane*) if* $\pi_n(X) = 0$ $\forall n \geq 2$. *Given a group G we say that X is $K(G,1)$ if X is an aspherical and* $\pi_1(X) = G$.

Remark 1.18. *It follows that X is aspherical if and only if every continous map $f\colon S^n \to X$ admits an extension $\bar{f}\colon E^{n+1} \to X$ (viewing $\partial E^{n+1} = S^n$ $\therefore$ $\bar{f}/\partial E^{n+1} = f$).*

Proposition 1.19. $\mathbb{R}^k$ *is aspherical* $\forall k \in \mathbb{N}$.

Proof. The image of a map $f\colon S^n \to \mathbb{R}^k$ is contained in a ball $B \subseteq \mathbb{R}^k$ and B is *contractible* (i.e. exists homotopy $H\colon B \times I \to \mathbb{R}^k$ $\quad H(x,1) = x, \quad \forall x \in B$, $H(x,0) = Q_0\,, \quad \forall x \in B$). We define $\bar{f}(z \in E^{n+1}) = H(f(\theta(z)), r(z))$ where $z = (\theta(z), r(z)) \in S^n \times (0,1]$ is the polar coordinate system of E^{n+1}). This map proves the assertion. ☐

Theorem 1.20 (Whitehead)**.** *If X, Y are $K(G,1)$, then X and Y are homotopically equivalent, i.e. there are maps*

$$f\colon X \to Y \qquad \text{and} \qquad f'\colon Y \to X$$

such that

$$\begin{cases} \bullet\, f \circ f' \simeq \; Id_x \\ \bullet\, f' \circ f \simeq \; Id_x \end{cases}$$

A consequence of the above theorem is that if X, Y are both $K(G,1)$ then

$$H_i(X) = H_i(Y) \qquad \forall i \in \mathbb{N}.$$

($H_i(\,\cdot\,)$ denote the i-homology group with coefficients in $\mathbb{Z}$). By this observation we can define the i-homology group of a group G as:

$$H_i(G) \overset{\text{def}}{=} H_i(X)$$

for some $K(G,1)$ space X.

An example is as follows. Let $\mathbb{Z}_n = \mathbb{Z}/n\cdot\mathbb{Z}$ be the finite cyclic group of order n. Then,

$$H_i(\mathbb{Z}_n) = \begin{cases} \mathbb{Z}_n & \text{if } i \quad \text{odd} \\ 0 & \text{if } i \quad \text{even.} \end{cases}$$

In particular, $H_i(\mathbb{Z}_n) \neq 0$ for intimitely many i's.

Remark 1.21. *If M is a n-dimensional manifold then, $H_i(M) = 0$ for all $i > n$. Here is a Proof: By* the de Rham's theorem *we have that $H^i(M,\mathbb{R}) = 0$, $\forall i > n$. Hence $H_i(M,\mathbb{R}) = 0$ by the universal coefficient theorem. Since $H_i(M) = H_i(M,\mathbb{Z}) \subseteq H_i(M,\mathbb{R}) = 0$ the result follows.*

Lemma 1.4. *If M is an aspherical manifold of dimension n, then $H_i(G) = 0$ for all subgroup G of $\pi_1(M)$ and $i > n$.*

Proof. Fix $G \leq \pi_1(M)$. Let $\widehat{M} \to M$ be the *Galois covering associated to G*, i.e.

$$\pi_1(\widehat{M}) = G.$$

Since M is aspherical we have that $\widehat{M}$ also does. If $P\colon \widehat{M} \to M$ is a covering map, then $P_*\colon \pi_k(\widehat{M}) \to \pi_k(M)$ is a isomorphism $\forall k > 1$ (this follows from the fact that S^k is simply connected $\forall k > 1$). The observation above implies $H_i(\widehat{M}) = 0$, $\forall i > n$. It follows from the definition of $H_i(G)$ that $H_i(G) = H_i(\widehat{M}) = 0$ for all $i > n$. □

Proposition 1.22. *If M is an aspherical manifold of finite dimension, then $\pi_1(M)$ is torsion-free.*

Proof. Suppose by contradiction that $\pi_1(M)$ has an element of finite order. Hence there is $n \in \mathbb{Z}$ such that $G := \mathbb{Z}_n \leq \pi_1(M)$. Now, since M is aspherical and $\dim(M) < \infty$, one has that $H_i(G) = 0 \quad \forall i > \dim(M)$. However, by the example, we have that $H_i(G) \neq 0$ for infinitely many i's. This is a contradiction which proves the result. □

1.5 Foliation preliminaires

In this section we give some foliation background (see [65] for details). Roughly speaking a *foliation of codimension q* of a n-manifold N is a partition $\mathscr{F}$ of N formed by immersed submanifolds of constant dimension $n-q$ (the *leaves of $\mathscr{F}$*) which locally have the form $\mathbf{R}^{n-q} \times \mathbf{R}^q$. The sets of the form $\mathbf{R}^{n-q} \times \{*\}$ are called *the plaques of $\mathscr{F}$*. A leaf of $\mathscr{F}$ is union of plaques of $\mathscr{F}$. The intrinsic topology of a leaf F of $\mathscr{F}$ is precisely the atlas of F formed by the plaques of $\mathscr{F}$ contained in F. If $x \in N$ we denote by $\mathscr{F}_x$ the leaf of $\mathscr{F}$ containing x. We say that $B \subset N$ is *$\mathscr{F}$-invariant* if every leaf of $\mathscr{F}$ intersecting B is contained in B. A foliation is *transversely orientable* if it has a complementary orientable plane field. The most classical example of foliation is the *standard Reeb foliation* of the solid torus $D^2 \times$

S^1. A *Reeb component* of a foliation $\mathscr{F}$ on a 3-manifold is a $\mathscr{F}$-invariant compact submanifold R diffeomorphic to the solid torus $D^2 \times S^1$ such that $\mathscr{F}/_R$ is equivalent to the standard Reeb foliation. A foliation on a 3-manifold is *Reebless* if it has no Reeb components. We shall use the following version of the classical Novikov's.

Theorem 1.23 (Novikov)**.** *If $\mathscr{F}$ is a C^0 transversely orientable codimension one Reebless foliation of a closed orientable 3-manifold S, then the leaves of $\mathscr{F}$ are incompressible in S.*

Now we describe a result obtained by J. Plante [126]. Let M be a Riemannian manifold endowed with a volume form Ω. We say that $\pi_1(M)$ (the fundamental group of M) has *exponential growth* it there are constans B, $c > 0$ such that

$$\#\left\{\gamma \in \pi_1(M) : \exists c \in \gamma \text{ of length}(c) \leq R\right\} \geq Be^{cR},\ \forall R > 0.$$

For instance, if $\pi_1(M)$ is either finite or abelian then $\pi_1(M)$ has no exponential growth. Hence $\pi_1(S^3)$, $\pi_1(S^2 \times S^1)$ and $\pi_1(T^3)$ have no exponential growth. Examples with exponential growth will be given later one.

Let $\mathscr{F}$ be a C^r codimension one foliation on M, $r \geq 1$. Let d_x , Ω_x be respectively the restriction of the metric and the volume element Ω of M in the leaf $L(x)$ of $\mathscr{F}$ containing x. Define

$$D_R(x) = \{y \in L(x) : d_x(x,y) \leq R\}$$

(This is the R-ball in $L(x)$ centered at x) and

$$G(x,R) = \int_{D_R(x)} \Omega(x).$$

(This is the measure of $D_R(x)$).

The leaf $L(x)$ has exponential growth if there are constants B', $c' > 0$ so that

$$G(x,R) \geq B' e^{c' \cdot R}, \quad \forall R > 0.$$

The following theorem relates these notions of exponential growth:

Theorem 1.24 (Plante)**.** *Let $\mathscr{F}$ be a C^1 codimension one foliation on a compact manifold M. Suppose that $\exists x \in M$ such that*

1. *$L(x)$ does not intersect a null-homotopic closed transversal of $\mathscr{F}$.*
2. *$L(x)$ has exponential growth.*

Then, $\pi_1(M)$ has exponential growth.

Before the proof we establish the following lemma.

Lemma 1.5. *Let $P\colon \widetilde{M} \to M$ be the universal covering space of M endowed with the induced metric. Let $x \in M$ be a base point for $\pi_1(M)$ and fix $\tilde{x} \in P^{-1}(x)$ (the fibre of x). Let $V(R)$ denote the volume of the ball $B_R(\tilde{x})$ of radius R centered at $\tilde{x}$ w.r.t. the induced metric. Then, $\pi_1(M)$ has exponential growth if and only if $V(R)$ does, i.e. $\exists B'', C'' > 0$ s.t.*

$$V(R) \geq B'' e^{c'' \cdot R}, \quad \forall R > 0$$

Proof. Fix a fundamental domain K of the covering $P\colon \widetilde{M} \to M$. Since M is compact we have that $\mathrm{diam}(K) = \delta$ is finite. Recall that $\pi_1(M)$ can be identified with the deck transformations of M, i.e. there is an action by isometries

$$\pi_1(M) \times \widetilde{M} \to \widetilde{M}$$

such that $\pi_1(M)\backslash\widetilde{M} = M$. Fix $\bar{\gamma} \in \pi_1(M)$ such that $\gamma(K) \cap B_R(\tilde{x}) \neq \phi$. Then $\exists \tilde{y} \in \bar{\gamma}(K) \cap B_R(\tilde{x})$ and so $\exists$ curve $\tilde{c}$ of length$(\tilde{c}) \leq R + \delta$ joining $\tilde{x}$ and $\gamma(\tilde{x})$.

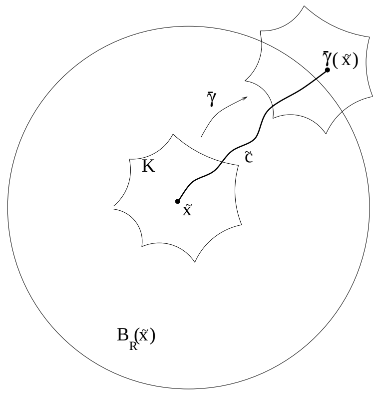


Fig. 1.5

Therefore, $c \overset{\text{def}}{=} P \circ \tilde{c}$ is a *closed* curve in $\bar{\gamma}$ of length $(c) \leq R + \delta$. This proves $\bar{\gamma} \in \{\gamma \in \pi_1(M) : \exists c \in \gamma \text{ of length}(c) \leq R + \delta\}$ and so the inclusion below holds:

(1) $\{\gamma \in \pi_1(M) : \gamma(K) \cap B_R(\tilde{x}) \neq \phi\} \subseteq \{\gamma \in \pi_1(M) : \exists c \in \gamma \text{ of length}(c) \leq R + \delta\}$

Next we fix $\gamma \in \pi_1(M)$ so that $\exists c \in \gamma$ of length$(c) \leq R$. Let $\tilde{c}$ be the lift of c to $\widetilde{M}$ with $\tilde{c}(0) = \tilde{x}$. Clearly length$(\tilde{c}) \leq R \therefore \tilde{c}(1) \in B_R(\tilde{x})$. By the definition of the action $\pi_1(M) \times \widetilde{M} \to \widetilde{M}$ one has $\tilde{c}(1) = \bar{\gamma}(\tilde{x})$ and so $\tilde{c}(1) \in \bar{\gamma}(K) \cap B_R(\tilde{x})$. We conclude that $\bar{\gamma}(K) \cap B_R(\tilde{x}) \neq \phi$ and then we have the following inclusion:

(2) $\{\gamma \in \pi_1(M) : \exists c \in \gamma \text{ of length}(c) \leq R\} \subset \{\gamma \in \pi_1(M) : \gamma(K) \cap B_R(\tilde{x}) \neq \phi\}$.

Denote by $N(R) = \#\{\gamma \in \pi_1(M) : \gamma(K) \cap B_R(\tilde{x}) \neq \phi\}$. By definition we have

$$V(R) = \text{ Volume of } B_R(\tilde{x}).$$

As $\bigcup_{\gamma(K)\cap B_R(\tilde{x})\neq\phi} \gamma(K)$ contains $B_R(\tilde{x})$ one has

$$V(R) = \text{ Volume of } B_R(\tilde{x}) \leq N(R)\cdot\delta \tag{3}$$

On the other hand, it is clear that

$$B_{R+2\delta}(\tilde{x}) \subseteq \bigcup_{\gamma(K)\cap B_R(\tilde{x})\neq\phi} \gamma(K)$$

(recall γ is isometry). Hence

$$V(R+2\delta) \geq N(R)\cdot\delta \tag{4}$$

If $\boxed{V(R) \text{ has exponential growth}}$ i.e. $V(R) \geq B''\, e^{c11\cdot R}$, then (3) implies $N(R) \geq B'''\cdot e^{c''\cdot R}$ with $B''' = B''/\delta$. Hence

$$\#\{\gamma\in\pi_1(M) : \exists c\in\gamma \text{ of length } (c) \leq R+\delta\} \geq B'''\cdot e^{c''\cdot R}$$

by (1) $\therefore$

$$\#\{\gamma\in\pi_1(M) : \exists c\in\gamma \text{ of length } (c) \leq R+\delta\} \geq B''''\cdot e^{c''\cdot R}$$

where $B'''' = B'''\cdot e^{-c''\cdot\delta}$

$\therefore$ $\boxed{\pi_1(M) \text{ has exponential growth}}$ conversely, if $\pi_1(B)$ has exponential growth then

$$\#\{\gamma\in\pi_1(M) : \exists c\in\gamma \text{ of length } (c) \leq R\} \geq B\cdot e^{c\cdot R}.$$

By (2) and the definition of $N(R)$ we get

$$N(R) \geq B\cdot e^{c\cdot R}$$

By (4) we have

$$V(R+1\delta) \geq B''\cdot e^{c\cdot R}$$

(where $B'' = B\cdot\delta$) and so

$$V(R) \geq B'\, e^{c'\cdot R} \text{ where } \begin{cases} B' = B''\cdot e^{-2c\delta} \\ c' = c \end{cases}$$

$\therefore V(R)$ has exponential growth. □

Proof of Theorem 1.24: $P\colon \widetilde{M}\to M$ be the universal cover of M. Let $\widetilde{\mathscr{F}}$ be the lift of $\mathscr{F}$ to $\widetilde{M}$. As note before $\widetilde{M}$ is equipped with the induced metric and there is an isometry action $\pi_1(M)\times\widetilde{M}\to\widetilde{M}$. Fix a base point $x\in M$ for $\pi_1(M)$, $\tilde{x}\in P^{-1}(x)$ and denote by $D_R(\tilde{x})$ the R-disk in $L(\tilde{x})$ (the leaf of $\widetilde{\mathscr{F}}$ containing $\tilde{x}$) centered at $\tilde{x}$.

We assume that x satisfies the hypotheses (1), (2) of the theorem. We can assume that $\mathscr{F}$ is $\pitchfork$ oriented by double covering. Hence $\exists c^1$ flow $\varphi_t \pitchfork \mathscr{F}$. Denote by $\tilde{\varphi}_r$ the lift of φ_t to $\widetilde{M}$.

Claim 1: There is $\lambda > 0$ fixed so that the set

$$\bigcup_{|t|\leq\lambda} \tilde{\varphi}_t(D_R(\tilde{x}))$$

consists only of points which are at distance $\leq R+1$ from $\tilde{x}$ (in the $\widetilde{M}$'s metric).

Proof of Claim 1: Clearly $D_R(\tilde{x}) \subseteq B_R(\tilde{x})$ since $D_R(\tilde{x})$ is the ball in $L(\tilde{x})$ when the induced metric.
Pick $\lambda > 0$ s.t.

$$\text{length}(\{\tilde{\varphi}_t(\tilde{y}) : |t| \leq \lambda\}) \leq 1 \quad \forall \tilde{y} \in \widetilde{M}$$

this λ exists since $||\dot{\varphi}_t(y)||$ is bounded in M (which is compact).
By the triangle inequality we get the result. □

Remark: $P\colon L(\tilde{x}) \to L(x)$ is a convering map.

Claim 2: Volume of $\bigcup_{|t|\leq\lambda} \tilde{\varphi}_t(D_R(\tilde{x}))$ growths at leat as fast as $G(\tilde{x}, R)$.

Proof of Claim 2: Recall that x satisfies the hypothesis (1) of the theorem. Consider the orbit segments

$$I(z) = \{\varphi_t(z) : z \in D_R(\tilde{x}); |t| \leq \lambda\}.$$

We have that such segments are *disjoint* for otherwise we get a picture as below

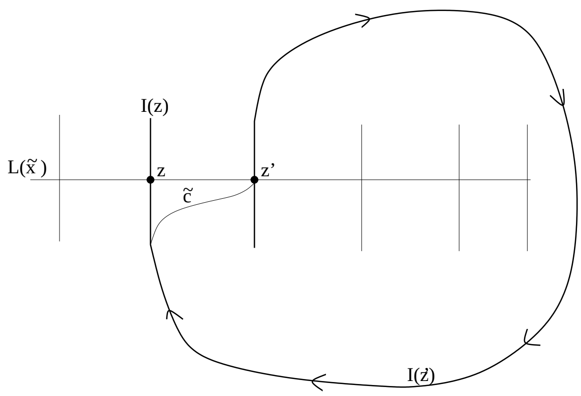


Fig. 1.6

This picture produces a closed transverse $\tilde{c}$ of $\widetilde{\mathscr{F}}$ intersecting $L(\tilde{x})$ (this is an standard trick in foliation's theory).

Since $\widetilde{M}$ is simply connected we have that $c = P \circ \tilde{c}$ is a null-homotopic closed transversal intersecting $L(x)$ contradicting (1). Hence the $I(z)$'s are disjoint.

Since the $I(z)$'s the disjoin we get

$$\text{Vol}\left(\bigcup_{|t|\leq\lambda} \tilde{\varphi}_t(D_R(\tilde{x}))\right) \geq K \cdot G(\tilde{x}, R)$$

by Fubini's theorem. The claim follow. □

Now we continue with the proof of Theorem 1.24. By Claim 1 we have

$$V(R) = \text{Vol}(B_R(\tilde{x})) \geq \text{Vol}\left(\bigcup_{|t|\leq\lambda} \tilde{\varphi}_t(D_R(\tilde{x}))\right)$$

and so

$$V(R) \geq K \cdot G(\tilde{x}, R)$$

by Claim 2. On the other hand,

$$L(\tilde{x}) \text{ is a covering of } L(x)$$

and so

$$G(\tilde{x}, R) \geq G(x, R) \quad \forall R > 0$$

(the volume of R-ball in $L(\tilde{x})$ is greather or equal than that of the R-ball in $L(x)$)

$\therefore$

$$V(R) \geq K \cdot G(x, R).$$

By the hypothesis (2) of the theorem we have that $\exists B', C' > 0$ s.t.

$$G(x, R) \geq B' e^{c' \cdot R}, \quad \forall R > 0.$$

From this we get

$$V(R) \geq B'' \cdot e^{c'' \cdot R}, \quad \forall R > 0$$

where

$$\begin{cases} B'' = KB' \\ C'' = C', \end{cases}$$

$\therefore$ $V(R)$ has exponential growth, and so, $\pi_1(M)$ has exponential growth by Lemma 1.5. □

To finish we study foliations which have a foliation-preserving transverse flow as in [127] Section 2 p. 736. Hereafter $\mathscr{F}$ is a C^1 codimension one foliation on a manifold M. We denote by $T\mathscr{F}$ the tangent plane field of $\mathscr{F}$ in M. We say that $\mathscr{F}$ is *tangent to a one form* ω *in* M if ω is non-singular and $T\mathscr{F} = Ker(\omega)$. For a proof of the lemma below see .....

Lemma 1.6. *Let $\mathscr{F}$ be a C^1 codimension one foliation on a manifold M. Suppose that there is a C^1 flow X on M such that if L is a leaf of $\mathscr{F}$, then $X_t(L)$ is a leaf of $\mathscr{F}$ for all $t \in \mathbb{R}$. Then, $\mathscr{F}$ is tangent a* closed *one form.*

1.6 Suspended flow

Let M be a Riemannian compact manifold with possibly non-empty boundary ∂M. We denote by $\mathscr{X}^r(M)$ be the space of C^r vector fields in M, inwardly transverse to ∂M (if $\partial M \neq \emptyset$), endowed with the C^r topology, $r \geq 1$. All manifolds M considered in this book will be connected. Given $X \in \mathscr{X}^1$ we denote by $Sing(X)$ the set of singularities of X.

This section is based on the following

Definition 1.25. *We say that $X \in \mathscr{X}^1(M)$ is* suspended *if there is a connected codimension one submanifold S in M transverse to X such that $M(X) \cap \partial S = \emptyset$ and $Sing(X) = \{x \in M(X) : X_t(x) \notin S, \forall t \in \mathbb{R}\}$. In such a case the submanifold S will be referred to as a* global cross section *of X.*

If X and M above satisfy that $Sing(X) = \partial M = \emptyset$ then S is a global cross section of X if and only if every orbit of X intersects S. From this we reobtain the classical definition of global cross section for vector fields [48]. The more general concept of singular partition will be given later one.

In general it is important to determinate whether a given flow is suspended or not. Obviously if M supports a suspented flow, then M fibers over S^1 $\therefore$ we have the following exact sequence

$$1 \to \pi_1(S) \to \pi_1(M) \to \pi_1(S^1) \to 1$$

where S is the fiber (global cross section) and S^1 is the base circle. It follows that all manifolds supporting suspended flows have infinite π_1.

We can construct many examples of suspended flows using suspension. To define it we let $f\colon M \to M$ be a diffeomorphism. The *suspension* of f is the flow X^f defined in the following way: In $[0,1] \times M$ we identify $(1,x)$ with $(0, f(x))$.
This yields the manifold M^f defined by

$$\boxed{M^f = [0,1] \times M \Big/ (1,x) \simeq (0, f(x))}$$

Observe that M^f fibers over S^1 with fibre M, and the trivial vector field $(t,x) \mapsto \partial/\partial t$ $(\equiv (1,0))$ in $I \times M$ gives rise to a vector field X^f in M^f. Another way to describe X^f is merely to say that it is the horizontal foliation defined by the suspension of

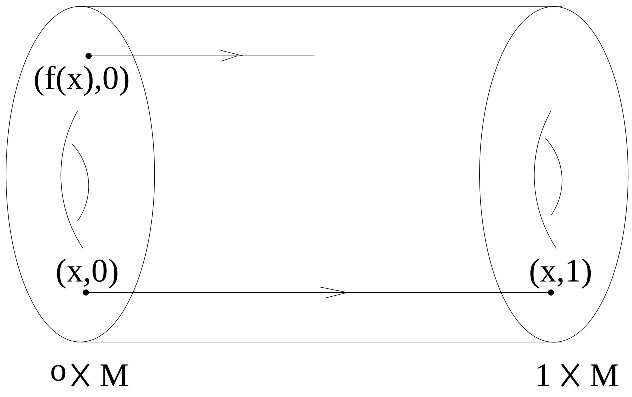


Fig. 1.7

the representation

$$\varphi : \mathbb{Z} = \pi_1(S^1) \to \mathrm{Diff}(M)$$

given by $\varphi(n) = f^n$.

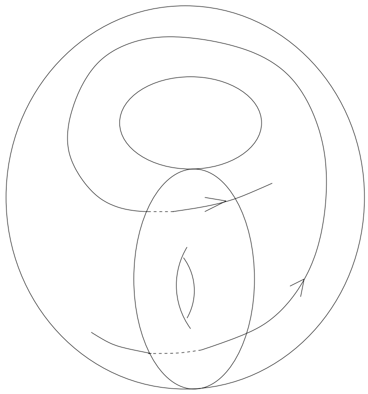


Fig. 1.8

We describe below a criterion due to Fried [48] for the existence of a cross section. Let M be a closed manifold. A $\mathbb{Z}$-covering of M is a regular covering $\widehat{M} \to M$ whose group of fiber-preserving diffeomorphisms is $\mathbb{Z}$.

Theorem 1.26. *A flow X on M has a global cross section if and only if there is a $\mathbb{Z}$-cover $\widehat{M} \to M$ such that if $\hat{X}$ is the lifted flow then*

$$\lim_{t\to\pm\infty} \hat{X}_t(\widehat{x}) = \pm\infty, \quad \forall \widehat{x} \in \widehat{M}.$$

Next we use the theory of closed forms to find global cross sections. This idea comes from J. Plante [127] and D. Tischler.

Definition 1.1. Let ω be a continuous one form on a manifold M. A two form $d\omega$ on M is the `exterior derivative` of ω if

$$\int_{\partial D}\omega = \int_D d\omega,$$

for every C^1 embedded 2-disk D in M whose boundary ∂D is piecewise C^1. A continuous one form ω is `closed` if it has zero exterior derivative (or equivalently $\int_\gamma \omega = 0$ for every piecewise smooth closed curve γ bounding a 2-disk in M). A continuous one-form ω on a manifold M is `locally closed` if for every $x \in M$ there is a neighborhood U of x such that the restricted one form $\omega/_U$ on U is closed.

Lemma 1.7. *A continuous one form is closed if and only if it is locally closed.*

Proof. Obviously closed implies locally closed. Conversely suppose that ω is a locally closed one-form on M. Then for all $x \in M$ there is a neighborhood U_x of x such that the one form $\omega/_{U_x}$ on U_x is closed. Let D be an embedded two disk in M with piecewise C^1 boundary ∂D. Since D is compact there are finitely many two disks $D_1, \cdots, D_k \subset D$ with piecewies C^1 boundaries $\partial D_1, \cdots, \partial D_k$ such that $D = \cup_{i=1}^k D_i$ and also, for all $1 \leq i \leq k$ there is $x_i \in M$ such that $D_i \subset U_{x_i}$. Then,

$$\int_{\partial D}\omega = \sum_{i=1}^{k}\int_{\partial D_i}\omega/_{U_{x_i}} = 0$$

proving the result. □

Definition 1.27. *A C^r one form ω is integrable if its kernel $ker(\omega)$ is tangent to a C^r foliation $\mathscr{F}_\omega$.*

Remark 1.28. *A C^r continuous closed one form is integrable. A non-singular continuous one form ω is integrable if $d\omega$ exists and satisfies $\omega \bigwedge d\omega = 0$.*

Definition 1.29. *Let* ω *be a closed one-form in a manifold* M*. The* `group of periods of` ω *is by definition the Image set of the homomorphism* $\pi_1(M) \to (\mathbb{R},+)$ *defined by*

$$[\gamma] \to \int_\gamma \omega$$

(such a homomorphism is well defined since ω *is closed). We say that a closed one form* ω *has* `rational` *or* `integer` *or* `trivial periods` *depending on whether its group of periods is contained in* $(\mathbb{Q},+)$ *or is* $(\mathbb{Z},+)$ *or is* 0.

Theorem 1.30. *Let* ω *be a non-singular continuous closed one form on a compact manifold* M*. If* ω *has rational periods, then* M *is a bundle over* S^1 *whose fibers are the leaves of* $\mathscr{F}_\omega$*. In particular, the leaves of* $\mathscr{F}_\omega$ *are compact.*

Proof. Pick a base point $x_0 \in M$. We can assume that ω has integer periods for, otherwise, we multiply ω by a suitable integer. This is possible since $\pi_1(M)$ is finitely generated (because M is compact) and ω has rational periods. Define the map $\pi : M \to S^1 = \mathbb{R}/\mathbb{Z}$ by

$$\pi(x) = \int_{x_0}^{x} \omega \pmod 1.$$

The map π is well defined (i.e. it does not depent on the path from x_0 to x) since ω has integer periods. In addition π is a C^1 map. If x, y belong to the same leaf of $\mathscr{F}_\omega$, then $\pi(x) = \pi(y)$ because ω vanishes in $T\mathscr{F}_\omega$. Hence $\pi^{-1}(\theta)$ is a finite union of leaves of $\mathscr{F}$ for all $\theta \in S^1$. Hence all leaves of $\mathscr{F}_\omega$ are compact (and without holonomy since $\mathscr{F}_\omega$ is defined by a closed one form). It follows from the Reeb Stability Theorem [65] that the leaves of $\mathscr{F}_\omega$ are the fibers of a fibration $M \to S^1$. This proves the result. □

Lemma 1.8. *A* C^r *closed one form* ($0 \le r \le \infty$) *on a compact manifold* M *can be* C^0 *approximated by a* C^r *closed one form with rational periods.*

Proof. Let $H^1(M,G)$ and $H_1(M,G)$ denote the first homology and cohomology groups of M with coefficients in the abelian group G respectively. Denote by V^* the dual of V.

The Universal Coefficient Theorem ([63] p. 198) gives $H^1(M,G) = H_1(M,G)^*$ for $G = \mathbb{R}$ or $\mathbb{Z}$. In the particular case $G = \mathbb{Z}$ we also know that $H_1(M,\mathbb{Z}) = \pi_1(M)/[\pi_1(M),\pi_1(M)]$. We have that $\pi_1(M)$ is finitely generated (because M is compact) and then so is $H_1(M,\mathbb{Z})$. Since $H_1(M,\mathbb{Z})$ is also abelian the Fundamental Theorem of finitely generated abelian groups [63] implies that $H_1(M,\mathbb{Z}) = \mathbb{Z}^k \oplus T$

for some $k \in I\!N$ and some finite abelian group T. $\mathbb{Z}^k$ is the free part of $H_1(M,\mathbb{Z})$. Clearly $H_1(M,\mathbb{Z})^* = (\mathbb{Z}^k)^*$ because $\mathbb{Z}$ is torsion free. Pick a basis $\{[\gamma_1],\cdots,[\gamma_k]\}$ of the free part $\mathbb{Z}^k \subset H_1(M,\mathbb{Z})$ (formed by homology classes of smooth curves $\gamma_1,\cdots,\gamma_k$) and the corresponding dual basis $\{\omega_1,\cdots,\omega_k\}$ in $(\mathbb{Z}^k)^* = \mathbb{Z}^k$. Since $(\mathbb{Z}^k)^* = H_1(M,\mathbb{Z})^* = H^1(M,\mathbb{Z})$ we have that $\{\omega_1,\cdots,\omega_k\}$ is a basis of the cohomology group $H^1(M,\mathbb{Z})$.

Let $H^1_{de\,R}(M)$ be the de Rham cohomology group of M which is the quotient between the closed and the exact one forms in M. The classical de Rham Theorem says that $H_1(M,\mathbb{R})^* = H^1_{de\,R}(M)$ via the bilinear map $H_{de\,R}(M)\times H_1(M,\mathbb{R}) \to \mathbb{R}$ given by

$$(\eta,[\gamma]) \mapsto \int_\gamma \eta.$$

As $H^1(M,\mathbb{Z}) \subset H^1(M,\mathbb{R}) = H_1(M,\mathbb{R})^* = H^1_{de\,R}(M)$ we have that $\{\omega_1,\cdots,\omega_k\}$ are closed one forms and, since they form the dual basis, we get

$$\int_{\gamma_i} \omega_j = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker's delta. In addition, each ω_i has integer periods.

Now, consider a closed one form ω in M and define the numbers

$$\alpha_i = \int_{\gamma_i} \omega, \quad 1 \le i \le k.$$

We claim that the closed one form

$$\omega - \sum_{j=1}^{k} \alpha_j \omega_j$$

satisfies

$$\int_\gamma \left(\omega - \sum_{j=1}^{k} \alpha_j \omega_j\right) = 0,$$

for all closed curve γ in M. Indeed, if γ is a closed curve then it represents the element $[\gamma] \in H_1(M,\mathbb{Z})$. Hence $[\gamma] = \sum_{i=1}^{k}(n_i\cdot[\gamma_i]) + t$ for some $\{n_1,\cdots,n_k\} \subset \mathbb{Z}$ and $t \in T$. As $\mathbb{Z}$ is torsion free (and the one form $\omega - \sum_{j=1}^{k}\alpha_j\omega_j$ is closed) we get

$$\int_t \left(\omega - \sum_{j=1}^{k} \alpha_j \omega_j\right) = 0.$$

Consequently

$$\int_\gamma \left(\omega - \sum_{j=1}^{k} \alpha_j \omega_j\right) = \int_{\sum_{i=1}^{k}(n_i\cdot[\gamma_i])} \left(\omega - \sum_{j=1}^{k} \alpha_j \omega_j\right) =$$

$$= \int_{\sum_{i=1}^{k}(n_i\cdot[\gamma_i])} \omega - \sum_{j=1}^{k}\left(\alpha_j \cdot \int_{\sum_{i=1}^{k}(n_i\cdot[\gamma_i])} \omega_j\right) =$$

$$= \sum_{i=1}^{k}\left(n_i \cdot \int_{\gamma_i} \omega\right) - \sum_{j=1}^{k}\sum_{i=1}^{k}\left(\alpha_j \cdot n_i \cdot \int_{\gamma_i} \omega_j\right) =$$

$$= \sum_{i=1}^{k} n_i \cdot \alpha_i - \sum_{j=1}^{k}\sum_{i=1}^{k}(\alpha_j \cdot n_i \cdot \delta_{i,j}) = \sum_{i=1}^{k} n_i \cdot \alpha_i - \sum_{i=1}^{k} \alpha_i \cdot n_i = 0.$$

This proves the claim.

Now pick a base point $x_0 \in M$. It follows from the claim that the map $f : M \to \mathbb{R}$ defined by

$$f(x) = \int_{x_0}^{x}\left(\omega - \sum_{i=1}^{k} \alpha_i \omega_i\right)$$

(where the integral is taken over a path joinning x_0 to x) is well defined i.e. does not depend on the path. This map satisfies

$$\omega = \left(\sum_{i=1}^{k} \alpha_i \omega_i\right) + df.$$

Given $\beta_1, \cdots, \beta_k \in \mathbb{Q}$ we define a new closed one-form η in M given by

$$\eta = \left(\sum_{i=1}^{k} \beta_i \omega_i\right) + df.$$

Clearly η is a closed one-form in M which is as smooth as ω. We have that η has rational periods since each ω_i has integer periods and $\beta_i \in \mathbb{Q}$ (note that df has trivial periods since it is exact). Moreover, if the β_i's are chosen close to the corresponding α_i's, then the one-form η is C^0 closed to ω. This proves the result. □

Definition 1.31. *Let X be a C^r flow on a manifold M and still denote by X the associated vector field. A one form ω on M is* `transverse` *to X if $\omega(x)(X(x)) \neq 0$ for all $x \in M$.*

Theorem 1.32. *A C^1 flow on a compact manifold M is suspended if and only if it is transverse to a continuous closed one form on M.*

Proof. Let X be a C^1 flow on a compact manifold M. Clearly if X is suspended, then X is transverse to a continuous closed one form on M. Conversely, assume that X is transverse to a continuous closed one form ω on M. Obviously ω is non-singular. Applying Lemma 1.8 we can assume that ω has rational periods. It follows from Lemma 1.30 that leaves of $\mathscr{F}_\omega$ are the fibers of a fibration of M over S^1. The hypothesis implies that X is transverse to the fibers, and so, since M is compact, each leaf of $\mathscr{F}_\omega$ is a global cross section of X. This finishes the proof. □

The following is a corollary of the above result.

Proposition 1.33. *Let X be a flow on a compact manifold M. Let $p\colon \hat{M} \to M$ be a finite covering. Let $\hat{X}$ be the lift of X to $\hat{M}$. Then, X is suspended if and only if $\hat{X}$ is.*

Proof. First assume that X is suspended. Then X is transverse to a closed one form ω. Let $P\colon \hat{M} \to M$ be the finite covering map. Then, the one form $p^*(\omega)$ is closed and transverse to the lifted flow $\hat{X}$. It follows that $\hat{X}$ is suspended by Lemma 1.32.

Now we assume that $\hat{X}$ is suspended. Then, by Lemma 1.32, $\hat{X}$ is transverse to a closed one form $\hat{\omega}$ on $\hat{M}$. Let G be the group of covering transformations of the covering $P\colon \hat{M} \to M$.

We have that G is a finite group. Indeed, consider the subgroup $H = P^*(\pi_1(\hat{M}))$ of $\pi_1(M)$. As the covering P is finite we have that the index $[\pi_1(M), H]$ is also finite. But G is isomorphic to the quotient $N(H)/H$ where $N(H) = \{\gamma \in \pi_1(M) : \gamma H \gamma^{-1} = H\}$ is the normalizer of H. Hence $|G| = |N(H)/H| \leq [\pi_1(M), H] < \infty$ proving that G is a finite set.

Since G is finite the one form $\hat{\omega}'$ on $\hat{M}$ given by

$$\hat{\omega}' = \sum_{g \in G} g^*(\hat{\omega})$$

is well defined. Clearly $\hat{\omega}'$ is closed and *G-invariant*, i.e. $g^*(\hat{\omega}) = \hat{\omega}'$ for all $g \in G$. As $\hat{X}$ is transverse to $\hat{\omega}$ we have that $\hat{X}$ is also transverse to $\hat{\omega}'$. Hence, without loss of generality, we can assume that $\hat{\omega}$ itself is G-invariant.

Let $k = |G|$ be the cardinality of G. Then, for all $x \in M$ the set $P^{-1}(x)$ is consists of k-points $\{\hat{x}_1, \cdots \hat{x}_k\}$. Moreover, for all $x \in M$ there is a neighborhood $U_x \subset M$ of x such that

$$P^{-1}(U_x) = V_{\hat{x}_1} \cup \cdots \cup V_{\hat{x}_k}$$

where $V_{\hat{x}_i}$ is a neighborhood of $\hat{x}_i$ in $\hat{M}$ such that $P/_{V_{\hat{x}_i}} : V_{\hat{x}_i} \to U_x$ is a diffeomorphism with inverse $\phi_i\colon U_x \to V_{\hat{x}_i}$ ($\forall i$). The $V_{\hat{x}_i}$'s are pairwise disjoint and also for every pair $i, j \in \{1, \cdots, k\}$ there is $g_{i,j} \in G$ such that

$$g_{i,j} \circ \phi_i = \phi_j.$$

In particular,

$$g_{i,j}(V_{\hat{x}_i}) = V_{\hat{x}_j}$$

for all i, j.

For all $x \in M$ and $i = 1, \cdots, k$ we define the one form ω_i^x on U_x by

$$\omega_i^x = \phi_i^*(\hat{\omega}/_{V_{\hat{x}_i}}).$$

If $i, j \in \{1, \cdots, k\}$ then

$$\omega_i^x = (\phi_i)^*(\hat{\omega}/_{V_{\hat{x}_i}}) = (\phi_i)^*(\hat{\omega}/_{g_{i,j}^{-1})^*(V_{\hat{x}_j})}) = (\phi_i)^*(g_{i,j}^*(\hat{\omega}/_{V_{\hat{x}_j}})) =$$

$$= (g_{i,j} \circ \phi_i)^*(\hat{\omega}/_{V_{\hat{x}_j}}) = \phi_j^*(\hat{\omega}/_{V_{\hat{x}_j}}) = \omega_j^x.$$

Consequently the value of ω_i^x does not depend on i.

Let $x_1, \cdots, x_r$ be finitely many points of M such that the collection

$$\{U_{x_1}, \cdots, U_{x_r}\}$$

is a covering of M. Take a partition of the unity $\delta_1, \cdots, \delta_r$ subordinated to such a covering and define the one form

$$\omega = \sum_{i=1}^{r} \delta_i \cdot \omega^{x_i}.$$

Hence ω is a one form on M which is transverse to X. Moreover, ω is locally closed and hence closed by Lemma 1.7. The existence of ω implies that X is suspended by Lemma 1.32. The proof follows. □

1.7 Algebraic diffeomorphisms and flows

Let us recall some basic concepts in Lie groups theory.

Definition 1.34. *A Lie group is a group G with a differentiable structure marking the maps*

$$\begin{array}{rcl} G \times G & \to & G \\ (x,y) & \to & x,y \end{array} \qquad \begin{array}{rcl} G & \to & G \\ x & \to & x^{-1} \end{array}$$

differentiable.

Examples of Lie groups are:

(1) $(\mathbb{R}^k, +)$ which is not finitely generated;
(2) $(\mathbb{Z}^k, +)$ as a discrete subgroup of $(\mathbb{R}^k, +)$;
(3) $(\mathbb{C}, +)$ and $(\mathbb{C}^*, \cdot)$;
(4) $(S^1, \cdot)$ viewed is a compact subgroup of $(\mathbb{C}^*, \cdot)$ (the product $\cdot$ is the complex product);
(5) $(S^1 \times \cdots \times S^1, \cdot) = (T^k, \cdot)$ viewed as the product of k-copies of $(S^1, \cdot)$;
(6) $\mathrm{GL}(n, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{Z}$ defined as the set of $n \times n$ matrices with non-zero determinant $\det \neq 0$ (the product is the matrix's product);
(7) $\mathrm{SL}(n, \mathbb{F}) = \{n \times n$ matrices with entries in $\mathbb{F} = \mathbb{R}, \mathbb{Z}$ and $det = 1\}$;

(8)$\mathrm{PSL}(n,\mathbb{F}) = \mathrm{SL}(n,\mathbb{F})/\mathbb{Z}_2$ (note that $\mathbb{Z}_2 = \{I, -I\} < \mathrm{SL}(n,\mathbb{F})$).

The *Lie algebra* of G is $\mathscr{Y} = Te\,G$ where e is the identity of G.
Every $g \in G$ defines a pair of diffeomorphisms L_g , R_g (left and right translations) which are defined by

$$L_g(h) = g \cdot h; \quad R_g(h) = h \cdot g.$$

A *vector field* $\widetilde{X}$ of G is *left* (*right*) *invariant* if $DL_g(h)$ $(DR_g(h)) = \widetilde{X}(L_g h)$ $(\widetilde{X}(R_g(h))$ $\forall g, h \in G$.
In particular, a left (right) invariant vector field $\widetilde{X}$ is well defined by its value at e, $\widetilde{X}(e)$. Hence we have a bijection

$$\{\text{Left invariant vector fields } \widetilde{X}\} \mapsto \mathscr{Y}$$

$$\widetilde{X} \mapsto X = \widetilde{X}(e).$$

Analogously for right invariant vector fields.
If $X \in \mathscr{Y}$ we denote by $\widetilde{X}$ the left invariant vector fields satisfying

$$\widetilde{X}(e) = X.$$

If $X, Y \in \mathscr{Y}$ then there is a well defined Lie bracket $[X,Y]$. One can to prove that $[\widetilde{X}, \widetilde{Y}]$ is left invariant if $\widetilde{X}$, $\widetilde{Y}$ are. This allows us to define the map

$$[\,\cdot\,,\,\cdot\,] : \mathscr{Y} \times \mathscr{Y} \to \mathscr{Y}$$
$$(X,Y) \mapsto [X,Y] = [\widetilde{X}, \widetilde{Y}]$$

The following properties hold:
- $[X,X] = 0 \quad \forall X \in \mathscr{Y}$.
- $[\,\cdot\,,\,\cdot\,]$ is bilinear.
- $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (*Jacobi identity*).

The map

$$ad\,X : \mathscr{Y} \to \mathscr{Y}$$
$$Y \mapsto [X,Y]$$

is called the *adjoint of* $X \in \mathscr{Y}$. We say that $\mathscr{Y}$ (or G) is *nilpotent* if $ad\,X$ is a nilpotent linear operator $\forall X \in \mathscr{Y}$. Given $X \in \mathscr{Y}$ we define

$$\boxed{\exp(X) = \theta(1)}$$

where $\theta_X : \mathbb{R} \to G$ is the *unique* C^1 homomorphism satisfying $\theta'_X(0) = X$. This map is called the exponential of X.

Remark: (1) $\exp(A)$ is the exponential of the matrix A, when $A \in (d(n,\mathbb{F}))$.
(2) exp is a diffeomorphism from some neighborhood N_0 of $0 \in \mathscr{Y}$ into some neighborhood N_e of $e \in G$.

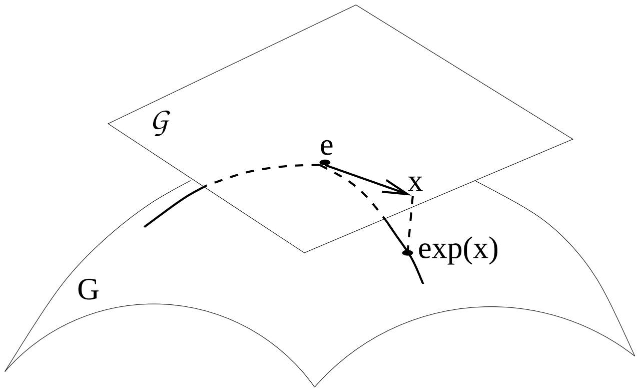


Fig. 1.9

Every homomorphism $\theta : \mathbb{R} \to G$ satisfies

$$\theta(t) = \exp(tX)$$

for some $X \in \mathscr{Y}$. Such homomorphismw are called *one-parameter subgroups of* G. A Lie subgroup of G is a submanifold which is also a subgroup of G.

Remark: A subgroup H of G is a *Lie subgroup* if and only if it is closed in G.
If $H < G$ is closed $\Rightarrow G/H$ has a manifold structure marking the canonical projection

$$\pi : g \in G \mapsto gH \in G/H$$

an onto open map.
Actually π defined a fibration $\pi : G \to G/H$ of G with fibre H and base G/H.

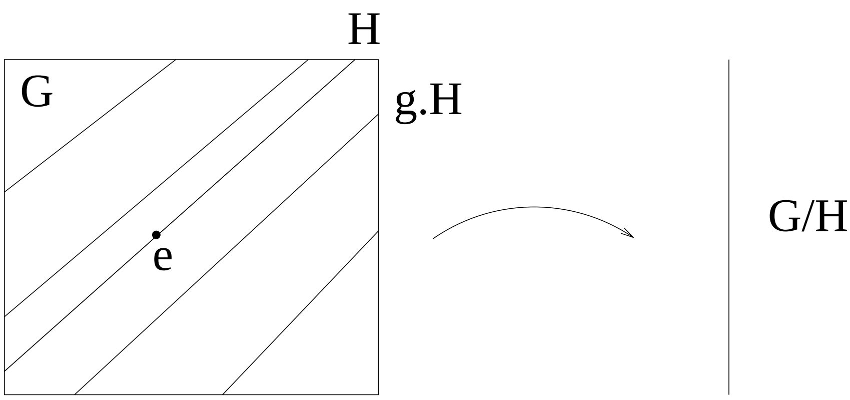


Fig. 1.10

Observe that G/H is *not* necessarily a Lie group (this happens if $H \lhd G$). When $\Gamma < G$ is another subgroup (for examploe $\Gamma = G$) then there is an action on the left

$$\Gamma \times G/H \to G/H$$

given by $\gamma \cdot (gH) = \gamma \cdot gH$ the orbit space of this action is denoted by $\Gamma \backslash G/H$, i.e.

$$\Gamma \backslash G/H = \{\Gamma \cdot g \cdot H : g \in G\}.$$

The adjoint representation is the representation

$$G \to L(\mathscr{Y}) \qquad \boxed{\text{the set of linear automorphis in } \mathscr{Y}}$$

given by $g \mapsto DI_g(e)$, where $I_g \colon G \to G$ is $I_g(h) = ghg^{-1}$.

Definition 1.35. *A diffeomorphism is called algebraic if it is conjugated to a diffeomorphism $A \colon M \to M$ with the following properties:*

(1)There are Lie groups G, Γ with G connected simply connected and Γ actions freely and property discontinuous in G such that $M = \Gamma \backslash G$.
(2)There is an automorphism $\widehat{A} \in \mathrm{Aut}(G)$ with $\widehat{A}(\Gamma) = \Gamma$ so that the diagram below commutes

$$\begin{array}{ccc} G & \xrightarrow{\widehat{A}} & G \\ \downarrow & & \downarrow \\ M = \Gamma \backslash G & \xrightarrow{A} & \Gamma \backslash G = M \end{array}$$

Remark: $\widehat{A}/\Gamma \in \mathrm{Aut}(\Gamma)$ *because* $\widehat{a}(\Gamma) = \Gamma$.

Remark: Under the condition above one has:
(•) the universal cover $\widetilde{M}$ of M is G
(•) $\pi_1(M) = \Gamma$

Definition 1.36. *A flow is called algebraic if it is topological equivalent to a flow $\phi_t \colon M \to M$ with the following properties:*

1. *There are Lie groups G, Γ, K with G connected simply connected $K < G$ is compact and Γ acting freely property discontinuous in G/K so that $M = \Gamma \backslash G/K$.*
2. *There is a element $\alpha \in \mathscr{Y}$ (the Lie algebra of G) such that ϕ if $\hat{\phi}_t$ is the flow*

$$\hat{\phi}_t(g) = g \cdot \exp(t\alpha)$$

(thus ($\hat{\phi}_t : G \to G$)), then the diagram below commutes

$$\begin{array}{ccc} G & \xrightarrow{\hat{\phi}_t} & G \\ \downarrow & & \downarrow \\ M = \Gamma\backslash G/K & \xrightarrow{\phi_t} & \Gamma\backslash G/K = M. \end{array}$$

(The vertical arrow corresponding to the proposition $g \mapsto \Gamma \cdot g \cdot K$).

Proposition 1.37. *The suspension of an algebraic diffeomorphism is an algebraic flow.*

Proof. It suffices to prove the result for a diffeomrphism $A : \Gamma\backslash G \to \Gamma\backslash G$ where:
- G is a connected simply connected Lie group;
- Γ is a Lie group acting locally properly discontinously in G such that there is $\widehat{A} \in \mathrm{Aut}(G)$ fro which the diagram below commutes

$$\begin{array}{ccc} G & \xrightarrow{\widehat{A}} & G \\ \downarrow & & \downarrow \\ M = \Gamma\backslash G & \xrightarrow{A} & \Gamma\backslash G. \end{array}$$

Let $M = \Gamma\backslash G$ and ϕ_t^A be the suspension of A defined on the suspended manifold M^A. Recall that $M^A =$ orbit space of the action $n \cdot (t,x) = (t+n, A^n(x))$ induce by the representation $\mathbb{Z} = \pi_1(S^1) \to \mathrm{Diff}(M)$ defined by $n \mapsto A^n = A \circ \overset{n}{\ldots} \circ A$. Applying the Seifert-Van Kampen Theorem we can compute $\pi_1(M^A)$ by the semidirect product

$$\pi_1(M^A) = \mathbb{Z} \underset{\widehat{A}_*}{\rtimes} \Gamma,$$

where $\widehat{A}_* : \mathbb{Z} \to \mathrm{Aut}(\Gamma)$ is the representation defined by

$$\widehat{A}_*(n) = \widehat{A}^n / \Gamma$$

(recall $\Gamma = \pi_1(M)$ and $\widehat{A}(\Gamma) = \Gamma$) *Recall that if G and H are groups and $\varphi : H \to \mathrm{Aut}(G)$ is a representation we define the semidirect product $H \underset{\varphi}{\rtimes} G = H \times G$ endowed with the product*

$$(h,g) \cdot (h',g') = (h \cdot h', g \cdot \varphi(h)g').$$

Example: $A \in \mathrm{GL}(2,\mathbb{Z}^2)$ and $\varphi\colon \mathbb{Z} \mapsto \mathrm{Aut}(\mathbb{Z}^2)$ is given by $A(n) = A^n$ then the product in $\mathbb{Z} \underset{A}{\rtimes} (\mathbb{Z}\times\mathbb{Z})$ is

$$(n,(u,v))\cdot(m,(a,b)) = (n+m,(u,v)+A^n(a,b))$$

It follows that $\widehat{\Gamma} = \pi_1(M^A)$ is a lie group. Now, $\widehat{G} = \mathbb{R}\times G$ is connected simply connected Lie group and there is a covery

$$\widehat{G} = \mathbb{R}\times G = \mathbb{R}\times\widetilde{M} \to \mathbb{R}\times M \to M^A$$

(the last covering exists by the definition of suspension). We conclude that the universal covering $\widetilde{M}^A$ of M^A is $\mathbb{R}\times G$. As it is well know $\widehat{\Gamma} = \pi_1(M^A)$ acts freely and properly discontinuous in $\widehat{G} = \widetilde{M}^A$ so that

$$\widehat{\Gamma}\backslash\widehat{G} = M^A.$$

On the other hand, the definition of the suspension ϕ_t^A of A says that the diagram below commutes

$$\begin{array}{ccc} \mathbb{R}\times M & \xrightarrow{\tilde{\phi}_t} & \mathbb{R}\times M \\ \downarrow & & \downarrow \\ M^A & \xrightarrow{\phi_t} & M^A \end{array}$$

where $\tilde{\phi}_t(s,x) = (t+s,x)$. This implies the existence of a commutative diagram

$$\begin{array}{ccc} \widehat{G} = \mathbb{R}\times G & \xrightarrow{\tilde{\phi}_t} & \widehat{G} = \mathbb{R}\times G \\ \downarrow & & \downarrow \\ \mathbb{R}\times M & \xrightarrow{\tilde{\phi}_t} & \mathbb{R}\times M \\ \downarrow & & \downarrow \\ \widehat{\Gamma}\backslash\widehat{G} = M^A & \xrightarrow{\phi_t} & M^A = \widehat{\Gamma}\backslash\widehat{G} \end{array}$$

where $\hat{\phi}_t(s,g) = (t+s,g)$. For $\hat{g} = (s,g)$ one has

$$\hat{\phi}_t(\hat{g}) = \hat{g}\cdot\overbrace{\exp(t,\alpha)}^{(t,0)} = (s+t,g)$$

where $\alpha = (1,0) \in T_{(0,e)}\widehat{G} = T_{(0,e)}(\mathbb{R}\times G)$. Setting $K = \{1\}$ in the definition of algebraic flow we get the result. $\square$

1.8 Triangular maps

In this section we examine certain maps defined in a finite disjoint union Σ of copies of $[-1,1]\times[-1,1]$. For simplicity we shall call them *triangular maps*. This name is deserved in the literature to properly continuous maps in $[-1,1]\times[-1,1]$ which are skew product, i.e., they preserve the constant vertical foliation (see for instance [13], [73]). In our context we shall consider discontinuous maps still preserving a continuous (but not necessarily constant) vertical foliation. We also assume two hypotheses **(H1)-(H2)** imposing certain amount of differentiability close to the points whose iterates fall eventually in the interior of Σ. They will be verified for the return maps associated to the cross-sections in Proposition 6.2.

The main result is Theorem 1.45 asserting the existence of periodic points for hyperbolic triangular maps that satisfy **(H1)-(H2)** and have large domain. Although this result is related to previous ones in the literature ([2], [75], [124], [3]) we cannot apply them to prove the ours because the maps we are going to deal with are not necessarily C^2, can have finite or infinitely many discontinuities or else no invariant measures (see for instance (H2) p. 125 in [124] or the proof of Theorem 11 in [124] p. 142). As we already saw along the book, hyperbolic triangular maps naturally appear as return maps nearby the singularities of sectional-Anosov flows (see Part II).

1.8.1 Definition

Let $I=[-1,1]$ denote the unit closed interval. Hereafter I_i will denote a copy of I and Σ_i will denote a copy of the square $I^2=I\times I$ for $i=1,2,...,k$. Denote by Σ the disjoint union of the squares Σ_i. Denote

$$L_{-i}=\{-1\}\times I_i\ ;\quad L_{0i}=\{0\}\times I_i\ ;\quad L_{+i}=\{1\}\times I_i\ ,$$

for $i=1,...,k$ and

$$L_{-}=\bigcup_{i=1}^{k}L_{-i}\ ;\quad L_0=\bigcup_{i=1}^{k}L_{0i}\ ;\quad L_{+}=\bigcup_{i=1}^{k}L_{+i}.$$

Given a map F we denote by $Dom(F)$ the domain of F.

The following is the standard definition of periodic point except by the fact that our maps F are not everywhere defined.

Definition 1.38. *Let $F:Dom(F)\subset\Sigma\to\Sigma$ be a map. A point $x\in Dom(F)$ is* periodic *for F if there is $n\geq 1$ such that $F^j(x)\in Dom(F)$ for all $0\leq j\leq n-1$ and $F^n(x)=x$.*

A *curve* c in Σ is the image of a C^1 injective map $c : Dom(c) \subset \mathbb{R} \to \Sigma$ with $Dom(c)$ being a compact interval. We often identify c with its image set. A curve c is *vertical* if it is the graph of a C^1 map $g : I_i \to I_i$, i.e., $c = \{(g(y), y) : y \in I_i\} \subset \Sigma_i$ for some $i = 1, ..., k$.

Definition 1.39. *A continuous foliation $\mathscr{F}_i$ on a component Σ_i of Σ is called* vertical *if its leaves are vertical curves and the curves L_{-i}, L_{0i}, L_{+i} are leaves of $\mathscr{F}_i$. A* vertical foliation *$\mathscr{F}$ of Σ is a foliation which restricted to each component Σ_i of Σ is a vertical foliation.*

It follows from the definition above that the leaves L of a vertical foliation $\mathscr{F}$ are vertical curves hence differentiable ones. In particular, the tangent space T_xL is well defined for all $x \in L$.

Definition 1.40. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ a map and $\mathscr{F}$ be a vertical foliation on Σ. We say that F* preserves *$\mathscr{F}$ if for every leaf L of $\mathscr{F}$ contained in $Dom(F)$ there is a leaf $f(L)$ of $\mathscr{F}$ such that*

$$F(L) \subset f(L)$$

and the restricted map $F/_L : L \to f(L)$ is continuous.

If $\mathscr{F}$ is a vertical foliation on Σ a subset $B \subset \Sigma$ is a *saturated set* for $\mathscr{F}$ if it is union of leaves of $\mathscr{F}$. We shall write $\mathscr{F}$-saturated for short.

Definition 1.41 (Triangular map). *A map $F : Dom(F) \subset \Sigma \to \Sigma$ is called* triangular *if it preserves a vertical foliation $\mathscr{F}$ on Σ such that $Dom(F)$ is $\mathscr{F}$-saturated.*

1.8.2 Hyperbolic triangular maps

It is easy to find examples of triangular maps without periodic points. On the other hand, the return map associated to the geometric Lorenz attractor [60] is a triangular map plenty of periodic points. This last example suggests the existence of periodic orbits for triangular maps with some hyperbolicity. The hyperbolicity will be defined through cone fields in Σ: Denote by $T\Sigma$ the tangent bundle of Σ. Given $x \in \Sigma$, $\alpha > 0$ and a linear subspace $V_x \subset T_x\Sigma$ we denote by $C_\alpha(x, V_x) \equiv C_\alpha(x)$ the cone around V_x in $T_x\Sigma$ with inclination α, namely

$$C_\alpha(x) = \{v_x \in T_x\Sigma : \angle(v_x, V_x) \leq \alpha\}.$$

Here $\angle(v_x, V_x)$ denotes the angle between a vector v_x and the subspace V_x. A *cone field* in Σ is a continuous map $C_\alpha : x \in \Sigma \to C_\alpha(x) \subset T_x\Sigma$, where $C_\alpha(x)$ is a cone with constant inclination α on $T_x\Sigma$. A cone field C_α is called *transversal* to a vertical foliation $\mathscr{F}$ on Σ if T_xL is not contained in $C_\alpha(x)$ for all $x \in L$ and all $L \in \mathscr{F}$.

Now we can define hyperbolic triangular map.

Definition 1.42 (Hyperbolic triangular map). *Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a triangular map with associated vertical foliation $\mathscr{F}$. Given $\lambda > 0$ we say that F is* λ-hyperbolic *if there is a cone field C_α in Σ such that:*

1. *C_α is transversal to $\mathscr{F}$.*
2. *If $x \in Dom(F)$ and F is differentiable at x, then*

$$DF(x)(C_\alpha(x)) \subset Int(C_{\alpha/2}(F(x)))$$

and

$$|| DF(x) \cdot v_x || \geq \lambda \cdot || v_x ||,$$

for all $v_x \in C_\alpha(x)$.

1.8.3 Existence of periodic points

In this section we give sufficient conditions for a hyperbolic triangular map to have a periodic point.

1.8.3.1 Hypotheses (H1)-(H2)

They impose some regularity around those leaves whose iterates *eventually fall into* $\Sigma \setminus (L_- \cup L_+)$. To state them we will need the following definition. If $\mathscr{F}$ is foliation we use the notation $L \in \mathscr{F}$ to mean that L is a leaf of $\mathscr{F}$.

Definition 1.43. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a triangular map such that $L_- \cup L_+ \subset Dom(F)$. For all $L \in \mathscr{F}$ contained in $Dom(F)$ we define the (possibly ∞) number $n(L)$ as follows:*

1. *If $F(L) \subset \Sigma \setminus (L_- \cup L_+)$ we define $n(L) = 0$.*
2. *If $F(L) \subset L_- \cup L_+$ we define*

$$n(L) = \sup\{n \geq 1 : F^i(L) \subset Dom(F) \text{ and }$$
$$F^{i+1}(L) \subset L_- \cup L_+, \forall 0 \leq i \leq n-1\}.$$

Essentially $n(L)+1$ gives the first non-negative iterate of L falling into $\Sigma \setminus (L_- \cup L_+)$. This number plays fundamental role in the following definition.

Definition 1.44 (Hypotheses (H1)-(H2)). *Let $F: Dom(F) \subset \Sigma \to \Sigma$ be a triangular map such that $L_- \cup L_+ \subset Dom(F)$. We say that F satisfies:*

(H1) If $L \in \mathscr{F}$ satisfies $L \subset Dom(F)$ and $n(L) = 0$, then there is a $\mathscr{F}$-saturated neighborhood S of L in Σ such that the restricted map $F/_S$ is C^1.

(H2) If $L_* \in \mathscr{F}$ satisfies $L_* \subset Dom(F)$, $1 \leq n(L_*) < \infty$ and

$$F^{n(L_*)}(L_*) \subset Dom(F),$$

then there is a connected neighborhood $S \subset Dom(F)$ of L_* such that the connected components S_1, S_2 of $S \setminus L_*$ (possibly equal if $L_* \subset L_- \cup L_+$) satisfy the properties below:

1. *Both $F(S_1)$ and $F(S_2)$ are contained in $\Sigma \setminus (L_- \cup L_+)$.*
2. *$\forall j \in \{1,2\}$ $\exists 1 \leq n^j(L_*) \leq n(L_*)+1$ such that if $y_l \in S_j$ is a sequence converging to $y \in L_*$, then $F(y_l)$ is a sequence converging to $F^{n^j(L_*)}(y)$. If $n^j(L_*) = 1$, then F is C^1 in $S_j \cup L_*$.*
3. *If $L_* \subset \Sigma \setminus (L_- \cup L_+)$ (and so $S_1 \neq S_2$), then either $n^1(L_*) = 1$ and $n^2(L_*) > 1$ or $n^1(L_*) > 1$ and $n^2(L_*) = 1$.*

These hypotheses present two alternatives for the image of a connected neighborhood S of L: It is either connected, and the restricted map is C^1 (Figure 1.11), or breaks in two pieces where the map is C^1 (Figure 1.12). The integers $n^1(L_*)$, $n^2(L_*)$ in **(H2)** appear because of the breaking at $F(L_*)$.

1.8.4 Statement of Theorem 1.45

The theorem will deal with the existence of periodic points for hyperbolic triangular maps satisfying **(H1)-(H2)**. The motivation is the Lorenz attractor's return map which: has a periodic point; and is a λ-hyperbolic triangular map satisfying **(H1)-(H2)** with λ large and $Dom(F) = \Sigma \setminus L_0$. Our theorem essentially says that the last property implies the first. More precisely, we have the following.

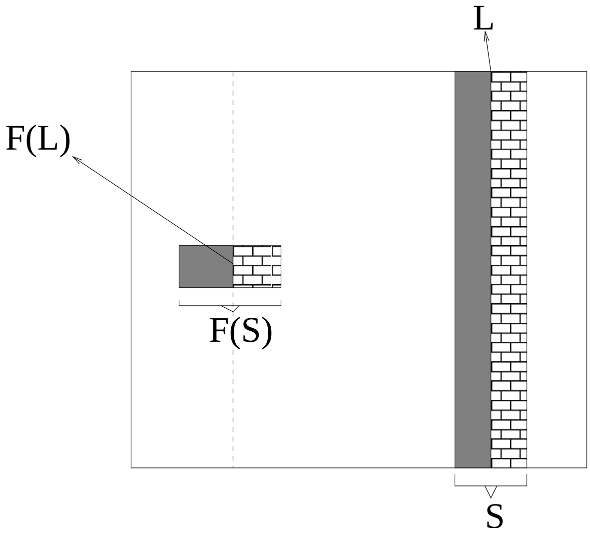


Fig. 1.11 Hypothesis (**H1**)

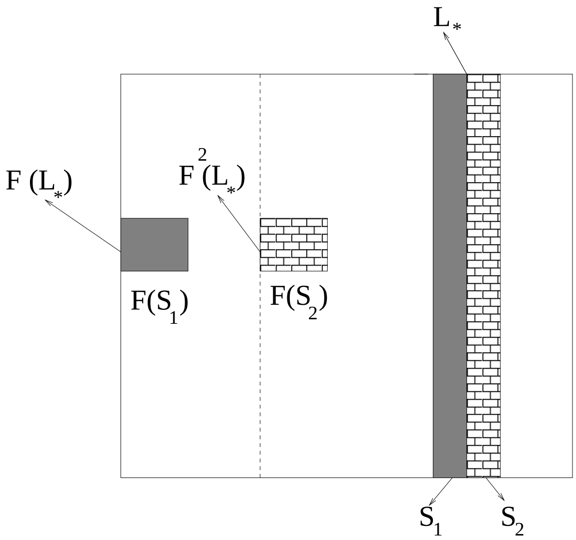


Fig. 1.12 Hypothesis (**H2**) with $n(L_*) = 1$, $n^1(L_*) = 1$ and $n^2(L_*) = 2$

Theorem 1.45. *Every* λ*-hyperbolic triangular map satisfying* **(H1)-(H2)** *with* $\lambda > 2$ *and domain* $\Sigma \setminus L_0$*, has a periodic point.*

We have two difficulties for the proof this result: The first one relies on the possible existence of finite or infinitely many discontinuities in a triangular map (this problem does not appear in the one-dimensional Lorenz map where the set of discontinuities $D(F)$ is empty, see (1.2) for the definition of $D(F)$). The second one is the lack of differentiability of the foliation $\mathscr{F}$ which makes the one-dimensional map f induced by F only C^0.

The first problem will be handled satisfactory with the hypotheses **(H1)-(H2)**. The second problem will be handled by just adapting the argument used by Guckenheimer and Williams to prove that one-dimensional Lorenz maps with derivative $> \sqrt{2}$ are *leo* (i.e. locally eventually onto, see [60]). The idea is to consider the "derivative" of f as being the derivative of F along the invariant cone field in Definition 1.42 (see Claim 1.8.5). This is the reason why we assume $\lambda > 2$ in Theorem 1.45. It seems that Theorem 1.45 holds not only with $\lambda > \sqrt{2}$ but also for all $\lambda > 1$.

1.8.5 Proof of Theorem 1.45

The proof will be divided in three parts. First we present some preliminary results.

Hereafter we fix Σ as in Subsection 1.8.1. Then k is the number of components of Σ. We shall denote by SL the leaf space of a vertical foliation $\mathscr{F}$ on Σ. It turns out that SL is a disjoint union of k-copies $I_1, \cdots, I_k$ of I. We denote by $\mathscr{F}_B$ the union of all leaves of $\mathscr{F}$ intersecting B. If $B = \{x\}$, then $\mathscr{F}_x$ is the leaf of $\mathscr{F}$ containing x. If $S, B \subset \Sigma$ we say that S *cover* B whenever $B \subset \mathscr{F}_S$.

The lemma below quotes some elementary properties of $n(L)$ in Definition 1.43.

Lemma 1.9. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a triangular map with associated vertical foliation $\mathscr{F}$. If $L \in \mathscr{F}$ and $L \subset Dom(F)$, then:*

1. *If F* has no periodic points *and $L_- \cup L_+ \subset Dom(F)$, then*
$$n(L) \leq 2k.$$
2. *$n(L) = 0$ if and only if $F(L) \subset \Sigma \setminus (L_- \cup L_+)$.*
3. *$F^i(L) \subset L_- \cup L_+$ for all $1 \leq i \leq n(L)$.*
4. *If $F^{n(L)}(L) \subset Dom(F)$, then $F^{n(L)+1}(L) \subset \Sigma \setminus (L_- \cup L_+)$.*

If $F : Dom(F) \subset \Sigma \to \Sigma$ is a triangular map with associated foliation $\mathscr{F}$, then we also have an associated one-dimensional map

$$f : Dom(f) \subset SL \to SL.$$

This map allows us to consider the lateral limits

$$f(L_{**}+) = \lim_{L \to L_{**}^+} f(L) \quad \text{and} \quad f(L_{**}-) = \lim_{L \to L_{**}^-} f(L)$$

for all $L_{**} \subset \mathrm{CL}(Dom(F))$ where they exist (as usual the notation $L \to L_{**}^{+}$ means $L \to L_{**}$ with $L > L_{**}$. Analogously for $L \to L_{**}^{-}$.)

We use this map in the definition below.

Definition 1.46. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ a triangular map with associated foliation $\mathscr{F}$ and $f : Dom(f) \subset SL \to SL$ its associated one-dimensional map. Then we define:*

1. $\mathscr{V} = \{f(B) : B \in \mathscr{F}, B \subset Dom(F)$ *and* $B \subset L_- \cup L_+\}$.
2. $\mathscr{L}_- = \bigcup \{f(L_{0i}-) : i \in \{1,...,k\}$ *and* $f(L_{0i}-)$ *exists*$\}$.
3. $\mathscr{L}_+ = \bigcup \{f(L_{0i}+) : i \in \{1,...,k\}$ *and* $f(L_{0i}+)$ *exists*$\}$.

The lemma below is a direct consequence of **(H2)**.

Lemma 1.10. *Let $F: Dom(F) \subset \Sigma \to \Sigma$ a triangular map satisfying* **(H2)** *and $\mathscr{F}$ be its associated foliation. If $L_* \in \mathscr{F}$, $L_* \subset Dom(F)$, $1 \le n(L_*) < \infty$ and*

$$F^{n(L_*)}(L_*) \subset Dom(F),$$

Then:

(1) If $L_* \subset L_-$, then $f(L_*+)$ exists.
(2) If $L_* \subset L_+$, then $f(L_*-)$ exists.
(3) If $L_* \subset \Sigma \setminus (L_- \cup L_+)$, then both $f(L_*+)$ and $f(L_*-)$ exist.

In each case the corresponding limits belong to

$$L_- \cup L_+ \cup \mathscr{V}.$$

In case (3) we have $f(L_*+) \neq f(L_*-)$ and just one them is $f(L_*)$.

Proof. The hypotheses imply that there is a neighborhood S of L_* as in **(H2)**. To prove (1) we observe that if $L_* \subset L_-$, then $S \setminus L_*$ has only one component S_1 (say) located at the right of L_* (in the natural order). So, $f(L_*+) = f^{n^1(L_*)}(L_*)$ where $n^1(L_*)$ is given in **(H2)**-(2). The conclusion follows because $1 \le n^1(L_*) \le n(L_*)+1$. Note in addition that $1 \le n^1(L_*) \le n(L_*)+1$ and so

$$f(L_*+) \subset L_- \cup L_+ \cup \mathscr{V}.$$

Analogously we can prove (2) and also

$$f(L_*-) \subset L_- \cup L_+ \cup \mathscr{V}.$$

(3) and the last part of the lemma follow from similar arguments considering the two components of $S \setminus L_*$. □

Given a map $F : Dom(F) \subset \Sigma \to \Sigma$ we define its *discontinuity set* $D(F)$ by

$$D(F) = \{x \in Dom(F) : F \text{ is discontinuous in } x\}. \tag{1.2}$$

In the sequel we derive useful properties of $Dom(F)$ and $D(F)$.

Lemma 1.11. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ a triangular map satisfying* **(H1)** *and $\mathscr{F}$ be its associated foliation. If $L \in \mathscr{F}$ and $L \subset D(F)$, then $F(L) \subset L_- \cup L_+$.*

Proof. Suppose by contradiction that $L \subset D(F)$ and $F(L) \subset \Sigma \setminus (L_- \cup L_+)$. These properties are equivalent to $n(L) = 0$ by Lemma 1.9-(2). Then, by **(H1)**, there is a neighborhood of L in Σ restricted to which F is C^1. In particular, F would be continuous in L which is absurd. □

Lemma 1.12. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ a triangular map satisfying* **(H1)-(H2)** *and $\mathscr{F}$ be its associated foliation. If F has no periodic points and $L_- \cup L_+ \subset Dom(F)$, then $Dom(F) \setminus D(F)$ is $\mathscr{F}$-saturated, open in $Dom(F)$ and $F/_{(Dom(F)\setminus D(F))}$ is C^1.*

Proof. It suffices to show that $\forall x \in Dom(F) \setminus D(F)$ there is a neighborhood S of $\mathscr{F}_x$ in Σ such that $F/_S$ is C^1. To find S we proceed as follows. Fix $x \in Dom(F) \setminus D(F)$. As $Dom(F)$ is $\mathscr{F}$-saturated, one has $\mathscr{F}_x \subset Dom(F)$ and so $n(\mathscr{F}_x)$ is well defined. Lemma 1.9-(1) implies

$$n(\mathscr{F}_x) < \infty.$$

If $n(\mathscr{F}_x) = 0$, then the neighborhood S of $L = \mathscr{F}_x$ in **(H1)** works.

If $n(\mathscr{F}_x) \geq 1$ we define $L_* = \mathscr{F}_x$. Clearly $1 \leq n(L_*) < \infty$ and Definition 1.43 of $n(L_*)$ implies $f^{n(L_*)}(L_*) \subset L_- \cup L_+$. By hypothesis $L_- \cup L_+ \subset Dom(F)$ and then

$$f^{n(L_*)}(L_*) \subset Dom(F).$$

So, we can choose S as the neighborhood of L_* in **(H2)**. Let us prove that this neighborhood works.

First we claim that $L_* \subset L_- \cup L_+$. Indeed, if $L_* \subset \Sigma \setminus (L_- \cup L_+)$, then $S \setminus L_*$ has two connected components S_1, S_2. By **(H2)**-(3) we can assume $n^1(L_*) > 1$ where $n^1(L)$ comes from **(H2)**-(2). Choose sequence $x_i^1 \in S_1 \to x$ then $F(x_i^1) \to F^{n^1(L_*)}(x)$ by **(H2)**-(2). As F is continuous in x we also have $F(x_i^1) \to F(x)$ and then $F^{n^1(L_*)}(x) = F(x)$ because limits are unique. Thus, $F^{n^1(L_*)-1}(x) = x$ because F is injective and so x is a periodic point of F since $n^1(L_*) - 1 \geq 1$. This contradicts the non-existence of periodic points for F. The claim is proved.

The claim implies that $S \setminus L_*$ has a unique component S_1 (say). For this component one has $n^1(L_*) = 1$ since F is continuous in $x \in L_*$. Then, $F/_S$ is C^1 by the last part of **(H2)**-(2). This finishes the proof. □

Lemma 1.13. *Let $F : Dom(F) \subset \Sigma \to \Sigma$ be a triangular map satisfying* **(H1)-(H2)**. *If F has no periodic points and $Dom(F) = \Sigma \setminus L_0$, then $Dom(F) \setminus D(F)$ is open in Σ.*

Proof. $Dom(F)$ is open in Σ because $Dom(F) = \Sigma \setminus L_0$ and L_0 is closed in Σ. $Dom(F) \setminus D(F)$ is open in $Dom(F)$ by Lemma 1.12 because F has no periodic points and $L_- \cup L_+ \subset \Sigma \setminus L_0 = Dom(F)$. Thus $Dom(F) \setminus D(F)$ is open in Σ. □

Now we start the second part of the proof.

Hereafter $F\colon Dom(F) \subset \Sigma \to \Sigma$ will denote a λ-hyperbolic triangular map satisfying **(H1)-(H2)** with $\lambda > 2$. We also assume that F has *large domain*, namely,

$$Dom(F) = \Sigma \setminus L_0.$$

The foliation and the cone field associated to F will be denoted by $\mathscr{F}$ and C_α respectively.

We shall denote by $<$ the natural order in the leaf space I_i of $\mathscr{F}_i$, where $\mathscr{F}_i$ is a vertical foliation in Σ_i ($i = 1,...,k$). A *vertical band* in Σ is nothing but a region in between two disjoint vertical curves L, L' in the same component Σ_i of Σ. The notation $[L,L']$ and (L,L') indicates closed and open vertical band respectively.

If c is a curve in Σ we denote by c_0, c_1 its end points. We denote $\mathrm{CL}(c) = c \cup \{c_0, c_1\}$ and $Int(c) = c \setminus \{c_0, c_1\}$. An open curve will be a curve without their end points. We say that c is *tangent* to C_α if $c'(t) \in C_\alpha(c(t))$ for all $t \in Dom(c)$. A C_α-*spine* of a vertical band $[L,L']$, (L,L'), $[L,L')$ or $(L,L']$ is a curve $c \subset [L,L']$ tangent to C_α such that $\{c_0, c_1\} \subset L \cup L'$ and $Int(c) \subset (L,L')$.

Definition 1.47. *A subset B of Σ is $\mathscr{F}$-*discrete *if it corresponds to a set of leaves whose only points of accumulations are the leaves in L_0.*

Lemma 1.14. *If F has no periodic points, then:*

(1) $D(F)$ is $\mathscr{F}$-discrete.
(2) If $i \in \{1, \cdots, k\}$ and $D(F) \cap [L_{0i}, L_{+i}]$ consists of finitely many leaves, then $f(L_{0i}+)$ exists.
(3) If $i \in \{1, \cdots, k\}$ and $D(F) \cap [L_{-i}, L_{0i}]$ consists of finitely many leaves, then $f(L_{0i}-)$ exists.

Proof. First we prove (1). By contradiction, suppose that $D(F)$ is not $\mathscr{F}$-discrete. Then, there is an open neighborhood U of L_0 in Σ such that $D(F) \setminus U$ contains infinitely many leaves L_n.

Lemma 1.12 implies that $D(F)$ is closed in $Dom(F) = \Sigma \setminus L_0$ so $D(F) \setminus U$ is closed in $Dom(F) \setminus U$. Since U is an open neighborhood of L_0 and $Dom(F) = \Sigma \setminus L_0$ we obtain that $Dom(F) \setminus U$ is compact in Σ. Henceforth $D(F) \setminus U$ is compact too. So, without loss of generality, we can assume that L_n converges to a leaf L_* of $\mathscr{F}$ *in* $D(F) \setminus U$ in a way that $L_n \cap L_* = \emptyset$ for all n.

Clearly $L_* \subset Dom(F)$. Since $L_n \subset D(F)$ we have $F(L_*) \subset L_- \cup L_+$ by Lemma 1.11. It follows that $n(L_*) \geq 1$. We also have $n(L_*) \leq 2k < \infty$ by Lemma 1.9-(2) since F has no periodic points and $L_- \cup L_+ \subset \Sigma \setminus L_0 = Dom(F)$. By Definition 1.43

we have $f^{n(L_*)}(L_*) \subset L_- \cup L_+ \subset Dom(F)$. Thus, we can choose the neighborhood $S \subset Dom(F)$ of L_* as in **(H2)**. As $L_n \to L_*$ and $L_n \cap L_* = \emptyset$ we can assume $L_n \subset S \setminus L_*$ for all n. As $L_n \cap L_* = \emptyset$ for all n we can further assume that $L_n \in S_1$ where S_1 is one of the (possibly equal) connected components of $S \setminus L_*$. As $F(S_1) \subset \Sigma \setminus (L_- \cup L_+)$ by **(H2)**-(1) we conclude that $F(L_n) \subset \Sigma \setminus (L_- \cup L_+)$ for all n. However, $F(L_n) \subset L_- \cup L_+$ by Lemma 1.11 since $L_n \subset D(F)$ a contradiction. This proves (1).

Now we prove (2). Fix $i \in \{1, \cdots, k\}$ such that $D(F) \cap [L_{0i}, L_{+i}]$ consists of finitely many leaves. As $Dom(F/_{\Sigma_i}) = \Sigma_i \setminus L_{0i}$ there is a leaf $B > L_{0i}$ in I_i such that $(L_{0i}, B] \subset Dom(F) \setminus D(F)$. This implies that f is continuous in $(L_{0i}, B]$. If $f/_{(L_{0i},B]}$ were not monotone, it would exist two different leaves $L, L' \subset (L_{0i}, B]$ bounding a band $[L, L']$ such that $f(L) = f(L') = L''$. Choose a C_α-spine c of $[L, L']$. Then $c \subset Dom(F) \setminus D(F)$ and by Lemma 1.12 we obtain that F is C^1 in c. Thereby, $F(c)$ is a curve transversal to $\mathscr{F}$ intersecting a leaf L'' at least twice. This is a contradiction. We conclude that $f/_{(L_{0i},B]}$ is monotone so $f(L_{0i}+)$ exists. This proves (2).

The proof of (3) is similar. □

Lemma 1.15. *Let $c \subset Dom(F) \setminus D(F)$ be an open curve transversal to $\mathscr{F}$. If there is $n \geq 1$ and a open C^1 curve c^* with closure $CL(c^*) \subset c$ such that $F^i(c^*) \subset Dom(F) \setminus D(F)$ for all $0 \leq i \leq n-1$ and $F^n(c^*)$ covers c, then F has a periodic point.*

Proof. By Lemma 1.12 we have that $Dom(F) \setminus D(F)$ is $\mathscr{F}$-saturated and $F/_{Dom(F)\setminus D(F)}$ is C^1. Then, c and c^* projects (via $\mathscr{F}$) into two intervals in SL still denoted by c and $c*$ respectively. The assumptions imply that $f^i(c^*)$ is defined for all $0 \leq i \leq n-1$ and $f^n(c^*) \supset c \supset Cl(c^*)$. Then, f^n has a fixed point L_{**}. As $F^n(L_{**}) \subset f(L_{**}) = L_{**}$ and $F^n/_{L_{**}}$ is continuous the Brower Fixed Point Theorem implies that F^n has a fixed point. This fixed point represents a periodic point of F. The result follows. □

Lemma 1.16. *F carries a curve $c \subset Dom(F) \setminus D(F)$ tangent to C_α (with length $\mid c \mid$) into a curve tangent to C_α (with length $\geq \lambda \cdot \mid c \mid$).*

Proof. Let $c : Dom(c) \to Dom(F) \setminus D(F)$ be a curve tangent to C_α. If $t \in Dom(c)$ and $c'(t) \in C_\alpha(c(t))$, then $DF(c(t))c'(t) \in C_\alpha(F(c(t))$, because

$$DF(c(t))(C_\alpha(c(t))) \subset Int(C_{\alpha/2}(F(c(t)))).$$

Also,

$$\mid F \circ c \mid = \int_{Dom(c)} \mid\mid DF(c(t))c'(t) \mid\mid dt \geq \int_{Dom(c)} \lambda \cdot \mid\mid c'(t) \mid\mid dt = \lambda \cdot \mid c \mid .$$

The proof follows. □

Lemma 1.17. *Suppose that F has no periodic points. Let L, L' be different leaves in $D(F)$ such that the open vertical band $(L, L') \subset Dom(F) \setminus D(F)$. If c is a C_α-spine of (L, L'), then $F(Int(c))$ covers a vertical band (W, W') with*

$$W, W' \subset L_- \cup L_+ \cup \mathscr{V}.$$

Proof. Without loss of generality we can assume $L < L'$ (in the natural order). Lemma 1.12 implies that $F/_{(L,L')}$ is C^1 because $(L,L') \subset Dom(F) \setminus D(F)$. And Lemma 1.11 implies

$$F(L), F(L') \subset L_- \cup L_+ \tag{1.3}$$

because $L, L' \subset D(F)$. Clearly $L, L' \subset Dom(F)$ and then $n(L), n(L')$ are defined.

By (1.3) we have $n(L), n(L') \geq 1$. Then, $1 \leq n(L), n(L') < \infty$ by Lemma 1.9-(1) since F has no periodic points and $L_- \cup L_+ \subset \Sigma \setminus L_0 = Dom(F)$. By the same reason

$$F^{n(L)}(L), F^{n(L')}(L') \subset Dom(F).$$

Since $L < L'$ we have either $L \subset L_-$ and $L' \subset \Sigma \setminus (L_- \cup L_+)$; or $L' \subset L_+$ and $L \subset \Sigma \setminus (L_- \cup L_+)$ or $L, L' \subset \Sigma \setminus (L_- \cup L_+)$. Then, Lemma 1.10 applied to $L_* = L, L'$ implies that $f(L+), f(L'-)$ exist and

$$f(L+), f(L'-) \subset L_- \cup L_+ \cup \mathscr{V} \tag{1.4}$$

Now, let c be a C_α-spine of (L,L'). To fix ideas we assume $c(0) \in L$ and $c(1) \in L'$. As $Int(c) \subset (L,L') \subset Dom(F) \setminus D(F)$ we have that $F(Int(c))$ is defined. As $F/_{(L,L')}$ is C^1 we have that $F(Int(c))$ is a curve whose boundary points are contained in $f(L+), f(L'-)$. Clearly $f(L+) \neq f(L'-)$ because F preserves $\mathscr{F}$. Then, $\mathscr{F}_{F(Int(c))} = (W, W')$ is an open vertical band with $W = f(L+)$ and $W' = f(L'-)$. Then, (1.4) applies. □

Lemma 1.18. *Suppose that F has no periodic points. For every open curve $c \subset Dom(F) \setminus D(F)$ tangent to C_α there are an open curve $c^* \subset c$ and $n'(c) > 0$ such that $F^j(c^*) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n'(c) - 1$ and $F^{n'(c)}(c^*)$ covers a band (W, W') with*

$$W, W' \subset L_- \cup L_+ \cup \mathscr{V} \cup \mathscr{L}_- \cup \mathscr{L}_+.$$

Proof. Let $c \subset Dom(F) \setminus D(F)$ be a curve tangent to C_α.

The proof is based on the following claim. This claim will be proved adapting the arguments used by Guchkenheimer and Williams to prove that Lorenz's maps with derivative $> \sqrt{2}$ are *leo* (see [60]).

Claim. There are an open curve $c^{**} \subset c$ and $n''(c) > 0$ such that $F^j(c^{**}) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq n''(c) - 1$ and $F^{n''(c)}(c^{**})$ covers an open vertical band

$$(L, L') \subset Dom(F) \setminus D(F),$$

where L, L' are different leaves in $D(F) \cup L_0$.

Proof. For every open curve $c' \subset Dom(F) \setminus D(F)$ tangent to C_α we define

$$N(c') = \sup\left\{n \geq 1 : F^j(c') \subset Dom(F) \setminus D(F), \forall 0 \leq j \leq n-1\right\}.$$

Note that $1 \leq N(c') < \infty$ because $\lambda > 1$ and Σ has finite diameter. In addition, $F^{N(c')}(c')$ is a curve tangent to C_α with

$$F^{N(c')}(c') \cap (D(F) \cup L_0) \neq \emptyset$$

because $Dom(F) = \Sigma \setminus L_0$.

Define the number β by

$$\beta = (1/2) \cdot \lambda.$$

Then, $\beta > 1$ since $\lambda > 2$. Define $c_1 = c$ and $N_1 = N(c_1)$.

If $F^{N_1}(c_1)$ intersects $D(F) \cup L_0$ in a unique leaf L_1, then $F^{N_1}(c_1) \cap L_1$ has a unique point p_1. In this case we define

- $c_2^* =$ the biggest component of $F^{N_1}(c_1) \setminus \{p_1\}$ and
- $c_2 = F^{-N_1}(c_2^*)$.

The following properties hold,

1) $c_2 \subset c_1$ and then c_2 is an open curve tangent to C_α.
2) $F^j(c_2) \subset Dom(F) \setminus D(F)$, for all $0 \leq j \leq N_1$.
3) $| F^{N_1}(c_2) | \geq \beta \cdot | c_1 |$.

In fact, the first property follows because $F^{N_1}/_{\mathscr{F}_{c_2}}$ is injective and C^1. The second one follows from the definition of $N_1 = N(c_1)$ and from the fact that $c_2^* = F^{N_1}(c_2)$ does not intersect any leaf in $D(F) \cup L_0$. The third one follows from Lemma 1.16 because

$$| F^{N_1}(c_2) | = | c_2^* | \geq (1/2) \cdot | F^{N_1}(c_1) | \geq$$
$$\geq (1/2) \cdot \lambda^{N_1} | c_1 | \geq (1/2) \cdot \lambda | c_1 | = \beta \cdot | c_1 |$$

since $\lambda > 2$ and $N_1 \geq 1$.

Next we define $N_2 = N(c_2)$. The second property implies $N_2 > N_1$. As before, if $F^{N_2}(c_2)$ intersects $D(F) \cup L_0$ in a unique leaf L_2, then $F^{N_2}(c_2) \cap L_2$ has a unique point p_2. In such a case we define $c_3^* =$ biggest component of $F^{N_2}(c_2) \setminus \{p_2\}$ and also $c_3 = F^{-N_2}(c_3^*)$. As before

$$| F^{N_3}(c_3) | = | c_3^* | \geq (1/2) \cdot | F^{N_2}(c_2) | \geq (1/2) \cdot \lambda^{N_2 - N_1} | F^{N_1}(c_2) | \geq \beta^2 | c_1 |$$

because of the third property. So,

1) $c_3 \subset c_2$ and c_3 is a curve tangent to C_α.
2) $F^j(c_3) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq N_2$.
3) $| F^{N_2}(c_3) | \geq \beta^2 \cdot | c_1 |$.

In this way we get a sequence $N_1 < N_2 < N_3 < \cdots < N_l < \cdots$ of positive integers and a sequence $c_1, c_2, c_3, \cdots c_l, \cdots$ of open curves (in c) such that the following properties hold $\forall l \geq 1$

1) $c_{l+1} \subset c_l$ and c_{l+1} is an open curve tangent to C_α.
2) $F^j(c_{l+1}) \subset Dom(F) \setminus D(F)$ for all $0 \leq j \leq N_l$.
3) $| F^{N_l}(c_{l+1}) | \geq \beta^l \cdot | c_1 |$.

The sequence c_l must stop by Property (3) since Σ has finite diameter. So, *there is a first integer l_0 such that $F^{N(c_{l_0})}(c_{l_0})$ intersects $D(F)\cup L_0$ in two different leaves L,L'*. Note that these leaves must be contained in the same component of Σ since $F^{N(c_{l_0})}(c_{l_0})$ is connected. Hence the vertical band (L,L') bounded by L,L' is well defined. We can assume that $(L,L')\subset Dom(F)\setminus D(F)$ because $D(F)$ is $\mathscr{F}$-discrete by Lemma 1.14-(1). Choosing $c^{**}=c_{l_0}$ and $n''(c)=N_{l_0}$ we get the result. □

Now we finish the proof of Lemma 1.18. Let $c^{**},n''(c)$ and $L,L'\subset D(F)\cup L_0$ be as in Claim 1.8.5. We have three possibilities: $L,L'\subset D(F)$; $L\subset L_0$ and $L'\subset D(F)$; $L\subset D(F)$ and $L'\subset L_0$. We only consider the two first cases since the later is similar to the second one.

First we assume that $L,L'\subset D(F)$. As $F^{n''(c)}(c^{**})$ is tangent to C_α, and covers (L,L'), we can assume that $F^{n''(c)}(c^{**})$ itself is a C_α-spine of (L,L'). Then, applying Lemma 1.17 to this spine, one gets that $F^{n''(c)+1}(c^{**})$ covers a vertical band (W,W') with

$$W,W'\subset L_-\cup L_+\cup\mathscr{V}$$

In this case the choices $c^*=c^{**}$ and $n'(c)=n''(c)+1$ satisfy the conclusion of Lemma 1.18.

Finally we assume that $L\subset L_0$ and $L'\subset D(F)$. As $L\subset L_0$ we have $L=L_{0i}$ for some $i=1,\cdots,k$. Without loss of generality we can also assume $L_{0i}<L'$.

On the one hand, $(L_{0i},L')=(L,L')\subset Dom(F)\setminus D(F)$ and then $D(F)\cap[L_{0i},L']=\emptyset$. So, Lemma 1.14-(1) implies that $D(F)\cap[L_{0i},L+i]$ consists of finitely many leaves. Then, $f(L_{0i}+)$ exists by Lemma 1.14-(2). Consequently

$$f(L_{0i}+)\in\mathscr{L}_+.$$

(Recall the definition of $\mathscr{L}_\pm$ in Definition 1.46).

On the other hand, $F(L')\subset L_-\cup L_+$ by Lemma 1.11 since $L'\subset D(F)$. It follows that $1\leq n(L')$ and also $n(L')\leq 2k$ by Lemma 1.9-(1) since F has no periodic points and $L_-\cup L_+\subset\Sigma\setminus L_0=Dom(F)$. Since $F^{n(L')}(L')\subset L_-\cup L_+$ by the definition of $n(L')$ we obtain

$$F^{n(L')}(L')\subset Dom(F).$$

Clearly $L'\not\subset L_-$ because $L_{0i}<L'$. Then, Lemma 1.10 applied to $L_*=L'$ implies that $f(L'-)$ exists and satisfies

$$f(L'-)\subset(L_-\cup L_+)\cup\mathscr{L}.$$

But $F((L_{0i},L'))$ (and so $F(F^{n''(c)}(c^{**}))$) covers $(f(L_{0i}+),f(L'-))$ since $(L_{0i},L')\subset Dom(F)\setminus D(F)$. Setting $W=f(L_{0i}+)$ and $W'=f(L'-)$ we get

$$W,W'\subset(L_-\cup L_+)\cup\mathscr{V}\cup\mathscr{L}_+.$$

(Recall the definition of $\mathscr{V}$ in Definition 1.46) Then, $F(F^{n''(c)}(c^{**}))$ covers (W,W') as in the statement. Choosing $c^*=c^{**}$ and $n'(c)=n''(c)+1$ we obtain the result. □

Finally we prove Theorem 1.45. Let F be a λ-hyperbolic triangular map satisfying **(H1)-(H2)** with $\lambda > 2$ and $Dom(F) = \Sigma \setminus L_0$. We assume by contradiction that the following property holds:

(P) F has no periodic points.

Since $L_- \cup L_+ \subset \Sigma \setminus L_0$ and $\Sigma \setminus L_0 = Dom(F)$ we also have

$$L_- \cup L_+ \subset Dom(F).$$

Then, the results in the previous subsections apply. In particular, we have that $Dom(F) \setminus D(F)$ is open in Σ (by Lemma 1.13) and that $D(F)$ is $\mathscr{F}$-discrete (by Lemma 1.14-(1)). All together imply that $Dom(F) \setminus D(F)$ is open-dense in Σ.

Now, let $\mathscr{B}$ be a family of open vertical bands of the form (W, W') with

$$W, W' \subset L_- \cup L_+ \cup \mathscr{V} \cup \mathscr{L}_- \cup \mathscr{L}_+.$$

It is clear that $\mathscr{B} = \{B_1, \cdots, B_m\}$ is a finite set. In $\mathscr{B}$ we define the relation $B \leq B'$ if and only if there are an open curve $c \subset B$ tangent to C_α with closure $\mathrm{CL}(c) \subset Dom(F) \setminus D(F)$, an open curve $c^* \subset c$ and $n > 0$ such that

$$F^j(c^*) \subset Dom(F) \setminus D(F), \qquad \forall 0 \leq j \leq n-1,$$

and $F^n(c^*)$ covers B'.

As $Dom(F) \setminus D(F)$ is open-dense in Σ, and the bands in $\mathscr{B}$ are open, we can use Lemma 1.18 to prove that for every $B \in \mathscr{B}$ there is $B' \in \mathscr{B}$ such that $B \leq B'$. Then, we can construct a chain

$$B_{j_1} \leq B_{j_1} \leq B_{j_2} \leq \cdots,$$

with $j_i \in \{1, \cdots, m\}$ ($\forall i$) and $j_1 = 1$. As $\mathscr{B}$ is finite it would exist a closed sub-chain

$$B_{j_i} \leq B_{j_{i+1}} \leq \cdots \leq B_{j_{i+s}} \leq B_{j_i}.$$

Hence there a positive integer n such that $F^n(B_{j_i})$ covers B_{j_i}. Applying Lemma 1.15 to suitable curves $c^* \subset Cl(c^*) \subset c \subset B_{j_i}$ we obtain that F *has a periodic point*. This contradicts **(P)** and the proof follows.

1.8.6 Homoclinic classes for triangular maps

In this subsection we describe a class of hyperbolic triangular maps with large domain where the conclusion of Theorem 1.45 can be improved. Indeed, to any triangular map F in Σ we can associate a sequence of compact sets $\Lambda_n(F)$ defined inductively by $\Lambda_0(F) = \Sigma$ and $\Lambda_n(F) = Cl(F(\Lambda_{n-1}(F) \cap Dom(F)))$ for $n \geq 1$. We can prove by induction that $\Lambda_{n+1}(F) \subset \Lambda_n(F)$ for all n thus $\Lambda(F)$ is a compact nonempty set unless $\Lambda_n(F) \cap Dom(F) = \emptyset$ for some n (of course this last situation can occur if we do not impose any restriction to F). This sequence allows us to define the *attracting set of* F,

$$\Lambda(F) = \bigcap_{n\geq 0} \Lambda_n(F).$$

It follows from the definition that the elements of $\Lambda(F)$ admit the following characterization:

(C) $y \in \Lambda(F) \iff \forall \varepsilon > 0$ and $\forall n \in \mathbb{N} \setminus \{0\}$ $\exists y_n \in \Sigma$ such that $F^i(y_n) \in Dom(F)$ for all $0 \leq i \leq n-1$ and $d(y, F^n(y_n)) \leq \varepsilon$.

Now we present a theorem saying that, under certain circumstances, the attracting set above is a homoclinic class, and so, the corresponding triangular map has not only one (as Theroem 1.45 says) but also infinitely many periodic points. Its proof relies also on generalization of the Guckenheimer and Williams's arguments in [60]. Hereafter Σ will be a single copy of the square $I^2 = I \times I$ with $I = [-1, 1]$.

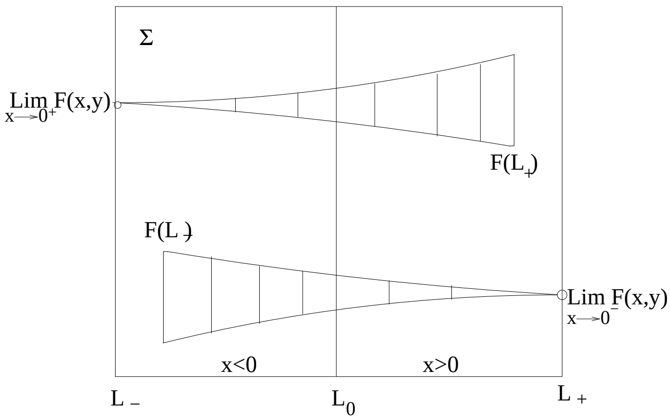
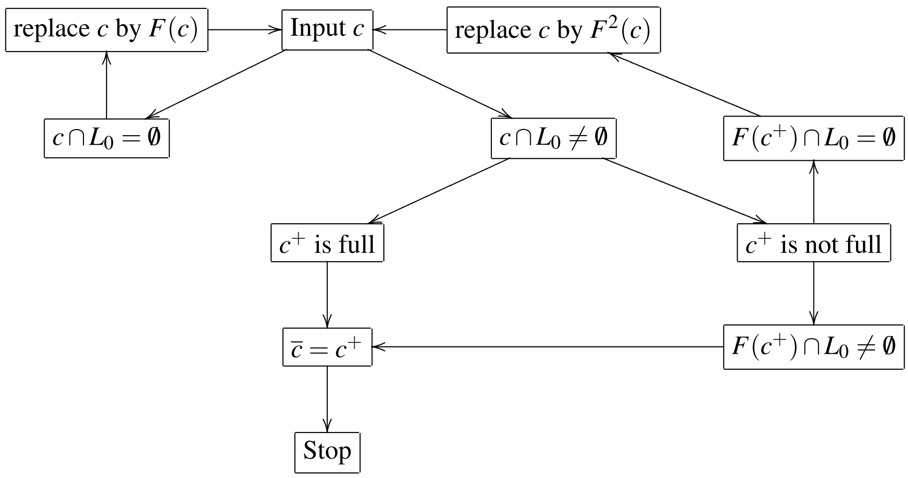


Fig. 1.13 Shape of F in Theorem 1.48.

Theorem 1.48. *Let F be a C^1 injective λ-hyperbolic triangular map with $\lambda > \sqrt{2}$ and large domain in Σ. If the lateral limits $\lim_{x\to 0^+} F(x,y)$ and $\lim_{x\to 0^-} F(x,y)$ exist, do not depend on y and belong to L^- and L^+ respectively, then $\Lambda(F)$ is a homoclinic class of F.*

Proof. See Figure 1.13 where the shape of F as in the statement of the theorem is described. Let $\mathscr{F}$ and C_α be the vertical foliation and the cone field associated to F respectively. We say that a curve c tangent to C_α is a *full curve* if its extreme points belong to $L_- \cup L_+$ and L_0 respectively. Since F is λ-hyperbolic with $\lambda > \sqrt{2} > 1$ we have $F(L_+) \subset Int(\Sigma \cap \{x > 0\})$ and $F(L_-) \subset Int(\Sigma \cap \{x < 0\})$. From this it is not difficult to see that if c is a full curve, then $c \subset Dom(F)$ and every leaf of $\mathscr{F}$ intersects $c \cup F(c)$.

We claim that for every curve c tangent to C_α there are a curve $\overline{c} \subset c$ and $n \in \mathbb{N}$ such that $F^i(\overline{c}) \subset Dom(F)$ for $0 \leq i \leq n$ and $F^n(\overline{c})$ is a full curve. The proof is summarized in the flow diagram below invented by zeze Pacifico:



Indeed, put the curve c into the diagram and ask if $c \cap L_0 = \emptyset$ or not. If yes then $c \subset Dom(F)$ so $F(c)$ exists and go to Input again but relacing c by $F(c)$ first. Since F is λ-hyperbolic we have that the new c has length at least $\lambda \cdot$Length(c). If $c \cap L_0 \neq \emptyset$ then L_0 cuts c in two connected components both contained in $Dom(F)$. Next we ask if the component c^+ with the biggest length is full or not. If it does then we define $\overline{c} = c^+$, $n = 0$ and go to Stop. Otherwise we ask if $F(c^+) \cap L_0 = \emptyset$ or not. If $F(c^+) \cap L_0 = \emptyset$ we have $F(c^+) \subset Dom(F)$ so $F^2(c^+)$ exists and go to Input again but now replacing c by $F^2(c^+)$. Since F is λ-hyperbolic, and the length of c^+ is at least $\frac{1}{2} \cdot$Length(c), we get in this alternative that the length of the new c is at least $\frac{\lambda^2}{2} \cdot$Length$(c)$. If $F(c^+) \cap L_0 \neq \emptyset$ then $F(c^+)$ is clearly a full curve and then we define $\overline{c} = c^+$, $n = 1$ and go to Stop. The diagram must eventually go to Stop since the gain of length at each replacement is at least $\min\{\lambda, \frac{\lambda^2}{2}\}$ which is bigger than 1 since $\lambda > \sqrt{2}$. This proves the claim

It follows from this claim that for every leaf L of $\mathscr{F}$ there is a periodic point of F close to L. Indeed, take a curve c close to L with $c \cap L \neq \emptyset$ and consider the vertical band $V = \{L' : L'$ is a leaf of $\mathscr{F}$ with $L' \cap c \neq \emptyset\}$. Since F contracts $\mathscr{F}$ the claim implies that there are a subband $\overline{V} \subset V$ and $n \in \mathbb{N}$ such that $F^n(\overline{V})$ crosses V as indicated in Figure 1.14.

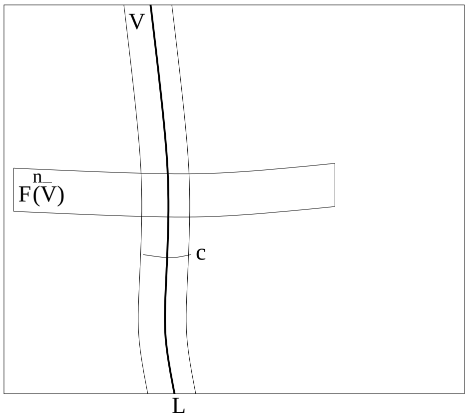


Fig. 1.14

It follows that F has a periodic point in V so the result follows. In particular, F has a periodic point x.

We finish the proof by proving that $\Lambda(F)$ is the homoclinic class of x. Take any $y \in \Lambda(F)$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $d(F^n(z), F^n(\overline{z})) < \frac{\varepsilon}{2}$ whenever $z, \overline{z} \in \Sigma$ satisfy $F^i(z) \in Dom(F)$, $\forall 0 \leq i \leq n$, and $\overline{z} \in \mathscr{F}_z$. Recalling the characterization (C) of $\Lambda(F)$ above we can select $y_n \in \Sigma$ with $F^i(y_n) \in Dom(F)$ for $0 \leq i \leq n$ and $d(y, F^n(y_n)) < \frac{\varepsilon}{2}$. Applying our first claim to a curve $c \subset W^u(x)$ we can choose $\overline{z} \in \mathscr{F}_{y_n} \cap W^u(x)$. By the choice of n with $z = y_n$ we get $d(F^n(y_n), F^n(\overline{z})) < \frac{\varepsilon}{2}$ and so $d(y, F^n(\overline{z})) < \varepsilon$ by the triangular inequality. This allows us to fix a second curve $c_{\overline{z}} \subset W^u(x)$ through $F^n(\overline{z})$ such that $d(y,k) < \varepsilon$ for all $k \in c_{\overline{z}}$. Applying again the claim to $c_{\overline{z}}$ there is $k \in c_{\overline{z}} \cap W^s(x)$. Thus, $k \in W^u(x) \cap W^s(x)$ and $d(y,k) < \varepsilon$ because $k \in c_{\overline{z}} \subset W^u(x)$. Since the last intersection is transversal and ε is arbitrary we conclude that y belongs to the homoclinic class of x. Since $y \in \Lambda(F)$ is arbitrary we get that $\Lambda(F)$ is the homoclinic class of x. The result is proved. □

1.9 Singular partition

In this section we introduce the concept of singular partitions which will be used later on. Afterwards we give properties and conditions for existence.

Let M be a compact manifold and $X \in \mathscr{X}^1(M)$. A *cross section* of X is a codimension one submanifold Σ transverse to X. The interior and the boundary of Σ as a submanifold are denoted by $Int(\Sigma)$ and $\partial\Sigma$ respectively. If $\mathscr{R} = \{S_1, \cdots, S_k\}$ is a collection of cross sections we still denote by $\mathscr{R}$ the union of its elements. We denote

$$\partial\mathscr{R} = \bigcup_{i=1}^{k} \partial S_i \quad \text{and} \quad Int(\mathscr{R}) = \bigcup_{i=1}^{k} Int(S_i).$$

The *size* of $\mathscr{R}$ will be the sum of the diameters of its elements.

Definition 1.49. *A* singular-partition *of an invariant set H of a vector field X is a finite disjoint collection $\mathscr{R}$ of cross sections of X such that $H \cap \partial \mathscr{R} = \emptyset$ and $H \cap Sing(X) = \{y \in H : X_t(y) \notin \mathscr{R}, \forall t \in \mathbb{R}\}$.*

This concept generalizes the notion of global cross section [48] to include invariant sets with singularities.

In the sequel we present existence results for singular partitions. We start with the following general proposition.

Proposition 1.50. *Let Λ be a compact invariant set all of whose singularities are hyperbolic of a C^1 vector field X. If for all $\delta > 0$ and all $z \in \Lambda \setminus Sin(X)$ there is a cross section Σ_z of diameter at most δ such that $z \in Int(\Sigma_z)$ and $\Lambda \cap \partial \Sigma_z = \emptyset$, then Λ has singular partitions of arbitrarily small size.*

Proof. Since every singularity in Λ is hyperbolic we can choose $\beta > 0$ such that

$$\Lambda \cap Sing(X) = \bigcap_{t \in \mathbb{R}} X_t \left(\bigcup_{\sigma \in \Lambda \cap Sing(X)} B_\beta(\sigma) \right). \tag{1.5}$$

For such a β we define

$$H = \Lambda \setminus \left(\bigcup_{\sigma \in \Lambda \cap Sing(X)} B_\beta(\sigma) \right).$$

We can assume that $H \neq \emptyset$ for, otherwise, (1.5) would imply $\Lambda = \Lambda \cap Sing(X)$ in whose case we are done. Clearly $H \subset \Lambda$ and $H \cap Sing(X) = \emptyset$ so Σ_z as in the statement exists for all $z \in H$. For all such z we define

$$V_z = \bigcup_{t \in (-1,1)} X_t(Int(\Sigma_z)).$$

Obviously $z \in V_z$ and then $\{V_z : z \in H\}$ is an open covering of H which is clearly compact. So, there is a finite subset $\{z_1, \cdots, z_r\} \in H$ such that

$$H \subset \bigcup_{i=1} V_{z_i}.$$

By moving the cross sections $\Sigma_{z_1}, \cdots, \Sigma_{z_r}$ along the flow as in [33] p.189 (say) we can assume that the collection

$$\mathscr{R} = \{\Sigma_{z_1}, \cdots, \Sigma_{z_r}\}$$

is pairwise disjoint. Moreover, since $\Lambda \cap \partial \Sigma_z = \emptyset$ we have

$$\Lambda \cap \partial \mathscr{R} = \emptyset.$$

If $z \in \Lambda \setminus Sing(X)$, then (1.5) implies that there is $t \in \mathbb{R}$ such that

$$X_t(z) \notin \bigcup_{\sigma \in \Lambda \cap Sing(X)} B_\beta(\sigma).$$

But $X_t(z) \in \Lambda$ since z does therefore $X_t(x) \in H$ by definition. Hence $X_t(z) \in V_{z_i}$ for some i and then the orbit of z intersects Σ_{z_i} by the definition of V_{z_i}. This proves that

$$\Lambda \cap Sing(X) = \{z \in \Lambda : X_t(z) \notin \mathscr{R}\}$$

hence the result follows. □

1.9.1 Properties

We present some topological properties of the singular partitions. The first one is a direct consequence of the definition (c.f. [109]).

The *return map* $\Pi_\Sigma : Dom(\Pi_\Sigma) \subset \Sigma \to \Sigma$ associated to a cross section Σ is defined by

$$Dom(\Pi_\Sigma) = \{x \in \Sigma : X_t(x) \in \Sigma \text{ for some } t > 0\}$$

and

$$\Pi_\Sigma(x) = X_{t_\Sigma(x)}(x)$$

where $t_\Sigma(x)$ is the return time

$$t_\Sigma(x) = \inf\{t > 0 : X_t(x) \in \Sigma\}.$$

For all compact invariant set Λ we define

$$W^s(\Lambda) = \{x \in M : \omega(x) \subset \Lambda\} \quad \text{and} \quad W^u(\Lambda) = \{x \in M : \alpha(x) \subset \Lambda\} \tag{1.6}$$

For all $H \subset M$ we denote

$$W^s(Sing(X) \cap H) = \bigcup_{\sigma \in Sing(X) \cap H} W^s(\sigma).$$

Lemma 1.19. *If $\mathscr{R}$ is a singular-partition of a compact invariant set H of X, then the following properties hold:*

1. *$(H \cap \mathscr{R}) \cap Dom(\Pi_{\mathscr{R}}) \subset Int(Dom(\Pi_{\mathscr{R}}))$ and $\Pi_{\mathscr{R}}$ is C^1 in a neighborhood of $H \cap \mathscr{R}$ in $\mathscr{R}$.*
2. *$(H \cap \mathscr{R}) \setminus Dom(\Pi_{\mathscr{R}}) \subset W^s(Sing(X) \cap H)$.*

Hereafter $B_\delta(p)$ denotes the open δ-ball in $\mathscr{R}$ centered at $p \in \mathscr{R}$. Recall that $O^+(q) = \{X_t(q) : t \geq 0\}$ denotes the positive orbit of $q \in M$.

Lemma 1.20. *Let M be a compact 3-manifold, $X \in \mathscr{X}^1(M)$ and $q \in M$ be such that every singularity in $\omega(q)$ is hyperbolic with one-dimensional unstable manifold. If $\omega(q)$ is not a singularity and $\mathscr{R}$ is a singular-partition of $\omega(q)$, then the following properties hold for $\Pi = \Pi_{\mathscr{R}}$:*

1. *$O^+(q) \cap \mathscr{R} = \{q_1, q_2, \cdots\}$ is an infinite sequence ordered in a way that $\Pi(q_n) = q_{n+1}$.*
2. *There is $\delta > 0$ such that if $n \in \{1, 2, \cdots\}$ then either $B_\delta(q_n) \subset Dom(\Pi)$ and $\Pi/B_\delta(q_n)$ is C^1 or there is a curve $c_n \subset W^s(Sing(X) \cap \omega(q)) \cap B_\delta(q_n)$ such that*

$$B^+_\delta(q_n) \subset Dom(\Pi) \quad \text{and} \quad \Pi/B^+_\delta(q_n) \text{ is } C^1,$$

where $B^+_\delta(q_n)$ denotes the connected component of $B_\delta(q_n) \setminus c_n$ containing q_n.

Proof. To prove Item (1) notice that $\omega(q)$ contains regular orbits as it is not a singularity. Hence $\omega(q) \cap \mathscr{R} \neq \emptyset$ because $\mathscr{R}$ is a singular-partition of $\omega(q)$. Since each component of $\mathscr{R}$ is a cross section of X we have that $O^+(q) \cap \mathscr{R} = \{q_1, q_2, \cdots\}$ is a sequence whose accumulation points belong to $\omega(q) \cap \mathscr{R}$. The sequence must be infinite for otherwise $\omega(q) \cap \mathscr{R} = \emptyset$ a contradiction. Thus $q_n \in Dom(\Pi)$ $(\forall n)$ and clearly we can order the sequence in a way that $\Pi(q_n) = q_{n+1}$ $(\forall n)$. This proves Item (1) of the lemma.

Now we prove Item (2). To simplify the notation we write

$$H = \omega(q) \quad \text{and} \quad H^0 = H \cap \mathscr{R}.$$

Then, $H^0 \neq \emptyset$. By Lemma 1.19 one has

(i) $H^0 \cap Dom(\Pi) \subset Int(Dom(\Pi))$ and Π is C^1 in a neighborhood of H^0 in $\mathscr{R}$.
(ii) $H^0 \setminus Dom(\Pi) \subset W^s(Sing(X) \cap H)$.

On the other hand, every singularity in $\omega(q)$ is hyperbolic with one-dimensional unstable manifold by hypothesis. It follows that the stable manifold of every $\sigma \in Sing(X) \cap H$ is two dimensional.

Now, we fix $x \in H^0 \setminus Dom(\Pi)$ then $x \in \mathscr{R} \cap W^s(Sing(X) \cap H)$ by (ii). As $\mathscr{R}$ and the stable manifolds of the singularities in $Sing(X) \cap H$ are two dimensional we have that x lies in a curve

$$c_x \subset \mathscr{R} \cap W^s(\sigma_x)$$

for some $\sigma_x \in Sing(X) \cap H$. By hypothesis we have that $W^u(\sigma_x)$ is one-dimensional, so $W^u(\sigma_x) \setminus \{\sigma_x\}$ consists of two connected components to be denote by W^+ and W^-. We have three possibilities for these components:

- $W^+ \subset H$ and $W^- \subset H$,
- $W^+ \subset H$ and $W^- \not\subset H$,
- $W^- \subset H$ and $W^+ \not\subset H$.

First suppose that $W^+ \subset H$ and $W^- \subset H$. It follows that $W^+ \cap Int(\mathscr{R}) \neq \emptyset$ and $W^- \cap Int(\mathscr{R}) \neq \emptyset$ since W^-, W^+ are regular orbits of H and $\mathscr{R}$ is a singular partition of $\omega(q) = H$. By using such non-empty intersections we can find $\delta_x > 0$ such that

(iii) $B_{\delta_x}(x) \setminus c_x \subset Dom(\Pi)$ and $\Pi/(B_{\delta_x}(x) \setminus c_x)$ is C^1.

Second suppose that $W^+ \subset H$ and $W^- \not\subset H$. As $W^+ \subset H$ and $\mathscr{R}$ is a singular partition of H we have

(A) $W^+ \cap Int(\mathscr{R}) \neq \emptyset$.

As $W^- \not\subset H$ we have

(B) $O^+(q)$ does not accumulate on W^-.

By using (A) and (B) we can find $\delta_x > 0$ such that the connected components

$$B^+_{\delta_x}(x) \quad \text{and} \quad B^-_{\delta_x}(x)$$

of $B_{\delta_x}(x) \setminus c_x$ are labeled in a way that

(iv) $B^+_{\delta_x}(x) \subset Dom(\Pi)$, $\Pi/_{B^+_{\delta_x}(x)}$ is C^1 and $B^-_{\delta_x}(x) \cap O^+(q) = \emptyset$.

Third suppose that $W^- \subset H$ and $W^+ \not\subset H$. In this case we can proceed as in the second case to find $\delta_x > 0$ satisfying (iv).

Summarizing, for all $x \in H^0 \setminus Dom(\Pi)$ we have found $\delta_x > 0$ satisfying either (iii) or (iv).

On the other hand, (i) implies that $H^0 \setminus Dom(\Pi)$ is compact. Hence there are $x_1, \cdots, x_l \in H^0 \setminus Dom(\Pi)$ such that

$$H^0 \setminus Dom(\Pi) \subset \bigcup_{i=1}^{l} B_{\delta_{x_i}/2}(x_i). \tag{1.7}$$

Because the union in the right-hand side of (1.7) is open one has that

$$H^1 = H^0 \setminus \bigcup_{i=1}^{l} B_{\delta_{x_i}}(x_i)$$

is compact. By (1.7) one has

$$H^1 \subset H^0 \cap Dom(\Pi).$$

By (i) we have that $\forall y \in H^1\ \exists \beta_y > 0$ such that

$$B_{\beta_y}(y) \subset Dom(\Pi) \quad \text{and} \quad \Pi/_{B_{\beta_y}(y)} \text{ is } C^1. \tag{1.8}$$

It follows from the compactness of H^1 that $\exists y_1, \cdots, y_r$ (for some $r > 0$) such that

$$H^1 \subset \bigcup_{j=1}^{r} B_{\beta_{y_j}/2}(y_j). \tag{1.9}$$

Define

$$\delta = \min\{\delta_{x_i}/8, \beta_{y_j}/8 : 1 \le i \le l, 1 \le j \le r\}.$$

Let us prove that this δ works.

By (1.7) and (1.9) we have that

$$\{B_{\delta_{x_i}}(x_i), B_{\beta_{y_j}}(y_j) : 1 \le i \le l, 1 \le j \le r\}$$

is an open covering of $H^0 = \omega(q) \cap \mathscr{R}$. Then

$$q_n \in \left(\bigcup_{i=1}^{l} B_{\delta_{x_i}/2}(x_i) \right) \cup \left(\bigcup_{j=1}^{r} B_{\beta_{y_i}/2}(y_i) \right)$$

for n large enough. Hence for all n large we have either

$$q_n \in B_{\delta_{x_i}/2}(x_i) \qquad \text{for some } 1 \le i \le l,$$

or

$$q_n \in B_{\beta_j/2}(y_j) \qquad \text{for some } 1 \le j \le r.$$

Then, by the triangle inequality and the choice of δ we obtain

$$B_\delta(q_n) \subset B_{\delta_{x_i}}(x_i) \quad \text{or} \quad B_\delta(q_n) \subset B_{\beta_j}(y_j).$$

If $B_\delta(q_n) \subset B_{\beta_{y_j}}(y_j)$, then $B_\delta(q_n) \subset Dom(\Pi)$ and $\Pi/_{B_\delta(q_n)}$ is C^1 by (1.8). In this case we are done.

If $B_\delta(q_n) \subset B_{\delta_{x_i}}(x_i)$ we define

$$c_n = c_{x_i} \cap B_\delta(q_n).$$

In this case we have two subcases, namely either (iii) or (iv) hold.

First assume that (iii) holds. Recalling that $B_\delta^+(q_n)$ is the connected component of $B_\delta(q_n) \setminus c_n$ containing q_n we have $B_\delta^+(q_n) \subset B_{\delta_{x_i}}(x_i) \setminus c_{x_i}$ therefore $B_\delta^+(q_n) \subset Dom(\Pi)$ and $\Pi/_{B_\delta^+(q_n)}$ is C^1 by (iii).

Finally, if (iv) holds then $B_\delta^+(q_n) \subset B_{\delta_{x_i}}^+(x_i)$ since $q_n \in O^+(q)$ and $B_{\delta_{x_i}}^-(x_i) \cap O^+(q) = \emptyset$. Then the result follows from (iv). The lemma is proved. □

Chapter 2
Hyperbolic and sectional-hyperbolic sets: definition and properties

In this chapter we define the hyperbolic and sectional-hyperbolic sets. In the sequel we study their basic properties.

2.1 Definition

Consider a Riemannian manifold M and $X \in \mathscr{X}^1(M)$ We say that $\Lambda \subset M$ is an *invariant set* if $\Lambda \subset M(X)$ and $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We say that $p \in M$ is a *nonwandering point* of X if for all neighborhood U of p and all $T > 0$ there is $t \geq T$ such that $X_t(U) \cap U \neq \emptyset$. Denote by $\Omega(X)$ the set of nonwandering points of X which is clearly a closed invariant set.

Given $p \in M$ we define its orbit $O(p) = \{X_t(p) : X_t(p) \text{ is defined}\}$. An orbit of X is a set equals to $O(p)$ for some p. The *positive orbit* of p is defined by $O^+(p) = \{X_t(p) : t > 0\}$ and if $p \in M(X)$ we define its *negative orbit* $O^-(p) = \{X_t(p) : t < 0\}$.

A *periodic orbit* of X is the orbit of some p for which there is a minimal $t > 0$ (called the period) such that $X_t(p) = p$. In such a case we say that p is a *periodic point*. A *singularity* of X is a zero of X. We denote by $Per(X)$ the set of periodic points and by $Sing(X)$ the set of singularities of X. Clearly $Per(X) \cup Sing(X) \subset \Omega(X)$.

A subset $\Lambda \subset M$ is *singular* if it has a singularity; *non-trivial* if Λ is not a single orbit; *isolated* if there is a compact neighborhood U of Λ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U)$$

(U is called *isolating block*); and *attracting* if it is isolated and has a positively invariant isolating block U, i.e., $X_t(U) \subset U$, $\forall t \geq 0$.

Given $x \in M$ we define the ω*-limit set*

$$\omega(x) = \{y \in M : y = \lim_{n\to\infty} X_{t_n}(x) \text{ for some sequence } t_n \to \infty\}$$

and if $x \in M(X)$ the α-*limit set*

$$\alpha(x) = \{y \in M : y = \lim_{n\to\infty} X_{t_n}(x) \text{ for some sequence } t_n \to -\infty\}.$$

We say that x is *recurrent* if $x \in \omega(x)$. Singularities and periodic points are examples of recurrent points but not conversely.

A compact invariant set Λ is *transitive* or *has dense periodic orbits* if $\Lambda = \omega(x)$ for some $x \in \Lambda$ or $Cl(Per(X) \cap \Lambda) = \Lambda$. An *attractor* is a transitive attracting set. This is not the standard usage as for instance [71] calls attractor what we call attracting set. Several definitions of attractor are considered in [88].

A vector field X is *transitive* or *has dense periodic orbits* depending on whether its maximal invariant set is. Notice that, in the boundaryless case, X is transitive if and only if X has a dense orbit (thus recovering the standard definition of transitive vector field). Given $X \in \mathscr{X}^r(M)$ we say that X is C^r *robustly transitive* or C^r *robustly periodic* depending on whether every vector field that is C^r close to X is transitive or has dense periodic orbits.

Definition 2.1. *A compact invariant set H is* hyperbolic *if there are positive constants K, λ and a continuous invariant tangent bundle decomposition $T_HM = E^s_H \oplus E^X_H \oplus E^u_H$ such that*

- E^s_H *is* contracting, *i.e.*

$$|| DX_t(x)/E^s_x || \leq Ke^{-\lambda t}, \qquad \forall t > 0, \forall x \in H.$$

- E^u_H *is* expanding, *i.e.*

$$|| DX_{-t}(x)/E^u_x || \leq Ke^{-\lambda t}, \qquad \forall t > 0, \forall x \in H.$$

- E^X_H *is the subbundle tangent to X.*

A hyperbolic set H is saddle-type *if its contracting and expanding subbundles E^s_H, E^u_H never vanish, i.e., $E^s_x \neq 0$ and $E^u_x \neq 0$ for every $x \in H$.*

A singularity or periodic orbit of X is hyperbolic if it does as a compact invariant set. We say that $p \in Per(X)$ is hyperbolic if its orbit is.

The Invariant Manifold Theory [68] asserts that if H is a hyperbolic set of X then there are submanifolds $W^{ss}(p)$, $W^{uu}(p)$, $W^s(p)$, $W^u(p)$ of M tangent at p to the subspaces E^s_p, E^u_p, $E^s_p \oplus E^X_p$, $E^X_p \oplus E^u_p$ for all $p \in H$. We call $W^s(p)$ (resp. $W^u(p)$) the *weak stable* (resp. *unstable*) *manifold* of X at p. Analogously $W^{ss}(p)$ (resp. $W^{uu}(p)$)) is called the *strong stable* (resp. *unstable*) *manifold* of X at p. These foliations have the following dynamical characterization:

$$W^{ss}(p) = \{y \in M : \lim_{t\to\infty} d(X_t(p), X_t(y)) = 0\},$$

$$W^{uu}(p) = \{y \in M : \lim_{t \to -\infty} d(X_t(p), X_t(y)) = 0\},$$

$$W^s(p) = \bigcup_{t \in \mathbf{R}} W^{ss}(X_t(p)) \quad \text{and} \quad W^u(p) = \bigcup_{t \in \mathbf{R}} W^{uu}(X_t(p)).$$

A closed orbit O of X is *hyperbolic* if it is hyperbolic as a compact invariant set. If $O = O(p)$ is a hyperbolic *periodic* orbit of X we say that $q \in M$ is a *homoclinic point associated to* O if $q \in W^s(O) \cap W^u(O)$.

If in addition q is a transverse intersection point between these manifolds then we say that q is a *transverse* homoclinic point associated to O. We shall denote by $W^s(p) \pitchfork W^u(p)$ the set of transverse homoclinic points associated to p.

The *homoclinic class* $H(O)$ associated to O (or p) is the closure $H(p) = Cl(W^s(p) \pitchfork W^u(p))$. A compact invariant set is a homoclinic class if it is equals to $H(p)$ (or $H(O)$) for some hyperbolic periodic point P (or some hyperbolic periodic orbit O). It follows from the Birkhoff-Smale Theorem [62] that every homoclinic class is a transitive set with dense periodic orbits.

Denote by $m(A) = \inf_{v \neq 0} \frac{\|Av\|}{\|v\|}$ the minimum norm of a linear operator A.

Definition 2.2. *A continuous invariant splitting* $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ *over a compact invariant set* Λ *is* dominated *if* E_Λ *and* F_Λ *never vanish and there are positive constants* K, λ *such that*

$$\frac{\|DX_t(x)/_{E_x}\|}{m(DX_t(x)/_{F_x})} \leq Ke^{-\lambda t}, \qquad \forall t > 0, \forall x \in \Lambda.$$

A compact invariant set Λ *is* partially hyperbolic *if it exhibits a dominated splitting* $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ *such that* E^s_Λ is contracting, *i.e. there are* $K, \lambda > 0$ *such that*

$$\|DX_t(x)/_{E^s_x}\| \leq Ke^{-\lambda t}, \qquad \forall t > 0, \forall x \in \Lambda.$$

Denote by $Det(T)$ the Jacobian of a linear operator T. Now we state the definition of sectional-hyperbolic set [85].

Definition 2.3. *A compact invariant set* Λ *is a* sectional-hyperbolic set *if its singularities are hyperbolic and it is a partially hyperbolic set with* sectionally expanding central subbundle E^c_Λ. *More precisely,* $dim(E^c_x) \geq 2$ *and there are* $K, \lambda > 0$ *such that*

$$|Det(DX_t(x)/L^c_x)| \geq K^{-1} e^{\lambda t},$$

for all $t > 0, x \in \Lambda$ *and all two-dimensional subspace* L^c_x *of* E^c_x.

There is also a stable manifold theorem for sectional-hyperbolic set Λ. Indeed, denote by $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the sectional-hyperbolic splitting over Λ. It follows from

[68] that through each $p \in \Lambda$ passes an inmersed submanifold $W^{ss}(p)$, as smooth as X that is tangent to E^s_p at p, consisting of points x which are asymptotic to p in the sense that $\lim_{t\to\infty} d(X_t(x), X_t(p)) \to 0$ as $t \to \infty$.

2.2 Properties of hyperbolic sets

In this section we present some properties of the hyperbolic sets.

2.2.1 Shadowing lemma for flows

We start with the following basic property of hyperbolic sets known as *shadowing lemma for flows* [62]. Consider a real number $\rho > 0$. We say that a differentiable curve $c : \mathbb{R} \to M$ is either a ρ*-orbit* or ρ*-shadowed by the orbit of* x depending on whether $\|\dot{c}(t) - X(c(t)\| < \rho$ for all $t \in \mathbb{R}$ or there is a differentiable function $s : \mathbb{R} \to \mathbb{R}$ such that $|\dot{s}(t) - 1| < \rho$ and $d(c(s(t)), X_t(x)) < \rho$ for all $t \in \mathbb{R}$. By a *closed* ρ*-orbit* we mean a ρ-orbit which is also a closed curve.

Theorem 2.4 (Shadowing lemma for flows)**.** *For every saddle-type hyperbolic set without singularities of a C^1 vector field on a compact Riemannian manifold there is a neighborhood U so that for every $\varepsilon > 0$ there is $\delta > 0$ such that every (resp. closed) δ-orbit in U can be ε-shadowed by an unique (resp. periodic) orbit.*

Proof. We only give an outline of the proof (see [62] for details).

Fix a saddle-type hyperbolic set without singularities Λ of a C^1 vector field X on a compact manifold M. Let U^0 be an open neighborhood of Λ where the hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ extends to a continuous semi-invariant splitting $T_{U^0} M = E^s_{U^0} \oplus E^X_{U^0} \oplus E^u_{U^0}$ in a way that $E^s_{U^0}$ and $E^u_{U^0}$ are still contracting and expanding respectively. Denote $s = dim(E^s_{U^0})$, $u = dim(E^u_{U^0})$ and D^k the unit disk in $\mathbb{R}^k$.

A *rectangle* will be a cross section $S \subset U^0$ diffeomorphic to $D^s \times D^u$ with D^s and D^u being parallel to $E^s_{U^0}$ and $E^u_{U^0}$ respectively. We say that a rectangle is around p if its interior (as a submanifold) contains p. By a *horizontal* (resp. *vertical*) *submanifold* of S we mean the graph of a C^1 map $\varphi^s : D^s \times 0 \to 0 \times D^u$ (resp. $\varphi^u : 0 \times D^u \to D^s \times 0$).

Since $E^s_{U^0}$ and $E^u_{U^0}$ are contracted and expanded by X respectively we can fix a neighborhood $U \subset Cl(U) \subset U^0$ of Λ such that if $\varepsilon > 0$ is small, then through each $p \in U$ passes a rectangle $S_p \subset U^0$ of diameter ε with the following property: There are $t_\varepsilon > 0$ (usually large) and $a > 0$ (usually small) such that if $p, q \in U$ and

$d(X_t(p),q) < a$ with $t \geq t_\varepsilon$, then there are a return map $\Pi_p : S_p \to S_q$ and a horizontal rectangle H_p around p in S_p whose image $\Pi_p(R) = V_p$ is a vertical rectangle around $\Pi_p(p)$ in S_q (see Figure 2.1).

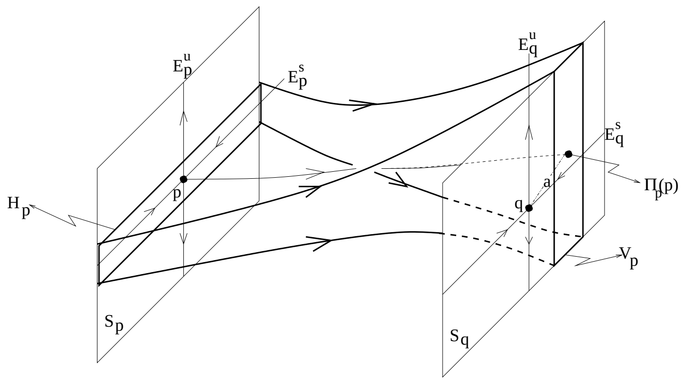


Fig. 2.1

Now fix $\varepsilon > 0$ (which can be assumed small), $t_\varepsilon > 0$ and $a > 0$ as above. Without loss of generality we can assume that $t_\varepsilon = 1$. Take $\delta = a$ and consider a δ-orbit $c(t)$ in U. By integrating $\|\dot{c}(t) - X(c(t))\|$ we have that $d(X_1(c(n)), c(n+1)) < \delta = a$ for all n. So, the vertical rectangles $V_n = V_{X_1(c(n-1))} \subset S_{c(n)}$ are well defined for all $n \geq 0$. Denote $\Pi_n = \Pi_{c(n)}$, $S_n = S_{c(n)}$ and $H_n = H_{c(n)}$ (see Figure 2.2).

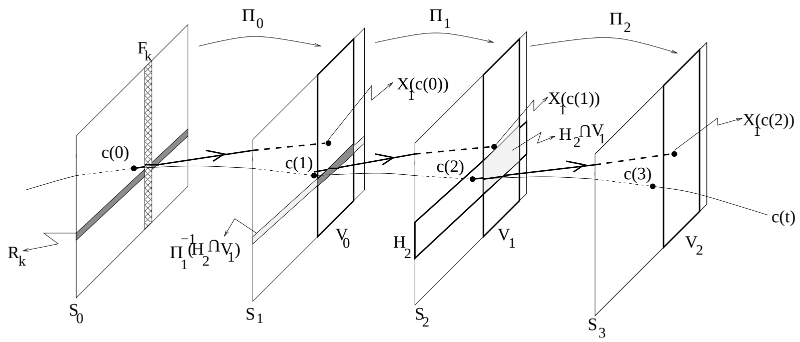


Fig. 2.2

We define

$$R_2 = \Pi_0^{-1}(\Pi_1^{-1}(H_2 \cap V_1) \cap V_0), \quad R_3 = \Pi_0^{-1}(\Pi_1^{-1}(\Pi_2^{-1}(H_3 \cap V_2) \cap V_1) \cap V_0),$$

$$R_4 = \Pi_0^{-1}(\Pi_1^{-1}(\Pi_2^{-1}(\Pi_3^{-1}(H_4 \cap V_3) \cap V_2) \cap V_1) \cap V_0),$$

and in general

$$R_k = \Pi_0^{-1}\left(\Pi_1^{-1}\left(\cdots\left(\Pi_{k-1}^{-1}(H_k \cap V_{k-1}) \cap V_{k-2}\right) \cap V_{k-3} \cap \cdots \cap V_1\right) \cap V_0\right).$$

(See Figure 2.2.) Making use of graph transformed techniques (e.g. [68]) we can see that the limit $R_\infty = \lim_{k\to\infty} R_k$ is a horizontal submanifold of S_0. Analogously but with negative values of k we construct a sequence F_k as in Figure 2.2 whose limit $F_\infty = \lim_{k\to-\infty} F_k$ is a vertical submanifold of $S_{c(0)}$. The intersection $R_\infty \cap F_\infty$ between these submanifolds is a unique point x whose orbit is the unique one which ε-shadows $c(t)$ (the reparametrization $s(t)$ appears since the flight time from x to S_n is not necessarily n). It follows from this unicity that if $c(t)$ is closed, then so does the orbit of x. □

2.2.2 Axiom A and spectral decomposition

A vector field X is *Axiom A* if its nonwandering set $\Omega(X)$ is a hyperbolic set with dense closed orbits.

The following is the so-called *Spectral Decomposition Theorem* due to S. Smale.

Theorem 2.5. *If X is an Axiom A vector field on a* compact *manifold, then there is a disjoint collection $H_1, \cdots, H_k$ of transitive isolated hyperbolic sets of X such that*

$$\Omega(X) = H_1 \cup \cdots \cup H_k.$$

Moreover, each H_i is either a singularity or a homoclinic class of X.

Proof. We shall give an outline of the proof leading the details to [123]. First one observes that the homoclinic classes of X are all transitive sets. Consequently all these classes are contained in $\Omega(X)$.

We claim that the collection of all homoclinic classes of X are pairwise disjoint. Indeed, suppose that $H_1 = H(O_1)$ and $H_2 = H(O_2)$ are homoclinic classes associated to some hyperbolic periodic orbits O_1, O_2 of X. If $H_1 \cap H_2 \neq \emptyset$, then we obtain transverse intersection points between $W^s(O_1)$ and $W^u(O_2)$ because these manifolds belong to the *same* hyperbolic set. Analoguously for $W^u(O_1)$ and $W^u(O_2)$. It follows from the Inclination Lemma [123] that the homoclinic classes H_1 and H_2 coincide. This proves the claim.

Next we observe that the collection of homoclinic classes of X is a finite set. The proof (by contradiction) is similar to the previous one by just taking an accumulation point of an possible infinite sequence of homoclinic classes.

The previous claims implies that the collection of all homoclinic classes of X is formed by a disjoint finite collection $H_1, \cdots, H_l$. Let $\sigma_1, \cdots, \sigma_r$ be the singularities of X. Define $k = l + r$, $H_{l+1}\{\sigma_1\}, \cdots H_k = \{\sigma_{k-r}\}$. Since the closed orbits are dense in $\Omega(X)$ we obtain the disjoint union $\Omega(X) = H_1 \cup \cdots \cup H_k$.

Finally one shows that each H_i is an isolated set of X. Fix H_i. since the collection is disjoint we can choose a compact neighborhood U_i of H_i such that $\Omega(X) \cap U_i = H_i$. Clearly

$$H_i \subset \bigcap_{t \in \mathbb{R}} X_t(U_i)$$

because H_i is an invariant set contained in U_i. On the other hand, if $x \in M$ is a point whose orbit is enterely contained in U_i. It follows that $\alpha(x)$ and $\omega(x)$ are both contained in U_i. Since both $\alpha(x)$ and $\omega(x)$ are also contained in $\Omega(X)$ we obtain

$$\alpha(x) \cup \omega(x) \subset \Omega(X) \cap U_i = H_i.$$

This implies that $x \in \Omega(X)$ and, since $x \in U_i$, we conclude that $x \in H_i$. This proves the reversed inclusion

$$\bigcap_{t \in \mathbb{R}} X_t(U_i) \subset H_i$$

from which the result follows. □

Let us state some consequences of the Spectral Theorem.

Corollary 2.6. *Let X be an Axiom A vector field on a compact manifold M.*

1. *X has a dense orbit if and only if $\Omega(X) = M$.*
2. *X has an attractor and a repeller.*

Proof. Clearly $\Omega(X) = M$ if X has a dense orbit (this implication does't require X to be Axiom A). Conversely if $\Omega(X) = M$ then M must be one of the sets H_i's in the Spectral Theorem since these sets are disjoint. It follows that X has a dense orbit and the proof follows.

Now we prove that X has an an attractor. For all compact invariant set Λ of X we define the stable set

$$W^s(\Lambda) = \{x \in M : \omega(x) \subset \Lambda\}.$$

With this definition in mind one sees that

$$M = \bigcup_{i=1}^{k} W^s(H_i).$$

Since the H_i's are disjoint one has that the above union is disjoint. Henceforth there is some $\Lambda = H_{i_0}$ whose stable set $W^s(H_i)$ has non-empty interior. Let us prove that Λ is an attractor of X.

Indeed, we can assume that Λ is a homoclinic class for otherwise it is a singularity of X which must be attracting and we are done. In such a case $\Lambda = H(O)$ for all periodic orbit $O \subset \Lambda$. Pick an open set $U \subset W^s(\Lambda)$. By the denseness of the periodic orbits in Λ and the continuity of the stable manifolds of Λ we can assume that $W^s(O_0) \cap U \neq \emptyset$ for some periodic orbit $O_0 \subset \Lambda$. Pick $q \in W^u(O_0)$ and let V be a fixed (but arbitrary) neighborhood of q in M. By the Inclination Lemma we can assume that $U \cap V \neq \emptyset$ and so $W^s(\Lambda) \cap V \neq \emptyset$. This implies $W^s(O_0) \cap V \neq \emptyset$ since $\Lambda = H(O_0)$. On the other hand, $W^u(O_0) \cap V \neq \emptyset$ since $q \in W^u(O_0)$ and V is an open neighborhood of q in M. Summarizing $W^s(O_0) \cap V \neq \emptyset$ and $W^u(O_0) \cap V \neq \emptyset$ for all neighborhood V of q in M. Hence $q \in \Omega(X)$ by the Inclination Lemma. Consequently $W^u(O_o) \subset \Omega(X)$ and so $W^u(O_0) \subset \Lambda$. The last inclusion implies that Λ is an attracting set. Since Λ is transitive we obtain that Λ is an attractor. The same argument applied to $-X$ implies that X has a repeller. $\square$

2.3 Properties of sectional-hyperbolic sets

Now we present some properties of the sectional-hyperbolic sets.

2.3.1 The splitting and the hyperbolic lemma

First we examine the sectional-hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of a sectional-hyperbolic set Λ of $X \in \mathscr{X}^1(M)$. Hereafter M will denote a compact manifold.

Lemma 2.1. *If $x \in \Lambda \setminus Sing(X)$, then $X(x) \notin E^s_x$.*

Proof. Suppose by contradiction that there is $x_0 \in \Lambda \setminus Sing(X)$ such that $X(x_0) \in E^s_{x_0}$. Then, $X(x) \in E^s_x$ for every x in the orbit of x_0 since E^s_Λ is invariant. So $X(x) \in E^s_x$ for every $x \in \alpha(x_0)$ by continuity. It follows that $\omega(x)$ is a singularity for all $x \in \alpha(x_0)$. In particular, $\alpha(x_0)$ contains a singularity σ which is necessary saddle-type. Now we have two cases: $\alpha(x_0) = \{\sigma\}$ or not. If $\alpha(x_0) = \{\sigma\}$ then $x_0 \in W^u(\sigma)$. For all $t \in \mathbb{R}$ define the unitary vector

$$v^t = \frac{DX_t(x_0)(X(x_0))}{|| DX_t(x_0)(X(x_0)) ||}.$$

It follows that

$$v^t \in T_{X_t(x_0)} W^u(\sigma) \cap E^s_{X_t(x_0)}, \quad \forall t \in \mathbb{R}.$$

Take a sequence $t_n \to \infty$ such that the sequence v^{-t_n} converges to v^∞ (say). Clearly v^∞ is an unitary vector and, since $X_{-t_n}(x_0) \to \sigma$ and E^s is continuous we obtain

$$v^\infty \in T_\sigma W^u(\sigma) \cap E^s_\sigma.$$

Therefore v^∞ is an unitary vector which is simultaneously expanded and contracted by $DX_t(\sigma)$ a contradiction. This contradiction shows the result when $\alpha(x_0)=\{\sigma\}$. If $\alpha(x_0)\neq\{\sigma\}$ then $(W^u(\sigma)\setminus\{\sigma\})\cap\alpha(x_0)\neq\emptyset$. Pick $x_1\in(W^u(\sigma)\setminus\{\sigma\})\cap\alpha(x_0)$. Clearly $X(x_1)\in E^s_{x_1}$ and then we get a contradiction as in the first case replacing x_0 by x_1. This contradiction proves the lemma in the second case. ☐

From this we have the following fundamental result.

Corollary 2.7. *If* $\sigma\in\Lambda\cap Sing(X)$*, then* $\Lambda\cap W^{ss}(\sigma)=\{\sigma\}$.

Proof. Notice that $E^s_x=T_xW^{ss}(\sigma)$ for all $x\in W^{ss}(\sigma)$. Moreover, $W^{ss}(\sigma)$ is an invariant manifold so $X(x)\in T_xW^{ss}(\sigma)$ for all $x\in W^{ss}(\sigma)$. We conclude that $X(x)\in E^s_x$ for all $x\in W^{ss}(\sigma)$ and now Lemma 2.1 applies. ☐

Lemma 2.2. *If* $x\in\Lambda$*, then* $X(x)\in E^c_x$.

Proof. The result is trivial if $x\in Sing(X)$ so we can assume that $x\in\Lambda\setminus Sing(X)$.

First suppose that x does not belong to $W^s(\sigma)$ for all $\sigma\in Sing(X)$. Then, $\alpha(x)$ has a regular point y. Take a sequence $t_n\to\infty$ such that $X_{-t_n}(x)\to y$. By Lemma 2.1 we have $X(y)\notin E^s_y$ so the angle between $X(y)$ and E^s_y is positive. On the other hand, $X(X_{-t_n}(x))\to X(y)$ and $E^s_{X_{-t_n}(x)}\to E^s_y$ by continuity so the angle between $X(X_{-t_n}(x))$ and $E^s_{X_{-t_n}(x)}$ is bounded away from 0 for n large. From this and the fact that E^s_Λ dominates E^c_Λ we conclude that the angle between $DX_{t_n}(X_{-t_n}(x))(X(X_{-t_n}(x)))$ and $DX_{t_n}(E^c_{X_{-t_n}(x)})$ converges to 0 as $n\to\infty$. But $DX_{t_n}(X_{-t_n}(x))(X(X_{-t_n}(x)))=X(x)$ and $DX_{t_n}(E^c_{X_{-t_n}(x)})=E^c_x$ therefore $X(x)\in E^c_x$ as desired.

Now assume that $x\in W^u(\sigma)$ for some $\sigma\in Sing(X)$. Clearly $T_\sigma W^u(\sigma)\cap E^s_\sigma=\{0\}$ hence $T_\sigma W^u(\sigma)\subset E^c_\sigma$ by dominance. So $T_xW^u(\sigma)\subset E^c_x$ and then $X(x)\in E^c_x$ since $X(x)\in T_xW^u(\sigma)$. This proves the lemma. ☐

Now we prepare three lemmas for the proof of the hyperbolic lemma.

Let M be a compact manifold and $X\in\mathscr{X}^1(M)$. Given $q\in M\setminus Sing(X)$ we define N_q as the orthogonal complement of E^X_q and denote by $O_q:T_qM\to N_q$ the orthogonal projection. The following linear algebra lemma will be useful.

Lemma 2.3. *If* $q\in M\setminus Sing(X)$ *and* L_q *is a subspace of* T_qM *such that* $E^X_q\subset L_q$*, then* $O_q(L_q)=N_q\cap L_q$.

Proof. Fix $v_q\in L_q$. Since $T_{M\setminus Sing(X)}M$ is a direct sum of $N_{M\setminus Sing(X)}$ and $E^X_{M\setminus Sing(X)}$ we have $v_q=v^N_q+v^X_q$ for some $v^N_q\in N_q$ and $v^X_q\in E^X_q$. On the other hand, since $E^X_q\subset L_q$ and $v^N_q=v_q-v^X_q$ we have $v^N_q\in L_q$ so $v^N_q\in L_q\cap N_q$. Therefore, $O_q(v_q)=v^N_q\in N_q\cap L_q$ proving $O_q(L_q)\subset N_q\cap L_q$. Conversely, if $v_q\in N_q\cap L_q$ then $O_q(v_q)=v_q$ so $v_q=O_q(v_q+X(q))$. Since $v_q\in L_q$ and $X(q)\in E^X_q\subset L_q$ we obtain $v_q\in O_q(L_q)$ therefore $N_q\cap L_q\subset O_q(L_q)$. This concludes the proof. ☐

If $v\in N_q$ we denote by $P_t(q)v$ the orthogonal projection of $DX_t(q)v$ over $N_{X_t(q)}$. In other words, $P_t(q)v=O_{X_t(q)}(DX_t(q)v)$. This defines a flow P_t in the fiber bundle $N\to M\setminus Sing(X)$ which is called the *linear Poincaré flow*.

Given a non-singular invariant set Λ we define the fiber bundle $N_\Lambda = \bigcup_{q\in\Lambda} N_q$. A subbundle G_Λ of N_Λ is called *invariant* if $P_t(q)G_q = G_{X_t(q)}$ for all $t \in \mathbb{R}$ and $q \in \Lambda$. Using Lemma 2.3 we obtain a simple criterium for the invariance of certain subbundles of N_Λ under the Poincaré flow P_t.

Lemma 2.4. *If L_Λ is an X_t-invariant subbundle of $T_\Lambda M$ containing E^X_Λ, then the induced subbundle $\overline{L}_\Lambda = N_\Lambda \cap L_\Lambda$ is P_t-invariant.*

Proof. Fix $q \in \Lambda$ and $t \in \mathbb{R}$. Since L_Λ is X_t-invariant we have from the hypothesis that $E^X_{X_t(q)} \subset L_{X_t(q)}$ so $O_{X_t(q)}(E^c_{X_t(q)}) = N_{X_t(q)} \cap L_{X_t(q)}$ by Lemma 2.3. Therefore,

$$
\begin{aligned}
P_t(\overline{L}_q) = & \ O_{X_t(q)}(DX_t(q)(N_q \cap L_q)) \\
= & \ O_{X_t(q)}(DX_t(q)(N_q) \cap L_{X_t(q)}) \\
= & \ P_t(q)(N_q) \cap O_{X_t(q)}(L_{X_t(q)}) \\
= & \ N_{X_t(q)} \cap (N_{X_t(q)} \cap L_{X_t(q)}) \\
= & \ N_{X_t(q)} \cap L_{X_t(q)} = \overline{L}_{X_t(q)}.
\end{aligned}
$$

□

We say that the Poincaré flow P_t *is hyperbolic over* Λ if there are a continuous splitting $N_\Lambda = G_\Lambda \oplus F_\Lambda$ and positive constants k, λ such that

- $N_\Lambda = G_\Lambda \oplus F_\Lambda$ *is P_t-invariant*, i.e., both G_Λ and F_Λ are P_t-invariant.
- P_t *contracts* G_Λ, i.e., $\|P_t(q)/G_q\| \le ke^{-\lambda t}$ for all $t \ge 0$ and $q \in \Lambda$.
- P_t *expands* F_Λ, i.e., $m(P_t(q)/F_q) \ge ke^{\lambda t}$ for all $t \ge 0$ and $q \in \Lambda$.

The following well-known lemma asserts the equivalence between the hyperbolicity of Λ as a non-singular compact invariant set of X and the hyperbolicity of the Poincaré flow P_t over Λ.

Lemma 2.5. *A necessary and sufficient condition for a non-singular compact invariant set to be hyperbolic is that the Poincaré flow be hyperbolic over it.*

Proof. To prove the necessity assume that Λ is a non-singular hyperbolic set of X with hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$. Defining $G_\Lambda = \overline{(E^s \oplus E^X)}_\Lambda$ and $F_\Lambda = \overline{(E^X \oplus E^u)}_\Lambda$ we obtain a continuous splitting $N_\Lambda = G_\Lambda \oplus F_\Lambda$ which is P_t-invariant by Lemma 2.4. Since Λ has no singularities we can easily see that G_Λ (resp. F_Λ) is contracted (resp. expanded) by P_t thus P_t is hyperbolic over Λ.

For the sufficiency, assume that P_t is hyperbolic over Λ and denote by $N_\Lambda = G_\Lambda \oplus F_\Lambda$ the corresponding splitting. Define the subbundles $A_\Lambda = G_\Lambda \oplus E^X_\Lambda$ and $B_\Lambda = E^X_\Lambda \oplus F_\Lambda$ over Λ. Let us see that both A_Λ and B_Λ are X_t-invariant. Indeed, if $x \in \Lambda$ and $v_x \in A_x$ then there is a unique $v^G_x \in G_x$ such that $v_x - v^G_x \in E^X_x$. If $t \in \mathbb{R}$ then $DX_t(x)v_x - DX_t(x)v^G_x \in E^X_{X_t(x)}$ since E^X_Λ is X_t-invariant so $O_{X_t(x)}(DX_t(x)v_x) = P_t(x)v^G_x$ from which we get $O_{X_t(x)}(DX_t(x)v_x) \in G_{X_t(x)}$ since G_Λ is P_t-invariant. As $DX_t(x)v_x - O_{X_t(x)}(DX_t(x)v_x) \in E_{X_t(x)}$ by the definition of $O_{X_t(x)}$ we obtain $DX_t(x)v_x \in G_{X_t(x)} \oplus E_{X_t(x)} = A_{X_t(x)}$ therefore A_Λ is X_t-invariant. Analogously for B_Λ. Finally, since Λ is compact and non-singular we can use conefields around G_Λ

to obtain a contracting subbundle E^s_Λ in A_Λ complementary to E^X_Λ. Analogously we obtain an unstable subbundle E^u_Λ in B_Λ complementary to E^X_Λ. Since $N_\Lambda = G_\Lambda \oplus F_\Lambda$ we obtain $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ thus we obtain a hyperbolic splitting over Λ. □

Now, let Λ be a sectional-hyperbolic set of X and define $\Lambda^* = \Lambda \setminus Sing(X)$. Then, there is a natural subbundle $F_{\Lambda^*} = \overline{(E^c)}_{\Lambda^*}$, i.e., $F_{\Lambda^*} = N_{\Lambda^*} \cap E^c_{\Lambda^*}$ where E^c_Λ is the central subbundle of Λ. We observe that if $q \in \Lambda^*$ then $E^X_{X_t(q)} \subset E^c_{X_t(q)}$ for all $t \in \mathbb{R}$ by Lemma 2.2 thus F_{Λ^*} is P_t-invariant by Lemma 2.4. Another property of F_{Λ^*} is the following one.

Lemma 2.6. *There are constants $K, \lambda > 0$ such that*

$$m(P_t(x)/F_x) \cdot m(DX_t(x)/E^X_x) \geq Ke^{\lambda t} \tag{2.1}$$

for all $x \in \Lambda^*$ and $t > 0$.

Proof. By sectional-hyperbolicity there are $K, \lambda > 0$ such that

$$|Det(DX_t(x)/L^c_x)| \geq Ke^{\lambda t}, \tag{2.2}$$

for all $t \geq 0$, $x \in \Lambda$ and all two-dimensional subspace L^c_x of E^c_x. Fix $x \in \Lambda^*$ and given $v, u \in T_xM$ we let $A(u,v)$ be the area of the parallelogram formed by u, v in T_xM. If $u \in F_x$ and $v \in E^X_x$ we have

$$A(u,v) = ||v|| \cdot ||v|| \tag{2.3}$$

by orthogonality. From this we get

$$A(DX_t(x)u, DX_t(x)v) = ||P_t(x)u|| \cdot ||DX_t(x)v||, \tag{2.4}$$

by the definition of P_t. But $A(DX_t(x)u, DX_t(x)v) = |Det(DX_t(x)/L^c_x)| \cdot A(u,v)$ where L^c_x is the subspace generated by $\{u,v\}$. Since $u \in E^c_x$ by definition and $v \in E^c_x$ by Lemma 2.2 we have $L^c_x \subset E^c_x$. So, applying (2.2), (2.3) and (2.4) we get

$$||P_t(x)u|| \cdot ||DX_t(x)v|| \geq Ke^{\lambda t}||u|| \cdot ||v||.$$

Since u, v are arbitrary we get the result. □

Now we have the following key result.

Lemma 2.7 (Hyperbolic lemma). *Every compact invariant set without singularities of a sectional-hyperbolic set is hyperbolic.*

Proof. Let Λ be a sectional-hyperbolic set of a vector field X on a compact manifold M. Then, there is a constant $A > 0$ such that $||X(x)|| \leq A$ for all $x \in M$ Let $H \subset \Lambda$

be a compact invariant set without singularities of X. Then, there is $B > 0$ such that $||X(x)|| \geq B$ for all $x \in H$. Therefore,

$$m(DX_t(x)/E^X_x) = \left|\left|DX_t(x)\frac{X(x)}{||X(x)||}\right|\right| = \frac{||X(X_t(x))||}{||X(x)||} \leq \frac{A}{B}, \;\; \forall x \in H, \forall t > 0.$$

Now applying (2.1) we get

$$m(P_t(x)/F_x)\cdot\frac{A}{B} \geq m(P_t(x)/F_x)\cdot m(DX_t(x)/E^X_x) \geq Ke^{\lambda t}$$

and so

$$m(P_t(x)/F_x) \geq C\cdot e^{\lambda t}, \quad \forall x \in H, \forall t > 0,$$

where $C = K\left(\frac{B}{A}\right) > 0$. Therefore P_t expands F_H.

Define the splitting $N_H = G_H \oplus F_H$ where F_H is as above and $G_H = N_H \cap (E^s_H \oplus E^X_H)$. As already seem F_H is P_t-invariant while it is clear from the definition that $G_H = \overline{(E^s \oplus E^X)}_H$ it is P_t-invariant by Lemma 2.4 since $E^X_H \subset E^s_H \oplus E^X_H$. Clearly P_t contracts G_H so Lemma 2.5 applies. □

2.3.2 The singularities and strong stable manifolds

Next we describe the singularities of a sectional-hyperbolic set. For this we introduce the following definition. Denote by $Re(\cdot)$ the real part operation.

Definition 2.8. *We say that singularity is* Lorenz-like *if it has three eigenvalues $\lambda^{ss}, \lambda^s, \lambda^u$ with λ^s, λ^u real and $Re(\lambda^{ss}) < \lambda^s < 0 < -\lambda^s < \lambda^u$ such that the real part of the remainder eigenvalues is outside $[\lambda^s, \lambda^u]$.*

In the present case of 3-manifolds M this definition reduces to say that the singularity has real eigenvalues which, up to some order $\lambda_1, \lambda_2, \lambda_3$, satisfy the eigenvalue relation $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

Hereafter Λ will denote a sectional-hyperbolic set of $X \in \mathscr{X}^1(M)$. For simplicity we restrict ourself to the case $dim(M) = 3$. We also assume that Λ is connected and non-trivial.

The following lemma presents an elementary dichotomy for the singularities of a sectional-Anosov flow.

Lemma 2.8. *Every singularity of X in Λ is Lorenz-like or has two positive eigenvalues.*

Proof. Denote by $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the sectional-hyperbolic splitting. By Lemma 2.2 we have $X(x) \in E^c_x$ and so $dim(E^s_x) = 1$ for every $x \in \Lambda$. In particular $dim(E^s_\sigma) = 1$

hence σ has a (strong) contracting eigenvalue $\lambda_2 < 0$. If σ has only one contracting eigenvalue, then σ has two positive eigenvalues and we are done in this case. So, we can assume that σ has another negative eigenvalue λ_3. Clearly one has $\lambda_2 < \lambda_3 < 0$ by dominance. Since $dim(M) = 3$ and no singularity in a singular-hyperbolic set can be attracting (by the sectional expanding condition) we have that there is a third eigenvalue $\lambda_2 < \lambda_3 < 0 < \lambda_1$ of σ. It follows from the volume expanding condition that $-\lambda_3 < \lambda_1$. In this case we have that σ is Lorenz-like and we are done. □

Afterward we study the local strong stable manifolds through a point x of a sectional-hyperbolic set. By a *local strong stable manifold* we mean an ε-ball $W^{ss}_{\varepsilon}(x)$ in $W^{ss}(x)$ centered at x for some $\varepsilon > 0$. These manifolds exists through any point of a sectional-hyperbolic set. The original proof of the results in this section can be found in [96]. We start with an useful lemma.

Lemma 2.9. *Let M be a compact manifold and $X \in \mathscr{X}^1(M)$. If $q \in M$ and $H \subset \omega(q)$ is a hyperbolic set containing a local strong stable manifold, then $q \in H$ and H is a hyperbolic repeller. In particular, q is recurrent.*

Proof. Let $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ be a hyperbolic set with hyperbolic splitting of H. Assume that H contains a local strong stable manifold $W^{ss}_{\varepsilon^*}(x^*)$ through some $x^* \in H$. The continuity of $x \mapsto W^{ss}_{\varepsilon^*}(x)$ implies

$$W^{ss}_{\varepsilon^*}(y) \subset H, \quad \forall y \in \alpha(x^*). \tag{2.5}$$

Fix $y \in \alpha(x^*)$ thus $W^{ss}_{\varepsilon^*}(y) \subset H$. According to Lemma 4.1 p. 127 in [69] there is $\Delta > 0$ satisfying

$$H \bigcap Int \left(\bigcup_{t \in [-\Delta,\Delta]} \quad \bigcup_{z \in X_t(W^{ss}_{\varepsilon^*}(y))} W^{uu}(z) \right) \neq \emptyset,$$

where $Int(\cdot)$ is the interior in M. But $H \subset \omega(q)$ so there are arbitrarily large numbers $t > 0$ satisfying

$$X_t(q) \in Int \left(\bigcup_{t \in [-\Delta,\Delta]} \quad \bigcup_{z \in X_t(W^{ss}_{\varepsilon^*}(y))} W^{uu}(z) \right).$$

It then follows that the negative orbit of $X_t(q)$ is asymptotic to the negative orbit of some point in H. Consequently $q \in H$ so $\omega(q) = H$.

Since H is hyperbolic, $\omega(q) = H$ and $y \in H$ we can apply the shadowing lemma for flows (Theorem 2.4) to a pseudo-orbit derived from the positive orbit of q to find a periodic orbit O with large unstable manifold $W^u(O)$ nearby $W^{ss}_{\varepsilon^*}(y)$. In particular, $W^u(O)$ intersects $W^{ss}_{\varepsilon^*}(y)$ transversally, so, $W^s(O) \subset H$ by the Inclination lemma [87] applied to the backward orbit of $W^{ss}_{\varepsilon^*}(q)$. From this we get $Cl(W^s(O)) \subset \omega(q)$ since $\omega(q)$ is compact invariant. Therefore $Cl(W^s(O))$ is a hyperbolic set contained in $\omega(q)$ which can be used to construct a hyperbolic repeller inside $\omega(q)$. So, $\omega(q) = Cl(W^s(O)) = H$ is a hyperbolic repeller containing q and the result follows. □

Theorem 2.9. *Let M be a compact manifold and $X \in \mathscr{X}^1(M)$. If $q \in M$ and $\omega(q)$ is a sectional-hyperbolic set with singularities of X, then $\omega(q)$ cannot contain any local strong stable manifold.*

Proof. For simplicity we write $\Lambda = \omega(q)$. Denote by $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the sectional-hyperbolic splitting over Λ. Denote also by $dim(E^s_\Lambda)$ and $dim(E^c_\Lambda)$ the dimension of the subbundles E^s_Λ and E^c_Λ respectively. We have that the set-valued map $x \in \Lambda \mapsto W^{ss}_\delta(x)$ is continuous for all fixed $\delta > 0$.

Assume by contradiction that Λ contains a local strong stable manifold $W^s_\varepsilon(x)$ where $x \in \Lambda$. Fix $0 < \varepsilon^* < \varepsilon$ and define

$$H = \left\{ y = \lim_{n\to\infty} X_{t_n}(z_n) \text{ for some sequences } t_n \to -\infty \text{ and } z_n \in W^{ss}_{\varepsilon^*}(x^*) \right\}.$$

Clearly H is compact invariant and $H \subset \Lambda$ since Λ is invariant and $W^{ss}_{\varepsilon^*}(x^*) \subset W^{ss}_\varepsilon(x^*) \subset \Lambda$. We shall obtain the contradiction depending on whether H contains a singularity or not.

If H contains a singularity σ, then there are sequences $t_n \to -\infty$ and $z_n \in W^{ss}_{\varepsilon^*}(x^*)$ such that

$$\sigma = \lim_{n\to\infty} X_{t_n}(z_n).$$

Since $0 < \varepsilon^* < \varepsilon$ we can choose $\delta > 0$ such that

$$W^{ss}_\delta(z_n) \subset W^{ss}_\varepsilon(x^*), \quad \forall n \in \mathbb{N}.$$

Then, since $W^{ss}(\Lambda)$ is contracting and $W^{ss}_\varepsilon(x^*) \subset \Lambda$ one has

$$W^{ss}_\delta(X_{t_n}(z_n)) \subset \Lambda, \qquad \forall n \in \mathbb{N}.$$

Taking limit as $n \to \infty$ and using the fact that Λ is closed and the map $z \to W^{ss}_\delta(z)$ is continuous one has $W^{ss}_\delta(\sigma) \subset \Lambda$. Therefore $\Lambda \cap W^{ss}_\delta(\sigma) = W^{ss}_\delta(\sigma)$ which together with Corollary 2.7 would imply $W^{ss}_\delta(\sigma) = \{\sigma\}$, an absurdity since $W^{ss}_\delta(\sigma)$ is a neighborhood of σ in $W^{ss}(\sigma)$. This gives the desired contradiction if H contains a singularity.

If H contains no singularities, then H is hyperbolic by the hyperbolic lemma. Consequently $q \in H$ by Lemma 2.9 and so $\Lambda = H$ which is absurd since Λ has singularities and H does not. This contradiction proves the result. □

Let us derive some corollaries of Theorem 2.9. The first one is motivated by Proposition 5.5 in [32] in which it is proved that a transitive, isolated, hyperbolic set containining a local strong stable manifold $W^{ss}_\varepsilon(x)$ is a hyperbolic repeller. Indeed, we obtain the same conclusion but for sectional-hyperbolic sets. More precisely we have the following.

Corollary 2.10. *A transitive, isolated, sectional-hyperbolic set containing a local strong stable manifold of a vector field on a compact manifold is a hyperbolic* *saddle-type repeller. Hence the hyperbolic saddle-type repellers are the sole repellers which are sectional-hyperbolic sets.*

Proof. If the set under consideration has a singularity, then it cannot contain any local strong stable manifold by Theorem 2.9. But this is against the hypothesis hence such a set is non-singular so hyperbolic. Then the conclusion follows from the aforementioned proposition in [32]. The last part follows immediately from the first. □

Recall that a subset of a manifold is *proper* if it is not the whole manifold.

Corollary 2.11. *A proper, transitive, sectional-hyperbolic set of a vector field on a compact manifold has empty interior.*

Proof. If a proper, transitive, sectional-hyperbolic set has no singularities, then it is hyperbolic by the hyperbolic lemma so it has no interior by well known result (e.g. Theorem 1 in [1]). If it has singularities and some interior point, then it would contain also some local strong stable manifold against Theorem 2.9. This contradiction proves the result. □

Remember that a metric space Λ *has topological dimension* $dim(\Lambda)$ if it is empty, and $dim(\Lambda) = -1$, or it is non-empty and $dim(\Lambda)$ is the last integer for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than $dim(\Lambda)$ (see [70]). The topological dimension of a compact invariant set without singularities on a transitive singular-hyperbolic set in dimension three has been computed to be 1 in [101].

Corollary 2.12. *A proper, transitive, sectional-hyperbolic set of a vector field on a compact n-manifold has topological dimension* $\leq n-1$.

Proof. The proof is a direct consequence of Corollary 2.11 and Corollary 1 p. 46 in [70]. □

2.3.3 Singular partitions for sectional-hyperbolic sets

We continue with existence results whose main ideas come from [103].

Let X denote a vector field on a compact 3-manifold M. A *rectangle of* X is a cross section D diffeomorphic to $[0,1]\times[0,1]$. In this case ∂D consists of two parts $\partial D = \partial^v D \cup \partial^h D$ in a way that $\partial^v D = D_v^l \cup D_v^r$ and $\partial^h D = D_h^t \cup D_h^b$, where the curves D_v^i ($i \in \{l,r\}$) and D_h^i ($i = t,b$) are as in Figure 2.3. We always assume that I_D, the image of $0\times[0,1]$, is a curve tangent to the centre direction $TD\cap E^c$. Here TD denotes the tangent space of D.

In the case when there is a stable foliation W^s for X (e.g. in a neighborhood of a sectional-hyperbolic set of X) we can define a continuous one-dimensional foliation $\mathscr{F}^s = \{\mathscr{F}^s(x,D) : x \in D\}$ in a rectangle D by intersecting the leaves of W^s with D. We then say that a rectangle D is a *foliated rectangle* if the leaves of $\mathscr{F}^s$ have the

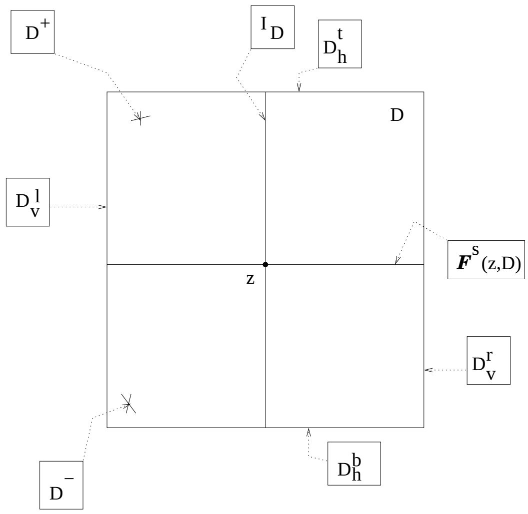


Fig. 2.3

form $* \times [0,1]$ up to identification. In such a case $\partial^v D$ is formed by leaves of $\mathscr{F}^s$ while $\partial^h D$ is transverse to $\mathscr{F}^s$.

The following result about existence of singular partitions is based on the following definition.

Definition 2.13. *Given $\Sigma \subset M$ we say that $q \in M$ satisfies* Property $(P)_\Sigma$ *if $Cl(O^+(q)) \cap \Sigma = \emptyset$ and there is open arc I in M with $q \in \partial I$ such that $O^+(x) \cap \Sigma \neq \emptyset$ for every $x \in I$.*

Again M will denote a compact 3-manifold and $X \in \mathscr{X}^1(M)$. The closure and boundary operations will be denoted by $Cl(\cdot)$ and $\partial(\cdot)$ respectively.

Theorem 2.14. *Let M be a compact 3-manifold, $X \in \mathscr{X}^1(M)$ and $q \in M$ be a point satisfying $(P)_\Sigma$ for some closed subset Σ. If $\omega(q)$ is sectional-hyperbolic, then $\omega(q)$ has singular partitions of arbitrarily small size.*

Proof. By Proposition 1.50 it suffices to show that for all $z \in \omega(q)$ regular there is a cross section Σ_z such that $z \in Int(\Sigma_z)$ and $\omega(q) \cap \partial \Sigma_z = \emptyset$. Fix $z \in \omega(q)$ regular.

We claim that $\omega(x) \cap W^{ss}(z)$ has empty interior in $W^{ss}(z)$. Indeed, if $\omega(q)$ has a singularity, then the result follows from Theorem 2.9. Otherwise $\omega(q)$ would contain a local strong stable manifold and so it would be a hyperbolic repeller by

Lemma 2.9. But since q satisfies $(P)_\Sigma$ we have that the hyperbolic repeller $\omega(q)$ accumulates $W^s(q)$ by one-side only. Therefore $W^s(q)$ is what is called a *stable boundary leaf* of $\omega(q)$ (see [23]). As such leaves are formed by stable manifolds of periodic orbits (e.g. Lemme 1.6 p. 129 in [23] applied to $-X$) we conclude that q belongs to the stable manifold of a periodic orbit. It then follows that $\omega(q)$ is a periodic orbit, a contradiction since it contains the two-dimensional manifold $W^s(q)$. This proves the claim.

By the claim we can fix a foliated rectangle of small diameter R_z^0 such that $z \in Int(R_z^0)$ and $\omega(x) \cap \partial^h R_z^0 = \emptyset$. If the positive orbit of x intersects $\mathscr{F}^s(z, R_z^0)$ infinitelty many times we would have as in Theroem 4.24 that $\omega(x)$ is a periodic orbit in whose case the result is trivial. Therefore, we can assume that the positive orbit of q does not intersect $\mathscr{F}^s(z, R_z^0)$. Then, it intersects either only one or the two connected components of $R_z^0 \setminus \mathscr{F}^s(z, R_z^0)$. We shall assume that it does in one component only. The proof for the other case is similar.

Let I be the interval in the definition of Property $(P)_\Sigma$. The positive orbit of q carries the positive orbit of I into that component. Hence we can assume that I itself is contained in R_z^0. We have that I defines a subrectangle R_I in R_z^0 formed by the stable leaves in R_z^o intersecting I. We clearly have that $\omega(q) \cap Int(R_I)$ for otherwise it would exist a point $x \in I$ whose positive orbit is asymptotic to that of q in whose case $O^+(x) \cap \Sigma$ contradicting $(P)_\Sigma$. Take $z' \in Int(R_I)$ and z'' in the component of $R_z^0 \setminus \mathscr{F}^s(z, R_z^0)$ which does not intersect the positive orbit of q. Hence the subrectangle Σ_z bounded by $\mathscr{F}^s(z', R_z^0)$ and $\mathscr{F}^s(z'', R_z^0)$ satisfies the required properties. This finishes the proof. □

Again M will denote a compact 3-manifold and $X \in \mathscr{X}^1(M)$. The length of an arc J will be denoted by Length(J).

Theorem 2.15. *Let $q \in M$ with sectional-hyperbolic omega-limit set $\omega(q)$ and $T_U M = \hat{E}_U^s \oplus \hat{E}_U^c$ be a continuous extension of the sectional-hyperbolic splitting $T_{\omega(q)} M = E_{\omega(q)}^s \oplus E_{\omega(q)}^c$ of $\omega(q)$ to a neighborhood U of $\omega(q)$. Let I be an arc tangent to $\hat{E}_U^c$, transverse to X, with q as boundary point. If $\omega(q)$ is not a singularity, then for every singular partition $\mathscr{R}$ of H there are $S \in \mathscr{R}$, $\delta > 0$, a sequence $\hat{q}_1, \hat{q}_2, \cdots \in S$ of points in the positive orbit of q and a sequence of intervals $\hat{J}_1, \hat{J}_2, \cdots \subset S$ in the positive orbit of I with $\hat{q}_j$ as a boundary point of $\hat{J}_j$ $(\forall j)$ such that*

$$Length(\hat{J}_j) \geq \delta, \quad \forall j = 1, 2, 3, \cdots .$$

Proof. Since $\omega(q)$ is sectional-hyperbolic and not a singularity we have that every singularity in $\omega(q)$ is Lorenz-like, and so, they have one-dimensional unstable manifold. Then, Lemma 1.20 applied to $\mathscr{R}$ implies that the return map $\Pi = \Pi_{\mathscr{R}}$ associated to $\mathscr{R}$ satisfies the following properties:

(A) $O_X^+(q) \cap \mathscr{R} = \{q_1, q_2, \cdots\}$ is an infinite sequence ordered in a way that $\Pi(q_i) = q_{i+1}$.

(B) There is $\delta > 0$ such that if $n \in \{1,2,\cdots\}$ then either $B_\delta(q_n) \subset Dom(\Pi)$ and $\Pi/B_\delta(q_n)$ is C^1 or there is a curve $c_n \subset W_X^s(Sing(X) \cap \omega(q)) \cap B_\delta(q_n)$ such that

$$B_\delta^+(q_n) \subset Dom(\Pi) \quad \text{and} \quad \Pi/B_\delta^+(q_n) \text{ is } C^1,$$

where $B_\delta^+(q_n)$ denotes the connected component of $B_\delta(q_n) \setminus c_n$ containing q_n.

We shall assume the second alternative in (B) since the first one is easier to handle.

We can assume that there is i_0 large such that $q_i \in Int(\mathscr{R})$ for all $i \geq i_0$. Otherwise $\omega(q) \cap \partial\mathscr{R} \neq \emptyset$ and we get a contradiction because $\mathscr{R}$ is a singular-partition of $\omega(q)$ (see Definition 1.49). We can assume $i_0 = 1$ without loss of generality. By (A) there is a sequence $n_1, n_2, \cdots \in \{1, \cdots, k\}$ such that

$$q_i \in S_{n_i}, \qquad \forall i.$$

By using the positive orbit of I we can assume

$$I \subset S_{n_1} \cap Dom(\Pi).$$

By shrinking I if necessary we can further assume that $I_1 \subset Int(B_\delta^+(q_1))$, where δ comes from (B).

Define $I_1 = I$ and, inductively, the interval sequence $I_i = \Pi(I_{i-1}) = \Pi^i(I)$ as long as $I_{i-1} = \Pi^{i-1}(I) \subset B_\delta(q_{i-1})$.

Now we recall I is tangent to $\hat{E}_\Lambda^c$ and transverse to X by hypothesis. Then, the volume expansivity of E_Λ^c implies that Π is *expanding along I* (see [110] p. 370).

Therefore the sequence $I_i = \Pi(I_{i-1})$ satisfies Length$(I_i) \to \infty$ if $I_i \subset B_\delta^+(q_i)$ for all i. Since the elements of $\mathscr{R}$ have finite diameter we conclude that there is a first index i_1 such that

$$I_{i_1} \not\subset B_\delta^+(q_{i_1}).$$

On the other hand, the positive orbits starting in I_{i_1} meet Σ by $(\mathrm{P})_\Sigma$ while the ones in c_i do not for they go to $Sing(X) \cap \omega(q)$ by (B). From this we conclude that

$$I_{i_1} \cap c_{i_1} = \emptyset.$$

Therefore, the connected component J_{i_1} of $I_{i_1} \cap B_\delta(q_{i_1})$ containing q_{i_1} joints q_{i_1} to some point in $\partial B_\delta(q_{i_1})$. This last assertion implies

$$\text{Length}(J_{i_1}) \geq \delta.$$

In conclusion we have found an index i_1 and an interval $J_{i_1} \subset I_{i_1}$ (and so in the positive orbit of I) such that q_{i_1} is a boundary point of J_{i_1} and Length$(J_{i_1}) \geq \delta$.

Repeating the argument we get a sequence $i_1, i_2, \cdots \in \{1, \cdots, k\}$, a sequence of points $q_{i_1}, q_{i_2}, \cdots$ with $q_{i_j} \in S_{i_j}$, and a sequence of intervals $J_{i_j} \subset S_{i_j}$ in the positive orbit of I such that q_{i_j} is a boundary point of J_{i_j} and Length$(J_{i_j}) \geq \delta$.

As $\{1,\cdots,k\}$ is a finite set and contains i_j we can assume that $i_j = r$ for some fixed index $r \in \{1,\cdots,k\}$. Denoting $S = S_r$, $\hat{q}_j = q_{i_j}$ and $\hat{J}_j = J_{i_j}$ we get the result.
$\square$

Chapter 3
Anosov and sectional-Anosov flows: definition and examples

In this chapter we introduce the definition of Anosov and sectional-Anosov flows. Actually we introduced the Anosov group actions including Anosov diffeomorphisms and flows. Afterward we present examples of Anosov systems which clarify the underliying definitions. We finish this chapter with some examples of sectional-Anosov flows.

3.1 Definition of Anosov flows

We start with the general definition of Anosov flows.

Definition 3.1. *An* Anosov flow *is a vector field for which the ambient manifold is a hyperbolic set.*

3.2 Examples

In this section we introduce some examples of Anosov flows.

3.2.1 Suspended Anosov flows

First we recall the definition of Anosov diffeomorphism.

Definition 3.2. *A diffeomorphism $f\colon M \to M$ of a manifold M is Anosov if there are a continuous splitting*

$$TM = E^s \oplus E^u,$$

such that the following properties hold:

1. *E^s and E^u are invariant for f, namely $Df(x)E^\sigma_x = E^\sigma_{f(x)}$ $\quad (\sigma = u, s)$, for all $x \in M$.*
2. *$\exists c$, $\lambda > 0$ such that $\forall x \in M$*

$$\begin{cases} \bullet\ ||Df^n(x)/E^s_x|| \leq c e^{-\lambda n}, & \forall n \in \mathbb{N} \\ \bullet\ ||Df^n(x)/E^u_x|| \geq c^{-1} e^{\lambda n}, & \forall n \in \mathbb{N} \end{cases}$$

An example of an Anosov diffeomorphism is as follows. Consider the linear map $\widehat{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$. Since $\det(\widehat{A}) = 1$ we have that $\widehat{A}(\mathbb{Z}^2) = \mathbb{Z}^2$ (note that $\widehat{A}$ has integer entries). So, there is a diffeomorphism $A = T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to T^2$ which makes the diagram below to commute

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\widehat{A}} & \mathbb{R}^2 \\ \pi \downarrow & & \downarrow \pi \\ T^2 & \xrightarrow{A} & T^2 \end{array}$$

Here $\pi : \mathbb{R}^2 \to T^2$ is the natural projection.

Proposition 3.3. $\boxed{\textit{A is Anosov}}$.

Proof. It suffices to show that $\widehat{A}$ is Anosov. The eigenvalues of $\widehat{A}$ are given by

$$\begin{aligned} X^2 - \mathrm{tr}\widehat{A}X + \det\widehat{A} = 0 &\Leftrightarrow X^2 - 3X + 1 = 0 \\ \Rightarrow X = \frac{3 \pm \sqrt{5}}{2} &\begin{cases} \lambda_u = \frac{3+\sqrt{5}}{2} \\ \lambda_s = \frac{3-\sqrt{5}}{2} \end{cases} \\ = &\boxed{0 < \lambda_s < 1 < \lambda_u} \end{aligned}$$

Let E^σ_0 be the eigenspace associate to λ_σ $\quad(\sigma = s, u)$. We set for every $z \in \mathbb{R}^2$

$$\boxed{E^\sigma_z = z + E^\sigma_0}, \quad \boxed{\sigma = s, u}$$

Since $\mathbb{R}^2 = E^s_0 \oplus E^u_0$ we have

$$T_z\mathbb{R}^2 = E^s_t \oplus E^u_z \quad \forall t \in \mathbb{R}^2$$

In addition,

$$\begin{aligned} D\widehat{A}(z)(E^\sigma_z) &= \widehat{A}(z+E^\sigma_0) = \widehat{A}(z) + \widehat{A}(E^\sigma_0) \\ &= \widehat{A}(z) + E^\sigma_0 = E^\sigma_{\widehat{A}(z)} \\ \therefore\ T\mathbb{R}^2 &= E^s \oplus E^u \text{ is invariant.} \end{aligned}$$

Onto finishes, fix $v^\sigma_z \in E^\sigma_z \therefore v^\sigma_z = z + v^\sigma_0$ for some $v^\sigma_0 \in E^\sigma_0$ and $||v^\sigma_z||_z = ||v^\sigma_0$ ($||\cdot||_z$ is the norm in $T_z\mathbb{R}^2$). Moreover,

$$\begin{aligned} D\widehat{A}(z)(v^\sigma_z) &= \widehat{A}(z+v^\sigma_0) = \widehat{A}(z) + \widehat{A}(v^\sigma_0) \\ &= \widehat{A}(z) + \lambda_\sigma \cdot v^\sigma_0 \\ &\therefore \\ ||D\widehat{A}(z)(v^\sigma_z)||_{\widehat{A}(z)} &= \lambda_\sigma ||v^\sigma_0|| = \lambda_\sigma ||v^\sigma_z||_z \end{aligned}$$

Hence

- $||D\widehat{A}(z)(v^s_z)||_{\widehat{A}(t)} = \lambda_s ||v^s_z||_z$
- $||D\widehat{A}(z)(v^u_z)||_{\widehat{A}(z)} = \lambda_u ||v^s_z||_t$

Proving that E^s (resp. E^u) is contracting (resp. expanding) since $0 < \lambda_s < 1 < \lambda_u$. □

Next we prove the following result.

Proposition 3.4. *If f is an Anosov diffeomorphism on a manifold M, then the suspended flow $\boxed{X^f \text{ is Anosov}}$.*

Proof. Denote by $TM = \widehat{E}^s \oplus \widehat{E}^u$ the splitting associated to f. Define the splitting

$$TM^t = E^s \oplus E^0 \oplus E^u.$$

For $(0,x) \in 0 \times M$ we set

$$E^s_{(0,x)} = \widehat{E}^s_x; \quad E^u_{0,x)} = \widehat{E}_x$$

(We use here the obvious identification $T(0 \times M) = TM$). For $(t,x) \in [0,1] \times M$ we set

$$\begin{cases} E^s_{(t,x)} = DX^f_t(0,x)(\widehat{E}^s_x); \\ E^u_{(t,x)} = DX^t_t(0,x)(\widehat{E}^u_x). \end{cases}$$

The splitting is well defined an invariant since f preserves the splitting $T_xM = \widehat{E}^u_x \oplus \widehat{E}^s_x$ (note that the return map of X^f in $0 \times M \simeq M$ is f). Let us prove that E^s is contracting. Observe that $E^s_{(t_0,x)} = DX^f_{t_0}(x)(\widehat{E}^s_x)$ by definition. Hence,

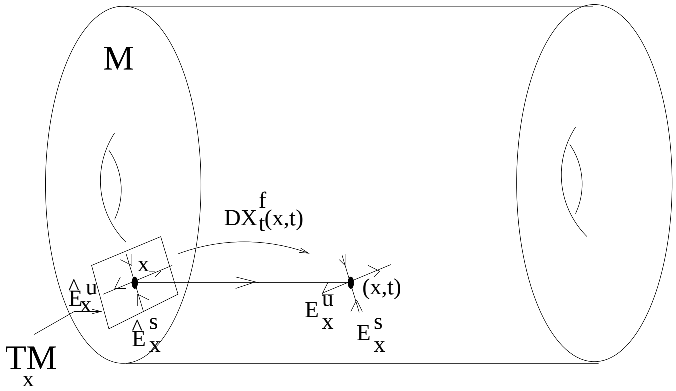


Fig. 3.1

$$DX^f_{-t_0}(t_0,x)(E^2_{(t_0,x)}) = \widehat{E}^s_x$$

by taking inverse. Fix $t > 0$ then

$$DX^t_t(t_0,x) = DX^f_{t+t_0-t_0}(t_0,x) = DX^f_{t_0+t}(x) \cdot DX^f_{-t_0}(t_0,x)$$

$$\therefore \boxed{DX^f_t(t_0,x)/E^s_{t_0,x)} = DX^f_{t+t_0}(x)/\widehat{E}^s_x}$$

By this formula we can assume that $t_0 = 0$ i.e. $(t_0,x) \in 0 \times M$. Set $t = [t] + r$, $r \in [0,1)$ where $[\cdot]$ denotes integer part. By definition we have

$$X^t_{[t]} = f^{[t]} \quad \text{in} \quad O \times M$$

then,

$$DX^f_t(x) = DX^f_{[t]}(x) = DX^f_r(f^{[t]}(x)) \cdot Df^{[y]}(x)$$
$$\therefore ||DX^f_t(x)/\widehat{E}^s_x|| \leq ||DX^f_r(f^{[t]}(x))|| \cdot ||Df^{[t]}(x)/\widehat{E}^s_x|| \leq K.c.e^{[t]\log\lambda} \quad \lambda \in (0,1)$$

$\therefore E^s$ is contracting. Analogously we prove that E^u is expanding. □

In analogy with the previous construction we can construct a suspension with "variable time". Set $\varphi : M \to \mathbb{R}$ be smooth positive bounded map we set

$$M(\varphi) = \{(t,x) : 0 \leq t \leq \varphi(x); x \in M\}.$$

The quocient space

$$M^f(\varphi) = M(\varphi)/(\varphi(x),x) \simeq (0, f(x))$$

is equipped with the vector field $X^{\varphi,f}$ induce by the trivial vector field $(t,x) \mapsto \dfrac{\partial}{\partial v} \cong (1,0)$ in $M(\varphi)$.

Proposition 3.5. *If f is Anosov, then $X^{\varphi,f}$ also is.*

Proposition 3.6. *An Anosov flow X on M is suspended if and only if there is a closed codimension 1 submanifold $S \pitchfork X$ in M intersecting every orbit of X. In particular, all suspended Anosov flows are topologically equivalent to $X^{\varphi,f}$ for some φ, f and viceversa.*

3.2.2 Geodesic flows

Let M be a closed manifold equipped with a Riemannian metric $\langle\cdot,\cdot\rangle$. Denote by ∇ the Riemannian connection of M, namely, the unique connection of the tangent bundle $\Pi : TM \to M$ which is symmetric and compatible with $\langle\cdot,\cdot\rangle$. A vector field along a curve $c(t)$ on M is a map V assigning to each t a tangent vector $V(t) \in T_{c(t)}M$. In particular, the derivative $c'(t)$ of $c(t)$ is a vector field along $c(t)$. A vector field $V(t)$ along $c(t)$ is parallel if $\frac{DV}{dt}(t) = 0$ for all t where $\frac{DV}{dt}(t) = \nabla_{c'(t)}V$ is the covariant derivative induced by ∇. As is well known for every curve $c : (-\varepsilon,\varepsilon) \to M$ and every tangent vector $v \in T_{c(0)}M$ there is a parallel vector field $W_v(t)$ along $c(t)$ with $W_v(0) = v$ called the *parallel transport of v along $c(t)$*. A *geodesic* of M is a curve whose derivative is a parallel vector field along itself. It is also well known that for every $\theta \in TM$ there is a unique geodesic $\gamma_\theta(t)$ with $\gamma'_\theta(0) = \theta$. When γ is a geodesic we obtain

$$\frac{\|\gamma'(t)\|}{dt} = 2\left\langle \frac{D\gamma}{dt}(t), \gamma'(t)\right\rangle = 0$$

thus $\|\gamma'(t)\| = \|\gamma'(0)\|$ for all t. In particular, $\|\gamma'_\theta(t)\| = \|\theta\|$ for all t.

Define the *geodesic flow* as the vector field G in TM with flow $G_t(\theta) = \gamma'_\theta(t)$. From the above remarks we see that G is tangent to the *unitary tangent bundle of M*, that is, the submanifold $T_1M = \{\theta \in TM : \|\theta\| = 1\}$ of TM.

Now we introduce the concept of sectional curvature of M. Denote by R the *Riemann tensor* of M which assigns to each triple of C^∞ vector fields X,Y,Z in M the C^∞ vector field $R(X,Y)Z$ defined by

$$R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z.$$

A direct computation shows that R is a tensor field, that is, R is additive in each variable separately and satisfies $R(fX,Y)Z = R(X,fY)Z = R(X,Y)fZ = fR(X,Y)Z$ for all X,Y,Z and all C^∞ real valued map f of M. From this it follows that for every triple of tangent vectors $X_p,Y_p,Z_p \in T_pM$ with $p \in M$ the number $R(X_p,Y_p)Z_p$ is well defined by $R(X_p,Y_p)Z_p = R(X,Y)Z(p)$ where X,Y,Z are vector fields with $X(p) = X_p, Y(p) = Y_p$ and $Z(p) = Z_p$. This allows us to define the *sectional curvature* of M as the map κ which assigns to each $p \in M$ and each plane $\sigma_p \subset T_pM$ the number

$$\kappa(\sigma_p) = \frac{\langle R(X_p,Y_p)X_p,Y_p\rangle}{\|X_p\|^2\|Y_p\|^2 - \langle X_p,Y_p\rangle^2}$$

where X_p,Y_p is a base of σ_p. It turns out that the value of $\kappa(\sigma_p)$ does not depend on the chosen base X_p,Y_p of σ_p.

We say that the Riemannain manifold M has *negative sectional curvature* if $\kappa(\sigma_p) < 0$ for all $p \in M$ and all plane $\sigma_p \subset T_pM$.

Now we state the following result which according to some authors is due to Cartan, Hadamard and Lobachevskii.

Theorem 3.7. *The geodesic flow restricted to the unitary tangent bundle of a closed manifold with negative sectional curvature is Anosov.*

Proof. To simplify the exposition we prove this result for manifolds M with constant sectional curvature, i.e., the value $\kappa(\sigma_p) = \kappa$ independs on $p \in M$ and $\sigma_p \subset T_pM$.

Consider the derivative $D\Pi : TTM \to TM$ of the tangent bundle projection $\Pi : TM \to M$. A first remark is that

$$D\Pi(G(\theta)) = \theta, \qquad \forall \theta \in TM. \tag{3.1}$$

Indeed, $D\Pi(G(\theta)) = (\Pi \circ z)(0)$ where $z(t)$ is a curve in TM with velocity $G(\theta)$ at $t = 0$. We can just choose $z(t) = G_t(\theta) = \gamma'_\theta(t)$ to obtain

$$D\Pi(G(\theta)) = \frac{d}{dt}\bigg|_{t=0} [\Pi(\gamma'_\theta(t)] = \gamma'_\theta(0) = \theta.$$

We shall call a vector $\xi \in TTM$ *horizontal* if $D\Pi(\xi) = 0$. Clearly the set of all vertical vectors H in TTM is a vector bundle H over TM with fiber $H(\theta) = \{\xi \in T_\theta TM : D\Pi(\xi) = 0\}$. Hereafter we write $D\Pi(\theta)\xi$ instead of $D\Pi(\xi)$ in order to emphasize the base point θ of ξ.

Now we use the Riemannian connection ∇ to define horizontal vectors in TTM. First define the *connection map* $K : TTM \to TM$ by

$$K(\xi) = \nabla_{(\Pi \circ z)'(0)} z,$$

where $z(t)$ is a curve in TM with velocity ξ at $t = 0$. It is a routine exercice to prove that the value of $K(\xi)$ independs on the chosen curve z. It turns out that $K(\theta) : T_\theta TM \to T_{\Pi(\theta)}M$ is linear. Now we say that ξ is *horizontal* if $K(\xi) = 0$. For instance, $G(\theta)$ is a horizontal vector for all $\theta \in TM$. Indeed, choosing $z(t) = G_t(\theta)$ as above we see that

$$K(G(\theta)) = \nabla_{\frac{d}{dt}|_{t=0}[\Pi(\gamma'_\theta(t))]}\gamma'_\theta = \nabla_{\gamma'_\theta(0)}\gamma'_\theta = 0$$

since $\gamma_\theta(t)$ is a geodesic. Again we write $K(\theta)\xi$ instead of $K(\xi)$ to emphasize the base point θ. We have a vector bundle $V \to TM$ with fiber $V(\theta) = \{\xi \in T_\theta TM : K(\theta)\xi = 0\}$ inducing the direct sum of vector bundles

$$TTM = H \oplus V.$$

Moreover, the linear maps $K(\theta) : V(\theta) \to T_{\Pi(\theta)}M$ and $D\Pi(\theta) : H(\theta) \to T_{\Pi(\theta)}M$ are linear isomorphisms.

We have the identity

$$T_\theta T_1M = \{\xi \in T_\theta TM : \langle K(\theta)\xi, \theta\rangle = 0\}, \qquad \forall\theta \in T_1M. \tag{3.2}$$

Indeed, if $\xi \in T_\theta T_1M$, then there is a curve $z(t)$ *in* T_1M with velocity ξ at $t = 0$. Since $z(t) \in T_1M$ we have $\langle z(t), z(t)\rangle = 1$ for all t and then

$$0 = \left.\frac{d}{dt}\right|_{t=0} \langle z(t), z(t)\rangle = 2\langle K(\theta)\xi, \theta\rangle.$$

Going into the reversed direction we get (3.2). Since $K(\theta)\xi = 0$ for horizontal vectors ξ we immediately get the inclusion

$$H(\theta) \subset T_\theta T_1M, \qquad \forall\theta \in T_1M. \tag{3.3}$$

Next we introduce the so-called *Sasaki metric* in TTM which is nothing but the metric that makes the above direct sum orthogonal, namely,

$$\langle\langle \xi, \eta\rangle\rangle = \langle D\Pi(\theta)\xi, D\Pi(\theta)\eta\rangle + \langle K(\theta)\xi, K(\theta)\eta\rangle, \qquad \forall\theta \in TM, \forall\xi, \eta \in T_\theta TM.$$

Define S as the orthogonal complement of the geodesic field G in T_1M with respect to the Sasaki metric. It turns out that S is a subbundle of TT_1M whose fiber $S(\theta)$ at $\theta \in T_1M$ is given by

$$S(\theta) = \{\xi \in T_\theta T_1M : \langle\langle \xi, G(\theta)\rangle\rangle = 0\}.$$

We clearly have

$$TT_1M = S \oplus E^G, \tag{3.4}$$

where E^G is the one-dimensional subbundle of TT_1M generated by the geodesic field G. Next we define the subbundles E^s, E^u of S by

$$E^s = \left\{\xi \in S : D\Pi(\xi) = -\frac{K(\xi)}{\sqrt{-\kappa}}\right\} \quad \text{and} \quad E^u = \left\{\xi \in S : D\Pi(\xi) = \frac{K(\xi)}{\sqrt{-\kappa}}\right\}.$$

Let us prove the direct sum decomposition

$$S = E^s \oplus E^u. \tag{3.5}$$

If $\xi \in E^s_\theta \cap E^u_\theta$ then

$$D\Pi(\theta)\xi = -\frac{K(\theta)\xi}{\sqrt{-\kappa}} = \frac{K(\theta)\xi}{\sqrt{-\kappa}}$$

yielding $K(\theta)\xi = 0$ and so $D\Pi(\theta)\xi = 0$. Therefore ξ is simultaneously horizontal and vertical hence $\xi = 0$. This proves $E^s_\theta \cap E^u_\theta = \{0\}$. It remains to prove $S(\theta) = E^s_\theta + E^u_\theta$. Take $\xi_h \in S(\theta) \cap H(\theta)$. In particular, $K(\theta)\xi_h = 0$. Since $K(\theta)$ restricted to $V(\theta)$ is an isomorphism onto $T_{\Pi(\theta)}M$ which in turn contains $\sqrt{-\kappa}D\Pi(\theta)\xi_h$ we have that there is a unique $\eta \in V(\theta)$ such that $K(\theta)\eta = \sqrt{-\kappa}D\Pi(\theta)\xi_h$. Since

$$\langle K(\theta)\eta,\theta\rangle = \sqrt{-\kappa}\langle D\Pi(\theta)\xi_h,\theta\rangle = \sqrt{-\kappa}\langle\langle \xi_h, G(\theta)\rangle\rangle = 0$$

for $\xi_h \in S(\theta)$ we get $\eta \in T_\theta T_1M$ (recall (3.2)). As

$$\langle\langle \eta, G(\theta)\rangle\rangle = \langle D\Pi(\theta)\eta,\theta\rangle = \langle 0,\theta\rangle = 0$$

for η is vertical we get $\eta \in S(\theta)$. If now $\xi^u = \frac{\eta_h+\eta}{2}$ then $\xi^u \in S(\theta)$ and

$$D\Pi(\theta)\xi^u = D\Pi(\theta)\frac{\eta_h+\eta}{2} = \frac{D\Pi(\theta)\xi_h}{2} = \frac{K(\theta)\eta}{2\sqrt{-\kappa}} = \frac{K(\theta)(\frac{\xi_h+\eta}{2})}{\sqrt{-\kappa}} = \frac{K(\theta)\xi^u}{\sqrt{-\kappa}}$$

proving $\xi^u \in E^u_\theta$. Setting $\xi^s = \frac{\xi_h-\eta}{2} \in S(\theta)$ an analogous computation shows $\xi^s \in E^s_\theta$ and then $\xi_h = \xi^s + \xi^u \in E^s_\theta + E^u_\theta$ proving

$$S(\theta) \cap H(\theta) \subset E^s_\theta + E^u_\theta.$$

Analogously we prove

$$S(\theta) \cap V(\theta) \subset E^s_\theta + E^u_\theta.$$

But now we observe that

$$S(\theta) = (S(\theta) \cap H(\theta)) + (S(\theta) + V(\theta))$$

for if $\xi \in S(\theta)$, then $\xi = \xi_h + \xi_v$ for unique $(\xi_h,\xi_v) \in H(\theta) \times V(\theta)$ and we have $\xi_h \in H(\theta) \subset T_\theta T_1M$ by (3.3). Moreover,

$$0 = \langle\langle \xi, G(\theta)\rangle\rangle = \langle\langle \xi_h, G(\theta)\rangle\rangle + \langle\langle \xi_h, G(\theta)\rangle\rangle = \langle\langle \xi_h, G(\theta)\rangle\rangle$$

since $G(\theta)$ is horizontal, so, $\langle\langle \xi_h, G(\theta)\rangle\rangle = 0$ proving $\xi_h \in S(\theta)$. Thus $\xi_h \in S(\theta) \cap H(\theta)$ and then $\xi_v = \xi - \xi_h \in S(\theta)$ yielding $\xi_v \in S(\theta) \cap V(\theta)$. This suffices. We therefore obtain

$$S(\theta) = (S(\theta) \cap H(\theta)) + (S(\theta) \cap V(\theta)) \subset E^s_\theta + E^u_\theta$$

yielding (3.5). Applying it and (3.4) we get the splitting

$$TT_1M = E^s \oplus E^G \oplus E^u.$$

The final step consists of proving that this is a hyperbolic splitting for the geodesic field restricted to T_1M. For this purpose we shall use the following definition. Given $\xi \in TTM$ we define the curve $J_\xi(t)$ by

$$J_\xi(t) = D\Pi(G_t(\theta))(DG_t(\theta)\xi) \quad \text{whenever} \quad \xi \in T_\theta TM$$

We shall analyze these curves when $\theta \in T_1M$. For simplicity we write J and γ isntead of J_ξ and γ_θ. First we remark that $J(t)$ is a vector field along the geodesic $\gamma(t)$. Moreover,

$$J(t) = \frac{\partial}{\partial s}\Big|_{s=0} (\Pi \circ G_t \circ z)(s),$$

where $z(s)$ is a curve in T_1M with velocity ξ at $s = 0$. So,

$$\frac{DJ}{dt}(t) = \frac{D}{dt}\frac{\partial}{\partial s}\Big|_{s=0} (\Pi \circ G_t \circ z)(s) = \frac{D}{ds}\Big|_{s=0} \frac{\partial}{\partial t}(\pi \circ G_t \circ z)(s) =$$

$$\frac{D}{ds}\Big|_{s=0} \frac{\partial}{\partial t}[\gamma_{z(s)}(t)] = \frac{D}{ds}\Big|_{s=0} G_t(z(s)).$$

Derivating again and taking into account well known properties of the Riemann tensor R we get

$$\frac{D^2J}{dt^2}(t) = \frac{D}{dt}\frac{D}{ds}\Big|_{s=0} G_t(z(s)) = \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}G_t(z(s)) + R\left(\frac{\partial}{\partial s}\Big|_{s=0} [\gamma_{z(s)}(t)], \gamma'(t)\right)\gamma'(t).$$

Now we observe that $G_t(z(s)) = \gamma_{z(s)}(t)$ is a geodesic, for fixed s, so $\frac{D}{dt}G_t(z(s)) = 0$. Replacing above we get

$$\frac{D^2J}{d^2}(t) = R\left(\frac{\partial}{\partial s}\Big|_{s=0} [\gamma_{z(s)}(t)], \gamma'(t)\right)\gamma'(t).$$

But $(\Pi \circ G_t \circ z)(s) = \gamma_{z(s)}(t)$ so $J(t) = \frac{\partial}{\partial t}[\gamma_{z(s)}(t)]$ and then $J(t)$ is the solution of following initial-value problem known as *Jacobi equation*:

$$\begin{cases} \frac{D^2J}{dt^2}(t) + R(\gamma'(t), J(t))\gamma'(t) = 0 \\ J(0) = D\Pi(\theta)\xi \\ \dot{J}(0) = K(\theta)\xi \end{cases} \tag{3.6}$$

where $\gamma = \gamma_\theta$ and $\dot{J}(0) \stackrel{def}{=} \frac{DJ}{dt}(0)$.

We claim that if $\xi \in S(\theta)$, then the solution $J(t)$ of (3.6) is *orthogonal to* $\gamma(t)$, i.e., $\langle J(t), \gamma'(t)\rangle = 0$ for all t. To see it define the auxiliary real-valued map

$$h(t) = \langle J(t), \gamma'(t)\rangle.$$

Clearly

$$h(0) = \langle J(0), \gamma'(0)\rangle = \langle D\Pi(\theta)\xi, \theta\rangle = \langle\langle \xi, , G(\theta)\rangle\rangle = 0$$

for $\xi \in S(\theta)$. Moreover,

$$h'(t) = \left\langle \frac{DJ}{dt}(t), \gamma'(t) \right\rangle$$

and so

$$h'(0) = \langle K(\theta)\xi, \theta \rangle = 0$$

for $\theta \in T_1M$ and $\xi \in T_\theta T_1M$ (recall (3.2)). Derivating again we get

$$h''(t) = \left\langle \frac{D^2J}{dt^2}(t), \gamma'(t) \right\rangle = -\langle R(\gamma'(t), J(t))\gamma'(t), \gamma'(t) \rangle$$

from (3.6). But now we make use of the fundamental identity

$$\langle R(X,Y)Z,T \rangle = \kappa \cdot (\langle X,Z \rangle \langle Y,T \rangle - \langle X,T \rangle \langle Y,Z \rangle) \tag{3.7}$$

in order to get

$$h''(t) = \kappa \cdot (\langle J(t), \gamma'(t) \rangle - \langle J(t), \gamma' \rangle) = 0.$$

We conclude that $h(t)$ solves $h'' = 0$ with initial conditions $h(0) = h'(0) = 0$ and so $h(t) = 0$ proving the desired orthogonality. It follows in particular that the vector field

$$T(t) = R(\gamma'(t), J(t))\gamma'(t) - \kappa \cdot J(t)$$

is orthogonal to $\gamma(t)$. Using again the fundamental identity (3.7) we obtain

$$\langle T(t), T(t) \rangle = \langle R(\gamma'(t), J(t))\gamma'(t), T(t) \rangle =$$

$$\kappa \cdot (\langle J(t), T(t) \rangle - \langle \gamma'(t), J(t) \rangle \langle \gamma'(t), T(t) \rangle) = \kappa \cdot \langle J(t), T(t) \rangle = \langle \kappa \cdot J(t), T(t) \rangle$$

so $\langle T(t), T(t) \rangle = 0$ and then $T(t) = 0$. Therefore,

$$R(\gamma'(t), J(t))\gamma'(t) = \kappa \cdot J(t)$$

and then Jacobi's reduces to

$$\begin{cases} \frac{D^2J}{dt^2}(t) + \kappa \cdot J(t) = 0 \\ J(0) = D\Pi(\theta)\xi \\ \dot{J}(0) = K(\theta)\xi. \end{cases} \tag{3.8}$$

There is a direct method to solve this equation. Take

$$J(t) = y(t)W_\xi(t)$$

where $y(t)$ is real valued and $W_\xi(t) = W_{\Pi(\theta)\xi}(t)$ is a parallel transport of $D\Pi(\theta)\xi$ along $\gamma(t)$. Replacing in (3.8) we get that y is the solution of the initial-value Problem

$$\begin{cases} y'' + \kappa \cdot y = 0 \\ y(0) = 1 \\ y'(0) = \frac{\langle K(\theta)\xi, D\Pi(\theta)\xi \rangle}{\|D\Pi(\theta)\xi\|^2}. \end{cases} \tag{3.9}$$

If ξ satisfies either

$$D\Pi(\theta)\xi = -\frac{K(\theta)\xi}{\sqrt{-\kappa}} \qquad \text{or} \qquad D\Pi(\theta)\xi = \frac{K(\theta)\xi}{\sqrt{-\kappa}}$$

we get the respective solutions

$$y(t) = e^{-\sqrt{-\kappa}t} \qquad \text{or} \qquad y(t) = e^{\sqrt{-\kappa}t}$$

of (3.9) corresponding to the solutions

$$J_\xi(t) = e^{-\sqrt{-\kappa}t} W_\xi(t) \qquad \text{or} \qquad J_\xi(t) = e^{\sqrt{-\kappa}t} W_\xi(t)$$

of (3.8). Consequently

$$E^s_\theta \subseteq \{\xi \in T_\theta T_1 M : J_\xi(t) = e^{-\sqrt{-\kappa}t} W(t) \text{ for some parallel field } W(t) \text{ along } \gamma(t)\}$$

and

$$E^u_\theta \subseteq \{\xi \in T_\theta T_1 M : J_\xi(t) = e^{\sqrt{-\kappa}t} W(t) \text{ for some parallel field } W(t) \text{ along } \gamma(t)\}.$$

But conversely, if $J_\xi(t) = e^{-\sqrt{-\kappa}t} W(t)$ for some parallel vector field $W(t)$ along $\gamma(t)$ then we have $D\Pi(\theta) = W(0)$ (taking $t = 0$) and so $K(\theta)\xi = \dot{J}(0) = -\sqrt{-\kappa} W(0)$ yielding $D\Pi(\theta)\xi = -\frac{K(\theta)\xi}{\sqrt{-\kappa}}$. Analogously for the unstable space so we have proved the identities

$$E^s_\theta = \{\xi \in T_\theta T_1 M : J_\xi(t) = e^{-\sqrt{-\kappa}t} W(t) \text{ for some parallel field } W(t) \text{ along } \gamma(t)\}$$

and

$$E^u_\theta = \{\xi \in T_\theta T_1 M : J_\xi(t) = e^{\sqrt{-\kappa}t} W(t) \text{ for some parallel field } W(t) \text{ along } \gamma(t)\}.$$

Let us use them to prove the invariance of E^s and E^u. Indeed, if $\xi \in E^s_\theta$ and $s \in \mathbb{R}$ the chain rule yields

$$J_{DG_s(\theta)\xi}(t) = D\Pi(G_t(G_s(\xi)))(DG_t(G_s(\theta))DG_s(\theta)\xi) =$$

$$D\Pi(G_{t+s}(\theta))(DG_{t+s}(\theta)\xi) = J_\xi(t+s) = e^{-\sqrt{-\kappa}(t+s)} W(t+s) = e^{-\sqrt{-\kappa}t} \overline{W}(t)$$

where $\overline{W}(t) = e^{-\sqrt{-\kappa}s} W(t+s)$ is clearly a parallel field along the geodesic $\gamma_{G_s(\theta)}(t)$. This proves $DG_s(\theta)\xi \in E^s_{G_s(\theta)}$ for all $\xi \in E^s_\theta$ and $s \in \mathbb{R}$ hence $DG_s(\theta)(E^s_\theta) = E^s_{G_s(\theta)}$. Analogously $DG_s(\theta)(E^u_\theta) = E^u_{G_s(\theta)}$ for all $s \in \mathbb{R}$. The invariance follows.

Finally we prove that G is contracting and expanding along E^s and E^u respectively. For this we can use any metric in T_1M since T_1M is compact. The one we are going to use is nothing but the metric induced by thet Sasaki metric of TM. Denote by $|||\cdot|||$ the norm induced by this metric. If $\xi \in E^u_\theta$ and $t \geq 0$ then $J_\xi(t) = e^{\sqrt{-\kappa}t}W(t)$ for some parallel field $W(t)$ so $\dot{J}_\xi(t) = \sqrt{-\kappa}J_\xi(t) = \sqrt{-\kappa}D\Pi(G_t(\theta))(DG_t(\theta)\xi)$ by the definition of $J_\xi(t)$. Using $\xi \in E^u$ we get

$$\dot{J}_\xi(t) = K(G_t(\theta))DG_t(\theta)\xi.$$

From this and the definition of $J_\xi(t)$ we obtain

$$|||DG_t(\theta)\xi|||^2 = \langle J_\xi(t), J_\xi(t))\rangle + (-\kappa)\langle J_\xi(t), J_\xi(t)\rangle = e^{2\sqrt{-\kappa}t}(1-\kappa)\|W(t)\|^2.$$

But

$$|||\xi|||^2 = \langle\langle \xi, \xi\rangle\rangle = \langle J_\xi(0), J_\xi(0)\rangle + (-\kappa)\langle J_\xi(0), J_\xi(0)\rangle = (1-\kappa)\|W(0)\|^2$$

and $\|W(0)\| = \|W(t)\|$ for $W(t)$ is parallel hence $|||\xi|||^2 = (1-\kappa)|||W(t)|||^2$. From this we get

$$|||DG_t(\theta)\xi||| = e^{\sqrt{-\kappa}t}|||\xi|||, \qquad \forall\theta \in T_1M, \forall\xi \in E^u_\theta, \forall t \geq 0$$

so E^u is expanding. Replacing E^u by E^s in the previous argument we get

$$|||DG_t(\theta)\xi||| = e^{-\sqrt{-\kappa}t}|||\xi|||, \qquad \forall\theta \in T_1M, \forall\xi \in E^s_\theta, \forall t \geq 0$$

so E^s is contracting. This proves the result. □

3.2.3 Algebraic Anosov systems

These are generalization the previous examples.

Definition 3.8. *An* algebraic Anosov flow *is an Anosov flow which is also algebraic.*

As hyperbolic toral diffeomorphisms as considered in Example (1) are algebraic we get the following result.

Corollary 3.9. *The suspension of a hyperbolic toral diffeomorphism is an algebraic Anosov flow.*

Proposition 3.10. *A geodesic Anosov flow on a closed 3-manifold is algebraic.*

Proof. Let $M = TS(1)$ the unitary tangent bundle of a negatively curved surface S. Let w_t be its corresponding geodesic flow. As is well known the universal cover of S is isometric to $\mathbb{H}$ and $\Gamma = \pi_1(M)$ is a discrete subgroup of $\mathbb{H}$ in a way that $S = \Gamma \backslash \mathbb{H}$ the action $\Gamma \times \mathbb{H} \to \mathbb{H}$ passed to an action $\Gamma \times T\mathbb{H}(1) \to T\mathbb{H}(1)$ via derivation. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{PSL}(2,\mathbb{R}) & \xrightarrow{g_t} & \mathrm{PSL}(2,\mathbb{R}) \\ \psi \downarrow & & \downarrow \psi \\ T\mathbb{H}(1) & \longrightarrow & T\mathbb{H}(1) \\ \downarrow & & \downarrow \\ M = TS(1) = \Gamma \backslash T\mathbb{H}(1) & \xrightarrow{w_t} & \Gamma \backslash T\mathbb{H}(1) = TS(1) = M \end{array}$$

since

$$\bar{g}_t(A) = A \exp(t \cdot \alpha)$$

where

$$\alpha = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

we obtain the result. □

We have proved the following

Corollary 3.11. *Suspended and geodesic Anosov flows on closed 3-manifolds are algebraic Anosov flows.*

A sort of converse of this corollary was proved by Tomter [143].

Theorem 3.12 (Tomter)**.** *If $\phi_t : M \to M$ is an algebraic Anosov flow on a closed 3-manifoled M, then there is a finite covering $\widetilde{M} \to M$ such that the lifted flow $\hat{\phi}_t$ of ϕ_t in $\widehat{M}$ is either geodesic or suspended.*

This theorem implies the following.

Corollary 3.13. *Algebraic Anosov flow on closed 3-manifolds are transitive.*

The next example describes a non-algebraic Anosov flow on certain closed 3-manifold.

3.2.4 Anomalous Anosov flows

In this example we describe an intransitive (and hence non algebraic) Anosov flow on certain closed 3-manifoled. More precisely we show the following result due to Franks and Williams [46]:

Theorem 3.14. *There is a closed 3-manifold supporting an intransitive Anosov flow.*

Proof. We start with the Anosov diffeomorphism A in T^2 induced by the linear map $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ in $\mathbb{R}^2$. This map has a fixed point p_0 corresponding to $(0,0) \in \mathbb{R}^2$. The portrait face of A around p_0 is as Figure 3.2.

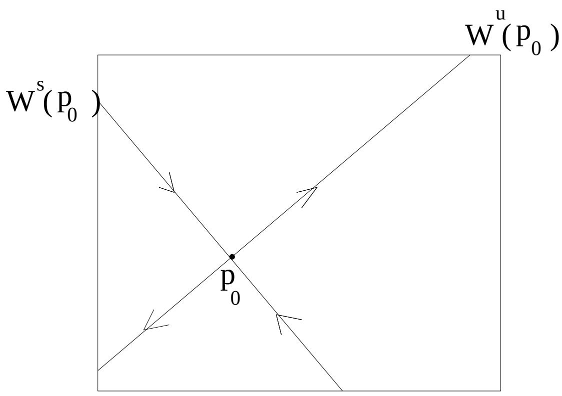


Fig. 3.2

The restriction $A/W^s(p_0)$ of A in the stable manifold $W^s(p_0)$ is as below:

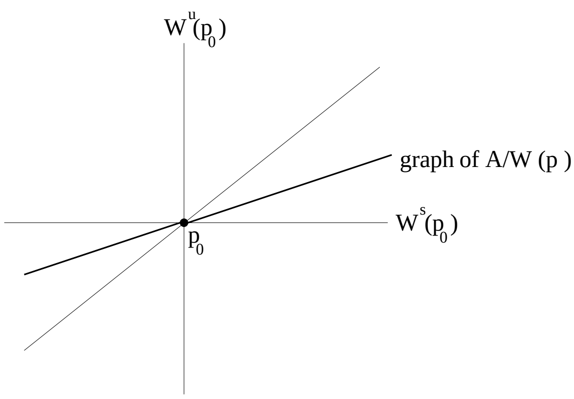


Fig. 3.3

In other words $A/W^s(p_0)$ is a contraction. We deform such a diffeomorphism in order to obtain a one-parameter family of diffeomorphism A_μ, $\mu \in [0,1]$, $A_0 = A$ so that the family $A_\mu/W^s p_0)$ bifurcates as in Figure 3.4.

Fig. 3.4

In T^2 the deformation looks as in Figure 3.5.

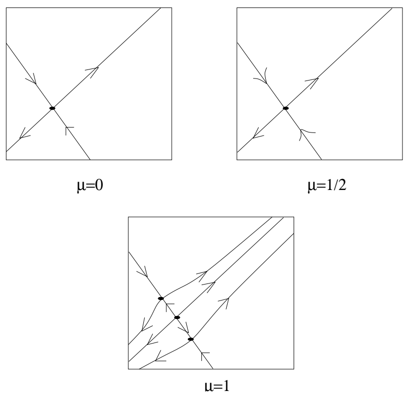


Fig. 3.5

The resulting diffeomorphism $A_1 = T^2 \to T^2$ satisfies the follows properties [149].

(1) $\Omega(A_1) = \{p_0\} \cup \Lambda$ where now p_0 is a repelling fixed point and A is a hyperbolic attractor.

(2) The stable manifold of A are the ones of $A_0 = A$ except for $W^s(p_0)$. The two connected components of $W^s(p_0)\backslash\{p_0\}$ are leaves of the stable foliation of Λ.

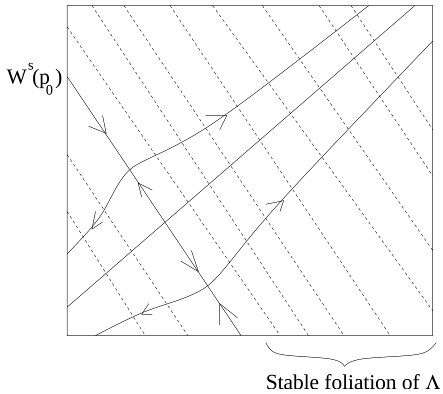


Fig. 3.6

The next step is to consider the suspension $X^1 = X^{A_1}$ of A_1 defined in the closed 3-manifold $M^1 = M^{A_1}$ as in Figure 3.7.

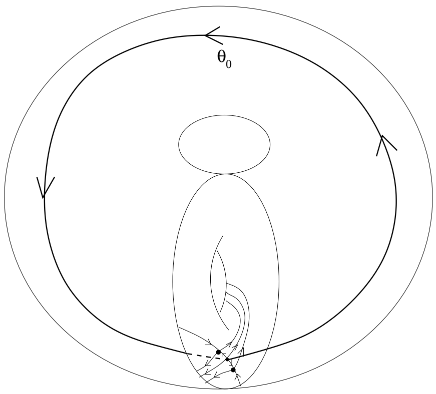


Fig. 3.7

The flow X^1 has a nonwandering decomposition

$$\Omega(X^1) = \theta_0 \cup \Omega^1$$

where θ_0 is a repelling closed orbit corresponding to p_0, and Ω^1 is a hyperbolic attractor corresponding to Λ.
Afterward we fix a torus $T^1 \pitchfork X^1$ which is the boundary of a solid torus neighborhood of θ_0 as in Figure 3.8.
Let us consider the stable foliation $\mathscr{F}^s$ of Δ the attractor Ω of X^1. Hence $\mathscr{F}^s \pitchfork T^2$ and the trace $\mathscr{F}^s \cap T^2$ of $\mathscr{F}^2$ in T^2 is as in Figure 3.9.

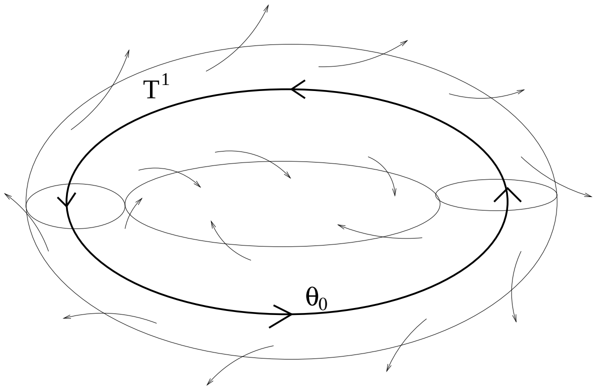


Fig. 3.8

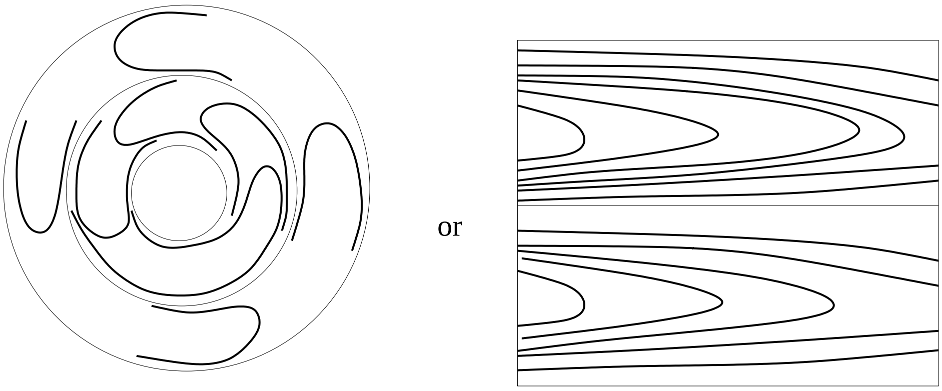


Fig. 3.9

Remove from M^1 the solid torus with boundary T^1 to obtain a compact 3-manifold $\widehat{M}^1$ whose boundary is T^1.
Observe that X^1 is a vector field pointing inward in $\partial\widehat{M}^1 = T^1$.
Consider now the reversed vector field $X^2 = -X^1$ defined in $\widehat{M}^2$ where is a copy of $\widehat{M}^1 \therefore \partial\widehat{M}^2 = T^2 \simeq T^1$ while X^1 contains an attractor Ω^1 in $\widehat{M}^1$ the flow X^2 contains a repeller Ω^2 in $\widehat{M}^2$ whose unstable manifold intersects $\partial\widehat{M}^2 = T^2$ as $\mathscr{F}^s$ does in T^1. Hence we have Figure 3.11 for $\mathscr{F}^s \cap T^1$ and $\mathscr{F}^u \cap T^2$ respectively:
We put a diffeomorphism $\varphi : T^1 \to T^2$ carrying $\mathscr{F}^s \cap T^1$ into T^2 as Figure 3.12.
Hence $\varphi(\mathscr{F}^s \cap T^2) \pitchfork \mathscr{F}^u \cap T^2$.
By gluing $M = \widehat{M}^1 \cup_\varphi \widehat{M}^2$ along $\partial\widehat{M}^1 = T^1$, $\partial\widehat{M}^2 = T^2$ using φ we get a vector field X in M (see Figure 3.13).
As $\varphi(\mathscr{F}^s \cap T^2) \pitchfork (\mathscr{F}^u \cap T^2)$ we have that the vector field X is Anosov. As

$$\Omega(X) = \Omega_0 \cup \Omega_1$$

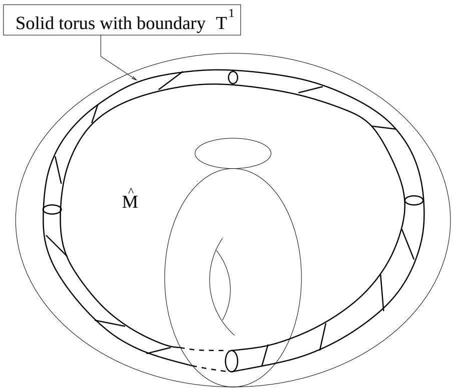


Fig. 3.10

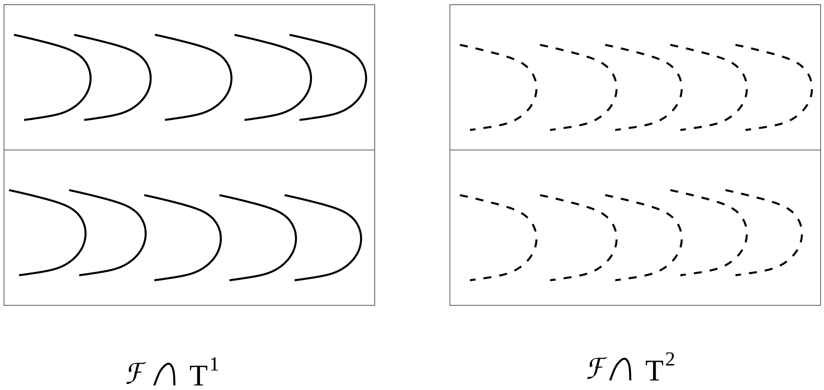


Fig. 3.11

and $\Omega_0, \Omega_1 \neq M$ we have that X is *not* transitive. This provides the result. □

Corollary 3.15. *There are non-algebraic Anosov flow on a certain closed 3-manifold.*

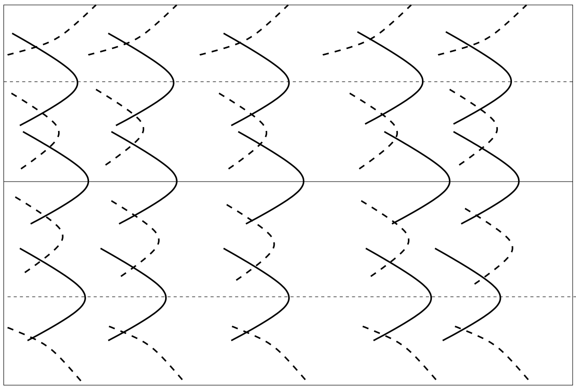


Fig. 3.12

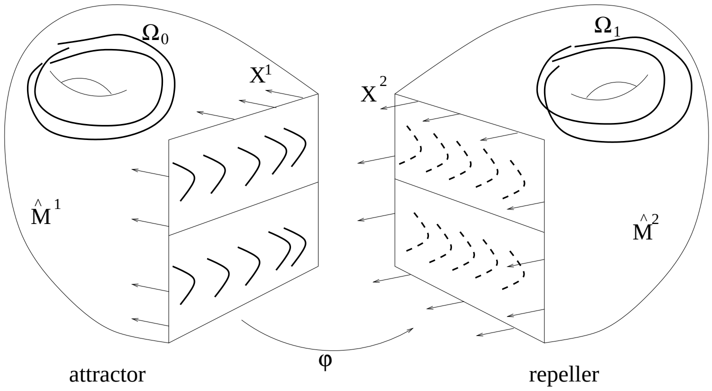


Fig. 3.13

3.2.5 Dehn surgery and Anosov flows

A *Dehn surgery* on a 3-manifold M is a procedure consisting of remove a solid torus in M and regluing it different in M. This procedure yields a new manifold $\widehat{M}$ which is general is *quite different* from M.

To fix ideas we have the example $M = S^3$ with the solid torus being an unknotted one.

Different "gluing" can produce: $\widehat{M} = S^2 \times S^1$ (the one sending meridians into meridians) or even $\widehat{M} =$ Lens space.

In particular the Dehn surgery transforms S^3 (which is simply connected) into $S^2 \times S^1$ (which has infinite cyclic π_1). Another procedure (still called Dehn surgery) can be obtained by considering a two-side embedded torus $T = T^2$ in a 3-manifold M: Let M_0 be the manifold obtained by *cutting open M along T^2*. M_0 may be connected or not. In the later case M_0 has just two connected components M_0^+, M_0^-

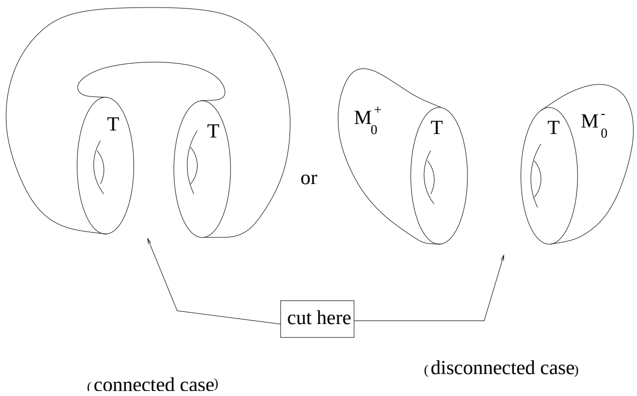


Fig. 3.14

In any case we have two boundary tori which are copies of T. By gluing the copies through suitable diffeomorphism we obtain a new manifold $\widehat{M}$ (in general $\neq M$).

Example: (Connected case) $M = S^1 \times T^2 = T^3$, the 3-torus and $T = 0 \times T^2$.

Cutting open M along T we get the manifold $I \times T^2$ with boundary tori $0 \times T^2$ and $1 \times T^2$ (here $I = [0,1]$). By gluing $0 \times T^2$ and $1 \times T^2$ with a diffeomorphism $\varphi : T^2 \to T^2$ we can get a manifold $\widehat{M}$ whose fundamental group may be non-abelian $\therefore \widehat{M} \neq M$. If now we suppose that M is equipped with a vector field X then we can obtain a new vector field $\widehat{X}$ in the surgered manifold $\widehat{M}$.

Example: M and T as before. $X =$ vector field $\dfrac{\partial}{\partial t}$ in $S^1 \times T^2 = T^3 = M$. If $\varphi : T^2 \to T^2$ (the gluing map) is Anosov, the $\widehat{X}$ is an Anosov flow. This procedure transforms X (which is not Anosov) into an Anosov flow $\widehat{X}$.

Example: Let $M = TS(1)$ the unitary tangent bundle on a negatively curved closed orientable surface S. Let g_t be its corresponding geodesic flow (g_t is Anosov). Pick a closed geodesic c in S.

Let $\pi : TS(1) \to S$ the projection induced by the fibration $TS \to S$. Hence $\pi^{-1}(c) = T$ is a torus in M. Note that c itself lift to a closed orbit $d = c'$ in T of $g_t \therefore T \not\pitchfork g_t$. Analogously $B = -c'$ is a closed orbit of g_t in T. The following result was proved by Handel and Thurston [61]:

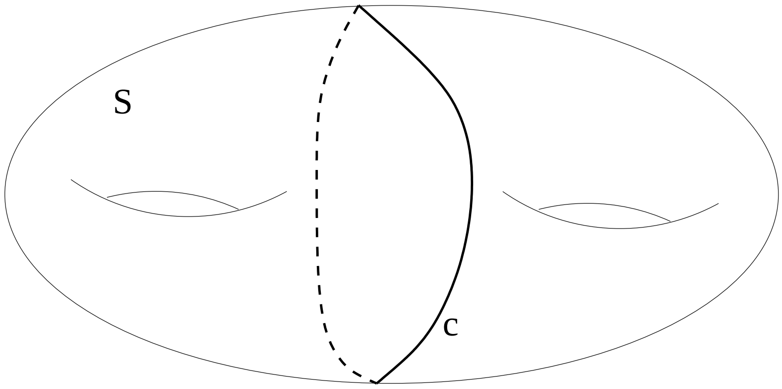


Fig. 3.15

Theorem 3.16. *There is a Dehn surgery on $M = TS(1)$ along $T = \pi^{-1}(c)$ which transforms the geodesic flow g_t (which is algebraic) into a transitive non-algebraic Anosov flow. In particular, there are transitive non-algebraic Anosov flows on certain closed* 3*-manifolds.*

More examples of this type are constructed as follows. Let M^0 be a closed 3-manifoled supporting a vector field X^0. Suppose that X^0 has an closed orbit A and an repelling closed orbit R both exhibiting solid tori neighborhoods ST^A, ST^R respectively. We can choose ST^A and ST^b such that X^0 points inward to ST^A and outward in ST^R. Removing from M^0 the solid tori ST^A, ST^R we get a manifold M^1 which is compact with boundary $\partial M^1 = T^A \cup T^R$ (disjoint union). Moreover, M^1 is equipped with a vector field X^1 (equals to X^0 in $M^1 \hookrightarrow M^0$) which points inward (resp. outward) to M^1 in T^R (resp. T^A)

Let $F : T^A \to T^R$ be a diffeomorphism (we call it *gluding map*). Let $M(F)$ be the manifold obtain from M^1 by identifying $X \in T^A$ with $F(x) \in T^R$.

By a simple modification we can see that the gluing map F induce a vector field XF for which the torus $T = T^A \simeq T^R$ is transverse. The following question is natural

Question: Is there a gluing map F such that XF is Anosov in $M(F)$?

The following necessary conditions for a positive answer hold:

(1) X^0 is non-singular Axiom A flow.

(2) Every closed orbit of X^0 different from A, R is saddle type (i.e. $(\dim E^u = \dim E^s = 1)$).

These conditions do not suffice for the existence of F such that XF is Anosov.

By (1) we have a spectral decomposition

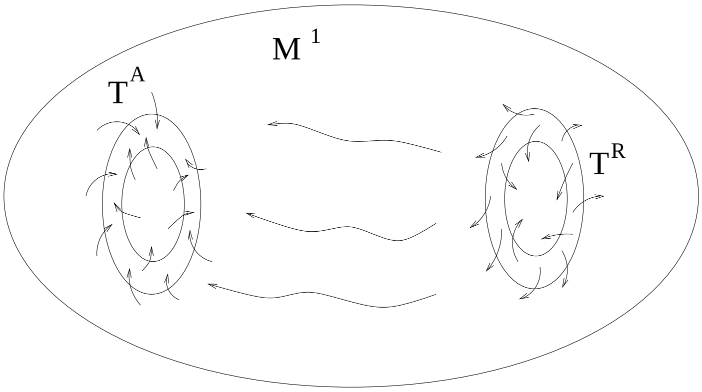


Fig. 3.16

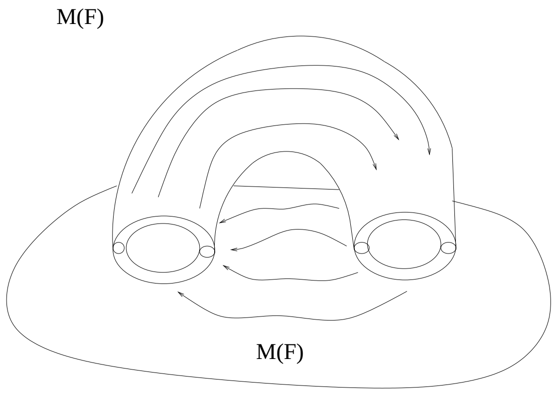


Fig. 3.17

$$\Omega(X^0) = A \cup R \cup \Lambda_1 \cup \cdots \cup \Lambda_R$$

where the Λ_i's are basic sets of X^0 *inside* M^1. Clearly such basic sets are basic sets of XF as well.

Problem: Is it true that if XF is Anosov and $k = 1$, then Λ_1 reduces to a closed orbit?

One can prove that if $k = 1$ and XF is Anosov, then XF is transitive. See [104] for a proof when Λ_1 reduces to a closed orbit. If $k = 0$ (i.e. $\Omega(X^0) = A \cup R$) it is possible to find M^0, X^0 and F such that the resulting flow XF is Anosov. On the other hand, there are examples with $\boxed{k = 1}$ where no gluing map F leads to XF Anosov. If $k = 1$, and Λ_k reduces to a closed orbit, then it is possible to find an "Anosov gluing map F" This was used by Bonatti and Langevin to prove the following

Theorem 3.17. *There are transitive Anosov flows with a transverse torus on certain closed 3-manifolds which are not suspended.*

Proof. Let us consider a vector field X^0 on the solid torus M^0 whose phase portrait is as in Figure 3.18.

The following "elements" are indicates in this figure:

$$\begin{cases} \bullet A^T = \text{top annulus in } \partial M^0 \\ \bullet A^B = \text{bottom annulus in } \partial M^0 \\ \bullet A^L = \text{left annulus in } \partial M^0 \\ \bullet A^R = \text{right annulus in } \partial M^0 \end{cases}$$

$$\begin{cases} \bullet \theta = \text{hyperbolic saddle-type periodic orbit of } X^0 \\ \qquad \text{(the core of } M^0\text{)} \\ \bullet \ell^T, \ell^B = \text{closed curves in } A^T, A^B \text{ contained in the local } \textit{stable} \\ \qquad \text{manifold } W^s_{\text{loc}}(\theta) \text{ of } \theta \text{ (resp.)} \\ \bullet \ell^L, \ell^R = \text{closed curves in } A^L, A^R \text{ contained in the local } \textit{unstable} \\ \qquad \text{manifold } W^u_{\text{loc}}(\theta) \text{ of } \theta \text{ (resp.)} \\ \bullet \mathscr{F}^{T,R} = \text{radial foliation in } A^T \\ \bullet \mathscr{F}^{B,R} = \text{radial foliation in } A^B. \end{cases}$$

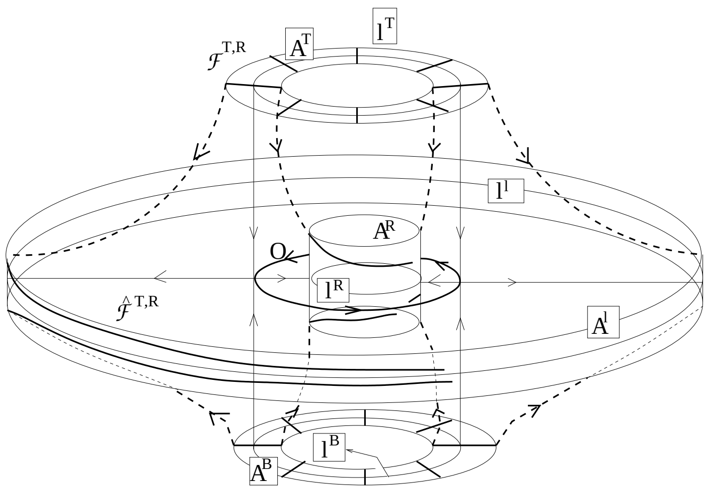


Fig. 3.18

We require the following properties:

(1) $(A^T \cup A^B) \pitchfork X^0$ (pointing inward to M_0)
and $(A^L \cup A^R) \pitchfork X^0$ (pointing outward to M_0)
(2) There is a holonomy map
$\pi:\ \mathrm{Dom}(\pi) = (A^T \cup A^B)\backslash(\ell^T \cup \ell^B) \to A^L \cup A^R$
induced by X^0 with image $\Im(\pi) = (A^L \cup A^R)\backslash(\ell^L \cup \ell^R)$.
Let $\widehat{\mathscr{F}}^{T,R}$ $(\widehat{\mathscr{F}}^{B,R})$ be the image by π of the radial foliation $\mathscr{F}^{T,R}$ $(\mathscr{F}^{B,R})$.
(3) $\widehat{\mathscr{F}}^{T,R}$ $(\widehat{\mathscr{F}}^{B,R})$ spirals toward $\ell^R \cup \ell^L$ as indicates in Figure 1.
(4) The leaves of $\widehat{\mathscr{F}}^{T,R}$ $(\widehat{\mathscr{F}}^{B,R})$ are almost parallel to the circles concentric to $\ell^R \cup \ell^L$.
(5) The holonomy π expands the leaves of $\mathscr{F}^{B,R} \cup \mathscr{F}^{T,R}$.

Figure 2 below give a better description of $\widehat{\mathscr{F}}^{T,R}$, $\widehat{\mathscr{F}}^{B,R}$:

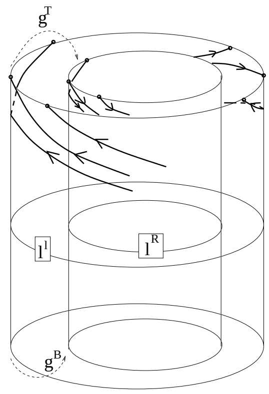


Fig. 3.19

The arrows indicate an orientation of the leaves in the radial foliations.

We glue the top (Bottom) boundary curves of A^L and A^R with an *Orientation-Reversing* map g_T (g_B) to obtain a torus which we denote by T^{out}

The foliations $\widehat{\mathscr{F}}^{T,R}$ $\widehat{\mathscr{F}}^{B,R}$ induce a foliaiton in T^{out} whose leaves spiral toward ℓ^R, ℓ^L as indicated in Figure 3 below

By the hypothesis (4) we have that the image's foliaiton depicted in Figure 3 is close to the parallel circles in T^{out}. Analogously we glue the boundary curve of A^T, A^B to obtain a torus T^{in}. Such a gluing is done in a way that the vector field X^0 induces a vector field X^1 in the esulting manifoled M^1, X^1 then have two transverse tori T^{in} (pointing inward) and T^{out} (pointing outward). By construction X^1 is Morse-Smale with $\Omega(X^1) = \theta$. In addition, there is a holonomy $\pi:\ T^{\mathrm{in}}\backslash(\ell^T \cup \ell^B) \to T^{\mathrm{out}}\backslash(\ell^L \cup \ell^R)$ carrying meridians to almost parallel curves in an expanding way. By gluing T^{in} and T^{out} with a map sending parallels in T^{out} have to meridians in T^{in} we get an Anosov flow X with the following properties:

(1) X has a transverse torus $T \simeq T^{\mathrm{out}} \simeq T^{\mathrm{in}}$

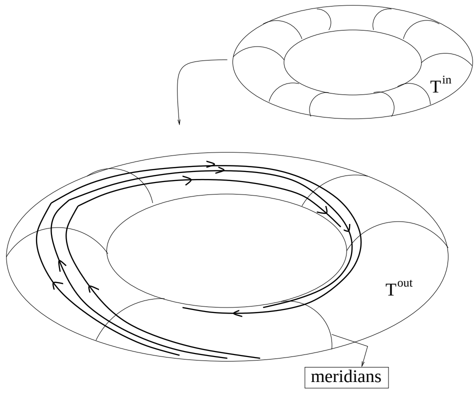


Fig. 3.20

(2)There is only one orbit of X which does not intersect T.

These properties imply that X is *transitive* and *non-suspended*. □

Remark 3.18. *Barbot used the name "Anosov BL-flows" for the example above. We shall see later along the course that Anosov BL-flows are not geodesic either. There are no Anosov BL-flows on closed* 4*-manifolds (see [37]).*

Corollary 3.19. *The Anosov BL-flow above is not algebraic.*

Proof. It follows from the Tomter's Theorem that any algebraic Anosov flow is (up to finite covering) either geodesic or suspended (see Example (4)). □

3.3 Definition of sectional-Anosov flows

We define sectional-Anosov flows.

Definition 3.20. *A* sectional-Anosov flow *is a vector field for which the maximal invariant set is a sectional-hyperbolic set.*

3.4 Examples

In this section we present examples of sectional-Anosov flows.

3.4.1 The geometric Lorenz attractor

In this section we give a presentation of a geometric Lorenz attractor which differs from the classical ones (e.g. Subsection 2.3.2 p. 71 of [6]). The construction is done through three steps the first of which consisting of the choice of certain constants.

In fact, we claim that for every $0 < c$, $0 < \beta < 1$ and $0 < \Delta < 1$ there is $\alpha^* > 0$ such that if $\alpha > \alpha^*$, then

$$\left(\frac{\sqrt{2}}{\beta}\right)^{\frac{1}{\beta-1}} < \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \qquad \text{and} \qquad \left(\frac{\alpha+c\Delta}{\beta}\right)\Delta^{(\alpha-\beta)} \leq \frac{1}{2}\cdot c. \tag{3.10}$$

Indeed, since $0 < \beta < 1$ we have $\beta < \sqrt{2}$ and then $\left(\frac{\sqrt{2}}{\beta}\right)^{\frac{1}{\beta-1}} < 1$. In addition $\left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \to 1$ as $\alpha \to \infty$ so there is $\alpha_0 > 0$ such that

$$\alpha > \alpha_0 \quad \Longrightarrow \quad \left(\frac{\sqrt{2}}{\beta}\right)^{\frac{1}{\beta-1}} < \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}.$$

On the other hand, since $0 < \Delta < 1$, β and c are fixed we have

$$\lim_{\alpha\to\infty}\left(\frac{\alpha}{c}+\Delta\right)\Delta^{(\alpha-\beta)} = 0$$

Therefore, since $\beta > 0$ there is $\alpha_1 > 0$ such that

$$\alpha > \alpha_1 \quad \Longrightarrow \quad \left(\frac{\alpha}{c}+\Delta\right)\Delta^{(\alpha-\beta)} \leq \frac{1}{2}\cdot\beta \quad \Longrightarrow \quad \left(\frac{\alpha+c\Delta}{\beta}\right)\Delta^{(\alpha-\beta)} \leq \frac{1}{2}\cdot c.$$

Then, $\alpha^* = \max\{\alpha_0, \alpha_1\}$ works.

To state the lemma below we need some short definitions. Given $\Delta > 0$ we define

$$\Sigma_\Delta = \{(x,y) \in \mathbb{R}^2 : |x| \leq \Delta, |y| \leq 1\} \quad \text{ and } \quad \Sigma^*_\Delta = \Sigma \setminus \{x = 0\}.$$

If additionally $\alpha, \beta, b \in \mathbb{R}$ we define the map $F_{\Delta,\alpha,\beta,b} : \Sigma^*_\Delta \to \mathbb{R}^2$ by

$$F_{\Delta,\alpha,\beta,b}(x,y) = \begin{cases} (|x|^\beta - \Delta, y|x|^\alpha + b) & \text{if } x > 0 \\ (\Delta - |x|^\beta, y|x|^\alpha - b) & \text{if } x < 0. \end{cases} \tag{3.11}$$

It is clear that $F_{\Delta,\alpha,\beta,b}$ is C^∞.

Given $c > 0$ we define the cone field $C_c^\Delta = \{C_c^\Delta(p) : p \in \Sigma\}$ by

$$C_c^\Delta(p) = \left\{ (u,v) \in T_p\Sigma_\Delta : \frac{|v|}{|u|} \le c \right\}.$$

We denote by $\|(u,v)\| = \max\{|u|,|v|\}$ the maximun norm in $T\Sigma_\Delta$.

Lemma 3.1. *Let $0 < c < 1$, $\alpha > 0$, $\Delta > 0$ and $\frac{\sqrt{2}}{2} < \beta < 1$ be satisfying*

$$2^{\frac{1}{\beta-1}} < \Delta < \min\left\{ \left(\frac{\sqrt{2}}{\beta}\right)^{\frac{1}{\beta-1}}, \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \right\} \qquad \text{and} \qquad \Delta^\alpha < 1 - \Delta^\alpha. \tag{3.12}$$

Then, $F_{\Delta,\alpha,\beta,b}$ satisfies the following properties for all $b \in (\Delta^\alpha, 1-\Delta^\alpha)$:

1. *$F_{\Delta,\alpha,\beta,b}$ is injective and $F_{\Delta,\alpha,\beta,b}(\Sigma_\Delta^*) \subset \Sigma_\Delta$.*
2. *$F_{\Delta,\alpha,\beta,b}$ preserves and contracts the foliation $\mathscr{F}$ in Σ_Δ whose leaves are the vertical straight lines.*
3. *If $p \in \Sigma_\Delta^*$, then $DF_{\Delta,\alpha,\beta,b}(p)(C_c(p)) \subset C_{\frac{1}{2}\cdot c}(F_{\Delta,\alpha,\beta,b}(p))$.*
4. *The lateral limits $\lim_{x\to 0^+} F_{\Delta,\alpha,\beta,b}(x,y)$ and $\lim_{x\to 0^-} F_{\Delta,\alpha,\beta,b}(x,y)$ exist, do not depent on y and belong to $\{x = -\Delta\}$ and $\{x = \Delta\}$ respectively.*
5. *If additionally*
$$\left(\frac{\alpha + c\Delta}{\beta}\right)\Delta^{(\alpha-\beta)} \le \frac{1}{2}\cdot c, \tag{3.13}$$
then there is a constant $\lambda > \sqrt{2}$ such that $\|DF_{\Delta,\alpha,\beta,b}(p)\cdot w\| \ge \lambda \cdot \|w\|$ for all $p \in \Sigma_\Delta^*$ and $w \in C_c(p)$.

Proof. We only prove these properties in the region $x > 0$ for the proof in $x < 0$ is analogous. For simplicity we write $\Sigma = \Sigma_\Delta$, $\Sigma^* = \Sigma_\Delta^*$ and $F = F_{\Delta,\alpha,\beta,b}$.

Proof of (1). It follows directly from (3.11) that F is injective. To prove $F(\Sigma^*) \subset \Sigma$ we need to verify the inequalities $|x^\beta - \Delta| \le \Delta$ and $|yx^\alpha + b| \le 1$. The first one is clearly equivalent to $0 \le x^\beta \le 2\Delta$. Since $x^\beta \le \Delta^\beta$ we have it as soon as $\Delta^\beta < 2\Delta$ which in turns follows from the first inequality of (3.12). For the second one we notice that $-\Delta^\alpha + b \le yx^\alpha + b \le \Delta^\alpha + b$. Since $\Delta^\alpha < b < 1 - \Delta^\alpha$ we have $0 < yx^\alpha + b < 1$ yielding the result.

Proof of (2). We first observe that F preserves $\mathscr{F}$ because of (3.11). In addition,

$$\|DF(0,v)\| = \|(0,x^\alpha v)\| \le \Delta^\alpha \|(0,v)\|$$

and certainly $\Delta^\alpha < 1/2$ because of the first inequality in (3.12).

Proof of (3). Notice that if $p = (x,y) \in \Sigma$ and $(u,v) \in T_p\Sigma$ then

$$\|DF(u,v)\| = \|(\overline{u},\overline{v})\| = \max\left\{1, \frac{|\overline{v}|}{|\overline{u}|}\right\} \cdot |\overline{u}| \ge |\overline{u}| = \beta x^{\beta-1}|u| \ge \beta\Delta^{\beta-1}|u|.$$

Now take $\lambda = \beta\Delta^{\beta-1}$. By (3.12) and $0 < \beta < 1$ we have

$$\Delta < \left(\frac{\sqrt{2}}{\beta}\right)^{\frac{1}{\beta-1}} \Longrightarrow \Delta^{-1} > \left(\frac{\sqrt{2}}{\beta}\right)^{\frac{1}{1-\beta}} \Longrightarrow \Delta^{\beta-1} > \frac{\sqrt{2}}{\beta} \Longrightarrow \beta\Delta^{\beta-1} > \sqrt{2}$$

proving $\lambda > \sqrt{2}$. But if $(u,v) \in C_c(p)$ we have from $c < 1$ that $\|(u,v)\| = |u|$ whence

$$\|DF(p)\cdot(u,v)\| \geq \lambda \cdot \|(u,v)\|.$$

Proof of (4). Just compute the lateral limits using (3.11).

Proof of (5). We notice that according to (3.11) the expression of DF at some tangent vector (u,v) of (x,y) is given by

$$DF(u,v) = \begin{cases} (\beta\,|x|^{\beta-1}u, \alpha y|x|^{\alpha-1}u + |x|^{\alpha}v) & \text{if } x > 0 \\ (-\beta\,|x|^{\beta-1}u, \alpha y|x|^{\alpha-1}u - |x|^{\alpha}v) & \text{if } x < 0. \end{cases} \tag{3.14}$$

Setting $F(u,v) = (\overline{u},\overline{v})$ we have for $(u,v) \in C$ that

$$\frac{|\overline{v}|}{|\overline{u}|} = \left|\frac{\alpha y x^{\alpha-1}u + x^{\alpha}v}{\beta x^{\beta-1}u}\right| = \left|\frac{\alpha}{\beta}yx^{(\alpha-\beta)} + \frac{1}{\beta}x^{(\alpha-\beta+1)}\frac{v}{u}\right| \leq$$

$$\frac{\alpha}{\beta}\Delta^{(\alpha-\beta)} + \frac{c}{\beta}\Delta^{(\alpha-\beta+1)} = \left(\frac{\alpha + c\Delta}{\beta}\right)\Delta^{(\alpha-\beta)}.$$

Now (3.13) applies. □ □

Denoting $\Sigma = \Sigma_\Delta$ and $\Sigma^* = \Sigma^*_\Delta$ we have that for $F = F_{\Delta,\alpha,\beta,b}$ as in (3.11) we can define the compact set sequence $\Lambda_n(F)$ inductively by $\Lambda_0(F) = \Sigma$ and $\Lambda_n(F) = Cl(F(\Lambda_{n-1}(F) \cap \Sigma^*))$ for $n \geq 1$. We define the attracting set of F by

$$\Lambda(F) = \bigcap_{n \geq 0} \Lambda_n(F).$$

The following corollary is a direct consequence of Theorem 1.48 in Appendix 1.8.

Corollary 3.21. *If F as in (3.11) satisfies (1) to (5) of Lemma 3.1, then $\Lambda(F)$ is a homoclinic class of F.*

By the *solid bitorus* we mean the handlebody of genus 2 (see Subsection 3.4.4).

Theorem 3.22. *There is a C^∞ suspended sectional-Anosov flow in the solid bitorus which is both C^r robustly transitive and C^r robustly periodic for all $r \geq 1$.*

Proof. Fix $0 < c < 1$ and $\frac{\sqrt{2}}{2} < \beta < 1$. Then,

$$2 > \frac{\sqrt{2}}{\beta} \implies 2^{\frac{1}{\beta-1}} < \left(\frac{\sqrt{2}}{\beta}\right)^{\frac{1}{\beta-1}}$$

so we can fix Δ satisfying

$$2^{\frac{1}{\beta-1}} < \Delta < \left(\frac{\sqrt{2}}{\beta}\right)^{\frac{1}{\beta-1}}.$$

In particular, $0 < \Delta < 1$. For such c, β, Δ we pick α^* such that (3.12) and (3.13) hold for all $\alpha > \alpha^*$. We fix $\alpha > \alpha^*$. In particular

$$\Delta < \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \implies \Delta^{\alpha} < \frac{1}{2} \implies \Delta^{\alpha} < 1 - \Delta^{\alpha}$$

therefore we can fix $b \in (\Delta^{\alpha}, 1 - \Delta^{\alpha})$. Hereafter we write $\Sigma = \Sigma_{\Delta}$, Σ^*_{Δ} and $F = F_{\Delta,\alpha,\beta,b}$.

Now take three real numbers $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$\lambda_2 < \lambda_3 < 0 < \lambda_1, \quad \beta = \frac{-\lambda_3}{\lambda_1} \quad \text{and} \quad \alpha = \frac{-\lambda_2}{\lambda_1}.$$

Consider the vector field X in $\mathbb{R}^3$ represented by the following ODE

$$\begin{cases} \dot{x} = \lambda_1 x \\ \dot{y} = \lambda_2 y \\ \dot{z} = \lambda_3 z. \end{cases} \tag{3.15}$$

Solving it we get the flow

$$X_t(x,y,z) = (xe^{\lambda_1 t}, ye^{\lambda_2 t}, ze^{\lambda_3 t}), \qquad \forall (x,y,z).$$

We shall use the identification $\Sigma \approx \{(x,y,1) : (x,y) \in \Sigma\}$ and fix the cross-sections $\Sigma^+ = \{x = 1\}$ and $\Sigma^- = \{x = -1\}$. Using the flow above we have the holonomy map $\Pi_{loc}(x,y) = X_{t_0}(x,y,1)$ from Σ^* to $\Sigma^- \cup \Sigma^+$ where t_0 is the flight time. Thus, $|x|e^{\lambda_1 t_0} = \pm 1$ hence $t_0 = \ln |x|^{-\frac{1}{\lambda_1}}$ which yields

$$\Pi_{loc}(x,y) = (y|x|^{\alpha}, |x|^{\beta}).$$

Next we introduce a global vector field outside the cube $[-1,1]^3$ whose flow carries a neighborhood of $\Pi_{loc}(\Sigma^*)$ in $\Sigma^- \cup \Sigma^+$ back into the section $\{z = 1\}$. A customary (but certainly not unique) way to do it is described in Figure 3.21-(a).

The resulting vector field L is then equipped with a second holonomy map $\Pi_{far} : \sigma^- \cup \Sigma^+ \to \Sigma$ for which the following formula is assumed:

$$\Pi_{far}(y,z) = \begin{cases} (z - \Delta, y + b) & \text{if} \quad (y,z) \in \Sigma^+ \\ (\Delta - z, y - b) & \text{if} \quad (y,z) \in \Sigma^-. \end{cases} \tag{3.16}$$

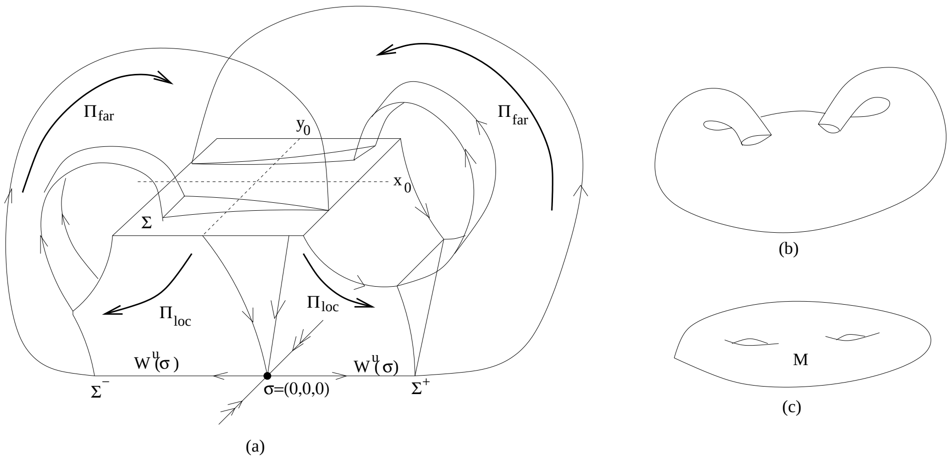


Fig. 3.21 Construction of L.

Then, there is a return map $\Pi_{far} \circ \Pi_{loc} : \Sigma^* \to \Sigma$ and it follows from (3.15) and (3.16) that $F = \Pi_{far} \circ \Pi_{loc}$. So, by Lemma 3.1 and Corollary 3.21, the attracting set $\Lambda(F)$ of F is a homoclinic class of F.

Now we define the attracting set of L,

$$A_L = W^u(\sigma) \cup \left(\bigcup_{t \geq 0} L_t(\Lambda(F)) \right),$$

where $\sigma = (0,0,0)$ is the equilibrium point of (3.15) (see Figure 3.21-(a)).

Since $\Lambda(F)$ is a homoclinic class of F which is a return map of L we have that A_L is a homoclinic class of L, and so, A_L is transitive with dense periodic orbits. It is not difficult to obtain a solid bitorus M in a way that $L \in \mathscr{X}^\infty(M)$ satisfying $A_L = M(L)$. This can be done by deforming Figure 3.21-(a) first into Figure 3.21-(b) and then into Figure 3.21-(c).

Now we prove that L is a sectional-Anosov flow. Since $M(L) = A_L$ we need to find a sectional-hyperbolic splitting $T_{A_L}M = E^s_{A_L} \oplus E^c_{A_L}$. We define E^s_p for $p \in A_L$ at once as the subbundle parallel to the y-axis (which corresponds to the eigenspace associated to λ_2). To define E^c_p we consider two cases, namely, either the negative orbit of p intersects Σ infinitely many times or not. In the first case we take t_p as the first non-negative number satisfying $L_{-t_p}(p) \in \Sigma$. Since the negative orbit intersects Σ infinitely many times we can arrange a sequence x_n with $x_0 = L_{-t_p}(p)$ in a way that $F^n(x_n) = x_0$. Then, we set

$$E^c_p = DL_{t_p}(x_0) \left(E^L_{x_0} \oplus \hat{E}^c_{x_0} \right),$$

where $C(x) = C \cap T_x\Sigma$ for all $x \in \Sigma$ and

$$\hat{E}^c_{x_0} = \bigcap_{n\geq 0} DF^n(C(x_n)).$$

Figure 3.22-(a) explain this construction.

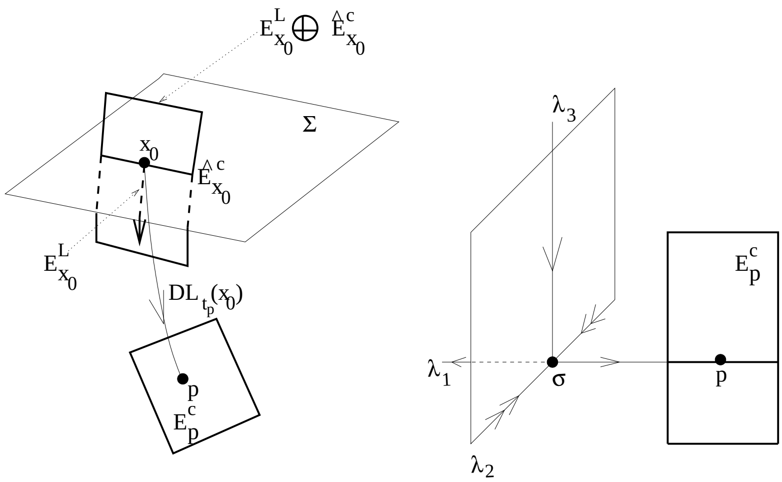


Fig. 3.22 Construction of E^c_p.

If the negative orbit of p intersects Σ finitely many times only we necessarily have $p \in W^u(\sigma)$. For such points we define E^c_p as the unique plane of T_pM containing E^L_p such that $DL_{-t}(p)(E^c_p)$ converges to the eigenspace of $T_\sigma M$ generated by the eigenvalues λ_1, λ_3 of $DL(\sigma)$. This is explained in Figure 3.22-(b).

The contraction of $E^s_{A_L}$ follows because any positive orbit spends a finite amount of time far from σ, and close to σ the contraction is clear since $E^s_{A_L}$ keeps parallel to the eigenspace associated to the negative eigenvalue λ_2.

The dominance of $E^s_{A_L}$ over $E^c_{A_L}$ is obtained from $\lambda_2 < \lambda_3 < 0 < \lambda_1$.

The sectional expansivity of $E^c_{A_L}$ is obtained depending on whether the positive orbit of p passes either far from σ or close to σ. In the first case we get sectional expansivity because DF expands the vectors in C and, in the second, we use the eigenvalue relation $-\lambda_3 < \lambda_1$ which follows from $\beta < 1$.

To prove that L is C^r robustly transitive and C^r robustly periodic for all $r \geq 1$ we just observe that for every C^r vector field Y that is C^r close to L there are continuations σ_Y and F_Y of the singularity σ and the return map F which satisfy the same properties of σ and F. Inparticular, the attracting set $\Lambda(F_Y)$ is still a homoclinic class of F_Y. It follows also that $M(Y) = A_Y$ where $A_Y = W^u(\sigma_Y) \cup \left(\bigcup_{t\geq 0} Y_t(\Lambda(F_Y))\right)$ thus Y is both transitive with dense periodic orbits. This ends the proof. □ □

The *geometric Lorenz attractor* is precisely the sectional-Anosov flow in the solid torus in Theorem 3.22.

3.4.2 The annular attractor

Next we present the *annular attractor* introduced in [116]. The motivation for this example is Corollary 4.10 which proves that every C^∞ transitive Anosov flow on a compact manifold is both C^r robustly transitive and C^r robustly periodic for all $r \geq 1$. The following result proved in shows that this is false for general sectional-Anosov flows.

Theorem 3.23. *There are C^∞ transitive suspended sectional-Anosov flows with dense periodic orbits on certain compact manifolds which are neither C^r robustly transitive nor C^r robustly periodic for all $r \geq 1$.*

Proof. Consider a two-dimensional annulus A and let X be the vector field whose flow is described in Figure 3.23-(a). Notice that there is a return map Π from $A \setminus l$ into A where l is a curve in the stable manifold of the singularity σ. It turns out that Π preserves and contracts the radial foliation in A and, moreover, it preserves and expands a cone field around the angular curves in A. The image $\Pi(A)$ of A is described in Figure 3.23.

One can be proved as in Theorem 1.48 that the attracting set of Π, $\bigcap_{n\geq 0} \Lambda_n(\Pi)$, where $\Lambda_n(\Pi) = A$ (for $n = 0$) or $Cl(\Pi^n(\Lambda_{n-1}(F) \setminus l))$ (for $n \geq 1$) is a homoclinic class of Π. From this we see that there is a compact 3-manifold with boundary M such that $X \in \mathscr{X}^\infty(M)$ is a transitive sectional-Anosov flow with dense periodic orbits. To prove that it is neither C^r robustly transitive nor C^r robustly periodic we consider the C^∞ perturbation of X whose return maps on A correspond to the botton picture in Figure 3.23-(b). It turns out that such a flow neither is transitive nor has dense periodic orbits for the curve C there belongs to the maximal invariant set (for it is part of the unstable manifold of the periodic orbit O in Figure 3.23-(a)) but not in the non-wandering set (for the nonwandering set is inside the cross section Σ in Figure 3.23-(b)). □

As a final remark let us mention that the compact 3-manifold supporting the sectional-Anosov flow in Theorem 3.23 is *not* the solid bitorus (although it is bounded by a bitorus). Indeed, such a manifold is nothing but $(T^2 \setminus D) \times [0,1]$ where T^2 is the two-dimensional torus and D is a two-disk in T^2.

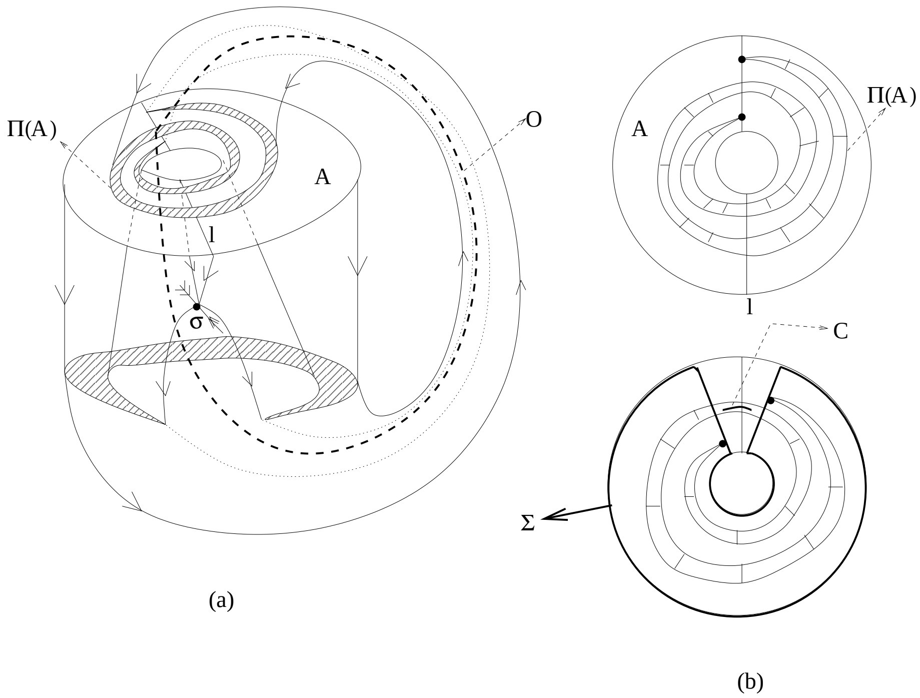


Fig. 3.23 Annular attractor.

3.4.3 Venice masks

It follows from Corollary 2.6 that every Anosov flow with dense periodic orbits on a *closed* manifold is transitive (recall that transitivity of X means that the maximal invariant set $M(X)$ is transitive). This fact is not true for sectional-Anosov flows due to the following example.

Theorem 3.24. *There are sectional-Anosov flows with dense periodic orbits on certain compact manifolds which are not transitive.*

A flow with three singularities satisfying the conclusion of this theorem is described in Figure 3.24.

One with a unique singularity can be obtained from the dual of Figure 3.25 (for a more accurate construction see [22]). Notice that this example of sectional-Anosov flow is suspended (indeed the double of the cross section Σ in Figure 3.25 is a global cross section of the flow). From this example we obtain that not every codimension one suspended sectional-Anosov flow on a compact manifold is transitive (as is

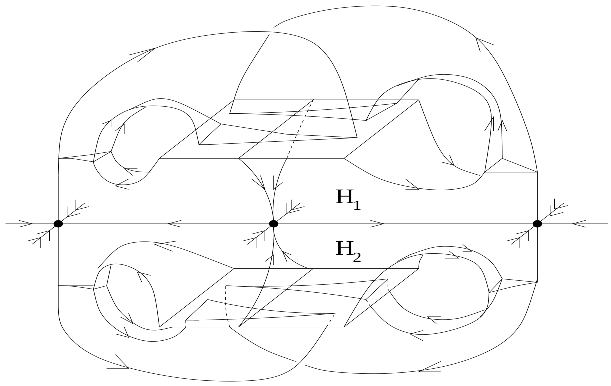


Fig. 3.24 A venice mask with three singularities

well known all suspended codimension one Anosov flows on closed manifolds are transitive).

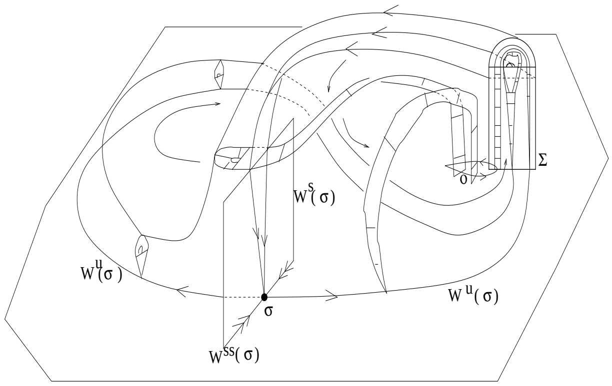


Fig. 3.25 A venice mask with a unique singularity

Motivated by Theorem 3.24 we introduce the following definition which will be analyzed later one.

Definition 3.25. *A* venice mask*([1]) is a sectional-Anosov flow with dense periodic orbits which is not transitive.*

It follows from this definition that all venice masks on compact manifolds have at least one singularity, and so, they do not exist on closed manifolds. We shall describe the dynamics and the perturbation of venice masks with only one singularity on compact 3-manifolds in chapters 6.2 and 6.3 respectively.

3.4.4 Sectional-Anosov flows on handlebodies

Here we present the results from [98].

A *handlebody of genus* $n \in \mathbb{N}$ (or cube with n-handles) is a compact 3-manifold with boundary V containing a disjoint collection of n properly embedded 2-cells such that the result of cutting V along these disks is a 3-cell ([66] p. 15). For example, the orientable Handlebodies of genus 0 and 1 are precisely the 3-ball and the solid torus respectively.

It possible to prove that a handlebody of genus ≤ 1 cannot support transitive sectional-Anosov flows. On the other hand, as alreadys saw in Theorem 3.22, the handlebody of genus 2 supports transitive sectional-Anosov flows. These remarks motivate the following result.

Theorem 3.26. *Every orientable Handlebody of genus $n \geq 2$ supports a transitive sectional-Anosov flow.*

Proof. The proof uses the following definition. A map $f : [0,1] \to [0,1]$ is called *n-Lorenz map* if there are $n \in \mathbb{N}$ and $C = \{c_0, \cdots, c_n\}$ with $0 < c_n < c_{n-1}, \cdots, c_1 < c_0 = 1/2$ such that the following properties hold.

1. C is the set of discontinuity points of f.
2. $\lim_{x \to c_i^-} f(x) = 1$ for all $i = \{1, \cdots, n\}$.
3. $\lim_{x \to c_i^+} f(x) = r_i^+$ exist for all $i = \{1, \cdots, n\}$.
4. f is C^1 on $[0,1] \setminus C$ and there is $\lambda > \sqrt{2}$ such that $f'(x) \geq \lambda$ if $x \in [0,1] \setminus C$.

In Figure 3.26 we describe the graph of f.

The n-Lorenz map is a direct generalization of the classical Lorenz maps in the interval $[0,1]$ (which corresponds to 0-Lorenz maps). Note that a n-Lorenz map has just $n+1$ discontinuity points. For every n-Lorenz map f we consider the $(n+1)$-vector $(f(0), r_1^+, \cdots, r_n^+)$. The *norm* of f is the Euclidian norm of such a vector. As usual we say that f is *transitive* if it has a dense forward orbit. Following the classical Guckenheimer-Williams argument for transitivity of Lorenz maps [60] it can be proved that every n-Lorenz map with sufficiently small norm is transitive (see also the proof of Theorem 1.48).

Next we describe a deformation first introduced in [120]. Start with the classical Cherry flow in the torus described in [87]. We cut open the torus along the trajectory in the stable manifold of the Cherry flow's singularity in order to obtain the small piece of the flow described below:

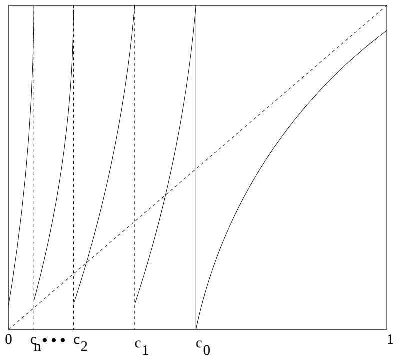


Fig. 3.26 *n*-Lorenz map

Fig. 3.27 Small piece of the Cherry flow

This piece is nothing but a rectangle equipped with the flow described in the figure. By increasing a strong contracting subbundle to this flow we obtain the three-dimensional flow in Figure 3.28-(b) that will be refered to as a *Cherry flow box*. It turns out that a Cherry flow-box has two singularities of saddle type: one with one-dimensional unstable manifold (say s_1) and another with two-dimensional stable manifold (say s_2). We can construct a Cherry flow box by deforming a tubular flow box around a regular orbit path R which we call *regular flow box* in Figure 3.28-(a). From any Cherry flow box we obtain a solid bitorus by just removing the cylinder-like 3-ball B centered at s_2 in Figure 3.28-(b).

Denote by X^2 the sectional-Anosov flow on the solid bitorus ST_2 given by the geometric Lorenz attractor described in Figure 3.29-(a). We deform this flow by inserting a Cherry flow box as described in Figure 3.29.

More precisely, we choose a path R contained in a regular orbit of X^2 as indicated in Figure 3.29-(a). By deforming a regular flow box around an orbit path R by a Cherry flow box we get a vector field as described in Figure 3.29-(b). We note that such a deformation produces a positively invariant Handlebody ST_3 which is obtained by removing a 3-ball B centered at s_2 from the original ST_2. Afterward we twist the cusp points of the resulting vector field as explained in Figure 3.29-(b). This produce yields the vector field X^3 described in Figure 3.29-(c). We observe that

Fig. 3.28 (a) regular flow box. (b) Cherry flow box

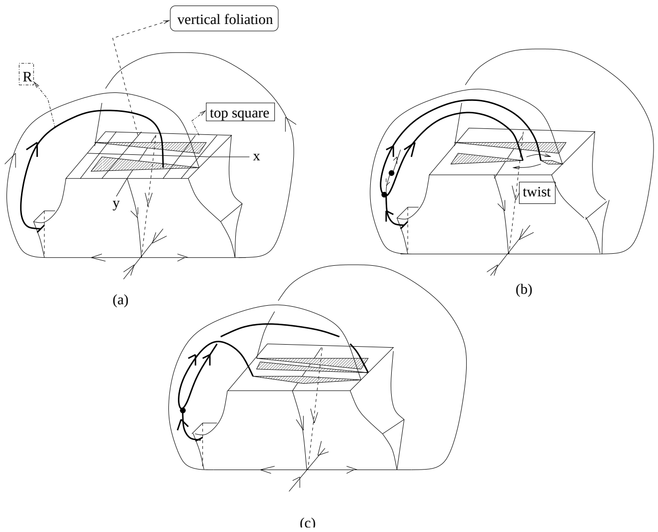


Fig. 3.29 Deforming the geometric Lorenz attractor

ST_3 is still a positively invariant compact neighborhood of X^3. The deformation is done in a way that the top square in Figure 3.29-(a) realizes as a cross section of X^3. The corresponding Poincaré map $\Pi(x,y)$ in the top square preserves the vertical foliation $\{x = cnt\}$, contracts such a foliation and induces a 2-Lorenz map with small norm in the verticals. In other words

$$\Pi(x,y) = (f(x), g(x,y)),$$

where f is a 2-Lorenz map with small norm and $\mid \partial_y g(x,y) \mid$ is uniformly small. We can show that the maximal invariant set $\bigcap_{t>0} X_t^3(ST_3)$ is sectional-hyperbolic exactly as in the Lorenz's case. Therefore X^3 is a sectional-Anosov flow on the Handlebody ST_3. By repeating this proceeding n times we get a sectional-Anosov flow with $n+1$ singularities X^{n+3} in the genus $n+2$ Handlebody ST_{n+3}. Indeed, for such flows, we still have that the same the top square is a cross section of X^3 but the return map now has the form $(f_n(x), g_n(x,y))$ where f_n is a n-Lorenz-like map with small norm and $|\partial_y g|$ is small. As already noticed f_n (and so X^{n+3}) are transitive. This finishes the proof. □ □

3.4.5 Sectional-Anosov flows without Lorenz-like singularities

We start with the following remark which was important in [21]:

Remark 3.27. *As we shall see later in Theorem 4.18 every singularity of a sectional-Anosov flow which is transitive or with dense periodic orbits on a compact 3-manifold is Lorenz-like. On the other hand, it would follow from Proposition 9.25-(1) in [28] that every singularity of a sectional-Anosov flow must be Lorenz-like. An analogous conclusion would follow from Lemma 3.45 p.153 of [6]. However, we can easily find examples of sectional-Anosov flows exhibiting non Lorenz-like singularities. Consequently the aforementioned proposition and lemma in [28] and [6] respectively are false.*

Since the counterexamples in the preceding remark also have Lorenz-like singularities we can ask if every sectional-Anosov flow with singularities on a compact 3-manifold has a Lorenz-like one. However, the answer is negative by the following result [97].

Theorem 3.28. *There is a compact* 3*-manifold supporting a sectional-Anosov flow with singularities none of which is Lorenz-like.*

Proof. Suspend the DA-diffeomorphism in the torus T^2 to obtain an Axiom A vector field X^0 on a closed manifold M^0. Its non-wandering set consists of a non-trivial hyperbolic attractor A and a repelling periodic orbit O. Pick a solid torus neighborhood U of O in M^0 such that the stable foliation of A intersects the boundary torus ∂U of U as depicted in Figure 3.30. Note that the intersection consists of two Reeb components. The core of the top component is denoted by C. Note that C is transverse to the intersection of the stable manifold of A with ∂U. Remove the interior of U from M^0 to obtain a compact manifold M^1 whose boundary is the torus ∂U. Denote by X^1 the restriction of X^0 to M^1. It turns out that X^1 is transverse to $\partial M^1 = \partial U$ pointing inward to M^1. All of this is similar to the proof of Theorem 3.14.

Next we consider the solid torus M^2 equipped with the vector field X^2 whose flow is depicted in Figure 3.31. In particular, X^2 has two hyperbolic singularities σ_1 (which is repelling) and σ_2 (which is a saddle with one-dimensional stable direction

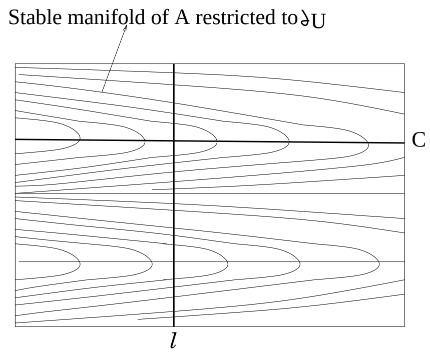


Fig. 3.30

$E^s_{\sigma_1}$). This vector field is also transverse to the torus boundary ∂M^2 of M^2 pointing outward. Note that there is a meridian curve C' in ∂M^2 which is the boundary of a meridian disk contained in the unstable manifold $W^u(\sigma_2)$ of σ_2.

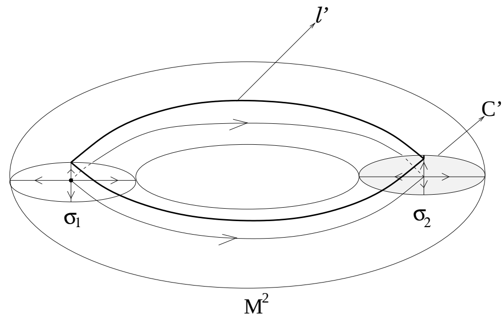


Fig. 3.31

As in [55] we perform a Dehn surgery between M^1 and M^2 by identifying $C \in \partial M^1$ with $C' \subset \partial M^2$. In this way we obtain a new 3-manifold M which is obviously closed. We can glue X^1 with X^2 to obtain a new vector field X in M because X^1 points inward to M^1 in ∂M^1 and X^2 points outward to M^2 in ∂M^2.

We claim that M and X satisfy the conclusion of Theorem 3.28. Indeed, define

$$A^* = W^u(\sigma_2) \cap A.$$

It follows from the construction that the nonwandering set of X is contained in the disjoint union of σ_1 and A^*. We have that A^* is attracting since we can obtain a positively invariant isolating block by just removing a small 3-ball centered at σ_1 whose S^2 boundary is transverse to X.

Now we prove that A^* is singular-hyperbolic. For this we need to find a dominated splitting $T_{A^*}M = E^s_{A^*} \oplus E^c_{A^*}$ so that $E^s_{A^*}$ is contracting and $E^c_{A^*}$ is volume expanding. To find $E^s_{A^*}$ we use the transversality between $C' \approx C$ and the stable manifold of A in ∂U. This transversality allows us to extend the stable direction to A to A^* via Inclination-lemma [87]. The resulting extension $E^s_{A^*}$ is continuous, contracting and contains $E^s_{\sigma_1}$. The central direction $E^c_{A^*}$ of A^* is defined to be $TW^u(\sigma_2)$ (in $W^u(\sigma_2)$) and the old stable direction of A (in A). The splitting $T_{A^*}M = E^s_{A^*} \oplus E^c_{A^*}$ is dominated by the Inclination-Lemma. The subbundle $E^c_{A^*}$ is volume expanding again by the Inclination-Lemma because A is hyperbolic and $C \approx C'$ is transverse to the stable foliation of A in ∂U. The proof follows since X has no Lorenz-like singularities. □ □

Remark 3.29. *Note the manifold M supporting the flow in Theorem 3.28 is not B^3. Indeed the meridian curve $l \subset \partial M_1 = \partial U$ in Figure 3.30, which is identified with the longitude curve $l' \subset \partial M_2$ in Figure 3.31, represents a non-trivial element of $\pi_1(M)$.*

3.4.6 Pathological examples

The purpose is to present the examples in [97] of sectional-Anosov flows on compact 3-manifolds whose maximal invariant sets accumulate on the singularities in a pathological way. The pathology will be described as follows.

Denote by $\mathscr{A}$ the set of (2×2)-matrices with entries in $\{0,1\}$, i.e.

$$\mathscr{A} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0,1\}, \forall i,j \in \{1,2\} \right\}$$

Denote by $F\colon \mathscr{A} \to \mathscr{A}$ (resp. $C\colon \mathscr{A} \to \mathscr{A}$) the bijection which interchanges the rows (resp. columns) of $A \in \mathscr{A}$, i.e.,

$$F\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} \quad \text{and} \quad C\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}.$$

Denote by $< F,G >$ the subgroup generated by $\{F,G\}$ in the group of bijections of $\mathscr{A}$. Then there is an action $< F,G > \times \mathscr{A} \to \mathscr{A}$. For simplicity *we identify $A \in \mathscr{A}$ with its own orbit*.

Consider a compact 3-manifold M and $X \in \mathscr{X}^1(M)$. Let σ be a Lorenz-like singularity of X with stable manifold $W^s(\sigma)$, unstable manifold $W^u(\sigma)$ and center-unstable manifold $W^{cu}(\sigma)$. The last manifold is divided by $W^u(\sigma)$ and $W^s(\sigma) \cap W^{cu}(\sigma)$ in the four *sectors* $s_{11}, s_{12}, s_{21}, s_{22}$ described in Figure 3.32. There is also a projection $\pi : V_\sigma \to W^{cu}(\sigma)$ defined in a neighborhood V_σ of σ via the strong stable foliation of Λ. We denote by $Cl(B)$ the closure of a subset B.

Definition 3.30. *We define*

$$M(\sigma) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{ij} = \begin{cases} 0 \text{ if } & \sigma \in Cl(\pi(M(X) \cap V_\sigma) \cap s_{ij}) \\ 1 \text{ if } & \sigma \notin Cl(\pi(M(X) \cap V_\sigma) \cap s_{ij}). \end{cases}$$

Define $\mathscr{M}(X) = \{M(\sigma) : \sigma \in \Lambda$ *is a Lorenz-like singularity of* $X\}$.

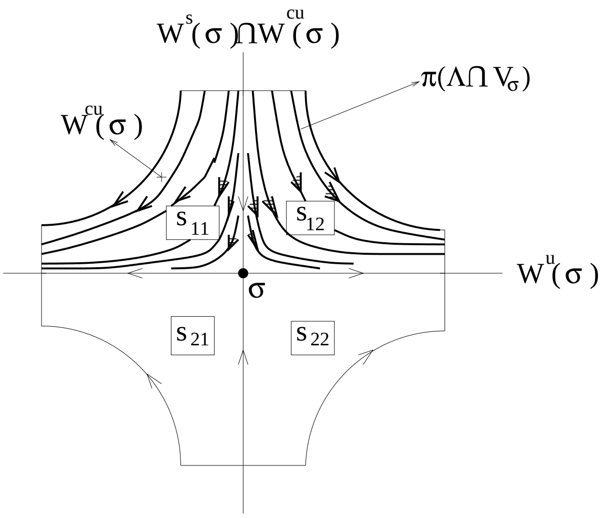


Fig. 3.32

Figure 3.32 describes an example with $M(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. It turns out that $M(\sigma)$ does not depend on the chosen center-unstable manifold $W^{cu}(\sigma)$.

We have that $\mathscr{M}(X)$ is not defined if X has no Lorenz-like singularities (c.f. Theorem 3.28), but it does for transitive sectional-Anosov flows with singularities. In the case of the geometric Lorenz attractor L we have

$$\mathscr{M}(L) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

See Figure 3.32. Analogously for the examples [116] and [113]. On the other hand, every transitive sectional-Anosov flow with singularities X satisfies

$$\mathscr{M}(X)\bigcap\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\end{pmatrix}\right\}=\emptyset.$$

In [21] there are examples of such flows where

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\in\mathscr{M}(X)$$

However, these examples have *more than one singularity*. Finally, if X is a transitive sectional-Anosov flow with a unique singularity σ, then

$$M(\sigma)\neq\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

As we shall see later this result is false for sectional-Anosov flows with more than one singularity (Theorem 3.31).

These examples motivates the question which matrices belong to $\mathscr{M}(X)$ for some sectional-Anosov flow X. The answer is given by the result below.

Theorem 3.31. *For every* $A\in\left\{\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$ *there is a transitive sectional-Anosov flow X on a compact 3-manifold such that $A\in\mathscr{M}(X)$. If* $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ *then X can be chosen with only one singularity.*

Proof. The proof follows from the three examples below.

The first one is a sectional-Anosov flow X^1 with two singularities σ_1,σ_2 derived from the *template* in Figure 3.33-(a) (see [52] or [39]). Actually, we get the flow from the template by multiplying it by a strong contracting direction. The singularity σ_1 in that example has the following property: There are cross-sections A,B,C,D arbitrarily close to σ_1 such that

- $M(X^1)\cap A=\emptyset$, $M(X^1)\cap B\neq\emptyset$, $M(X^1)\cap C\neq\emptyset$ and $M(X^1)\cap D=\emptyset$.

This property implies

$$M(\sigma_1)=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Notice from the figure that X^1 is supported on a Handlebody of genus three. The transitivity is proved as in the Lorenz attractor's case: The return map of the flow associated to the interval I is exactly the classical one-dimensional Lorenz map with a single discontinuity c and derivative $>\sqrt{2}$ (see Figure 3.33-(b)).

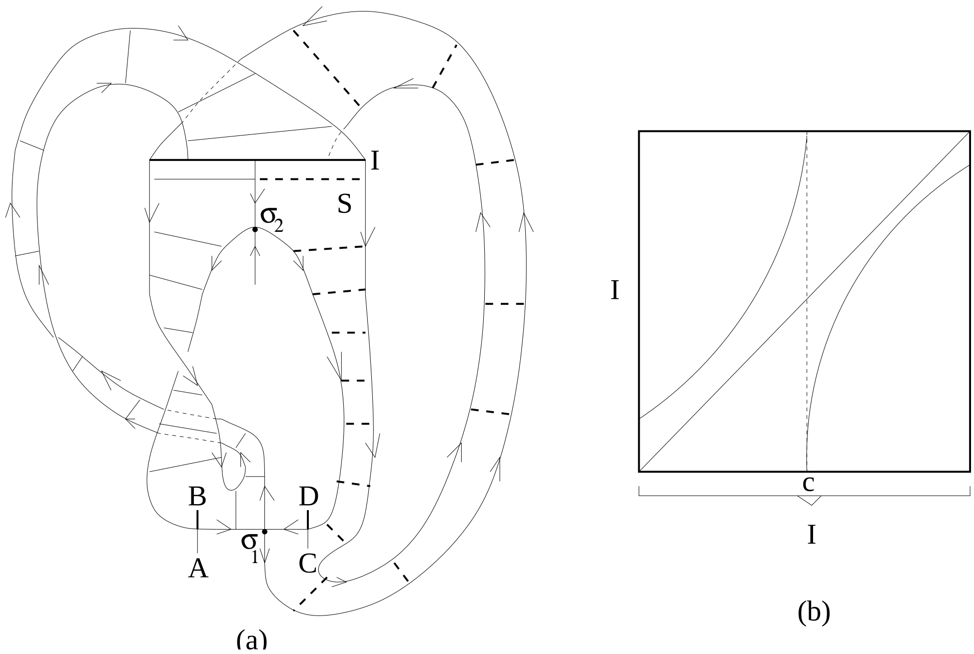


Fig. 3.33 The flow X^1

The second example is the geometric model X^2 with three singularities $\sigma_1, \sigma_2, \sigma_3$ depicted in Figures 3.34-(a). The singularity σ_1 has the following property: There are cross-sections A, B, C, D arbitrarily close to σ_2 such that

- $M(X^2) \cap A \neq \emptyset$, $M(X^2) \cap B \neq \emptyset$, $M(X^2) \cap C = \emptyset$ and $M(X^2) \cap D \neq \emptyset$.

This property implies

$$M(\sigma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

One sees that X^2 is supported on a Handlebody of genus three. The transitivity is proved again as in the Lorenz's case. Indeed, the return map corresponding to Figure 3.34-(b) has an invariant contracting foliation whose associated foliation map has derivative $> \sqrt{2}$. This lower bound implies that the positive iterates of any connected interval transverse to the foliation contain a connected interval of the form I_1, I_2, I_3, I_4 in Figure 3.34. The iterates of any of these intervals eventually intersects any stable leaf proving the desired transitivity.

The last example X^3 with only one singularity σ_3 outlined in Figure 3.35-(a). Note the similarity of this example with the well known *Plykin attractor* [52] p.36, [124], [122], [105]. The singularity σ_3 has the following property: There are cross-sections A, B, C, D arbitrarily close to σ_3 such that

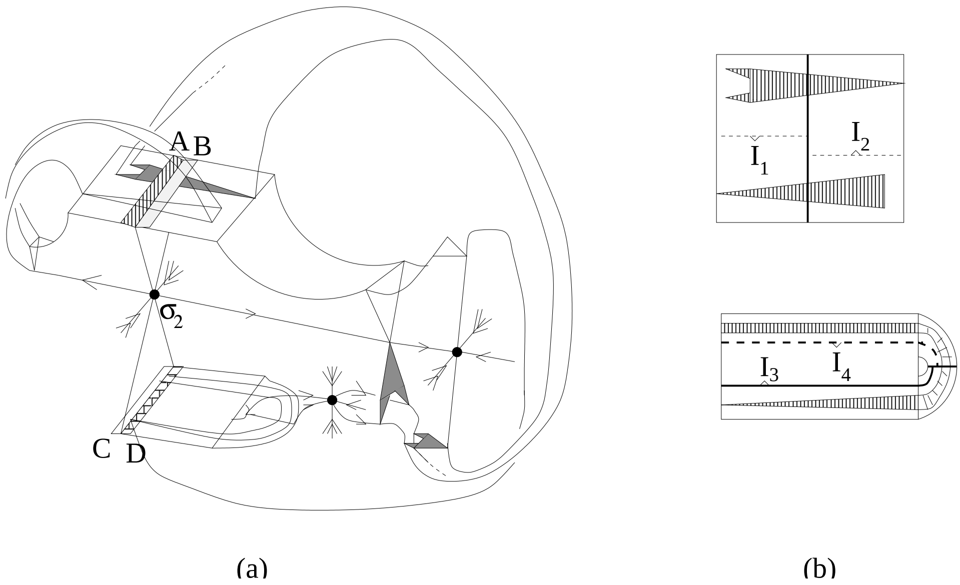


Fig. 3.34 The flow X^2

- $M(X^3) \cap A \neq \emptyset$, $M(X^3) \cap B \neq \emptyset$, $M(X^3) \cap C \neq \emptyset$ and $M(X^3) \cap D \neq \emptyset$.

  This property implies
$$M(\sigma_3) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

On the other hand, we can see from the figure that X^3 is supported on a Handlebody of genus two (a bitorus). The transitivity is proved as in the previous example but now using the form of the return map depicted in Figure 3.35-(b). This finishes the proof. □ □

Let us present some remarks relating the above examples to the works [110], [28], [11], [10] and [6].

The first one is that we don't know if there is a sectional-Anosov flow with only one singularity σ such that
$$M(\sigma) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

On the other hand, it would follow from Theorem 4.1 in [110] (or Theorem 3.61 p. 169 in [6]) that the unstable manifold of a periodic orbit a transitive sectional-Anosov flow with singularities on a compact 3-manifold intersects the stable manifold of *every* singularity of the flow. However, this is false because it would imply that *all* sectional-Anosov flows X on compact 3-manifolds satisfy

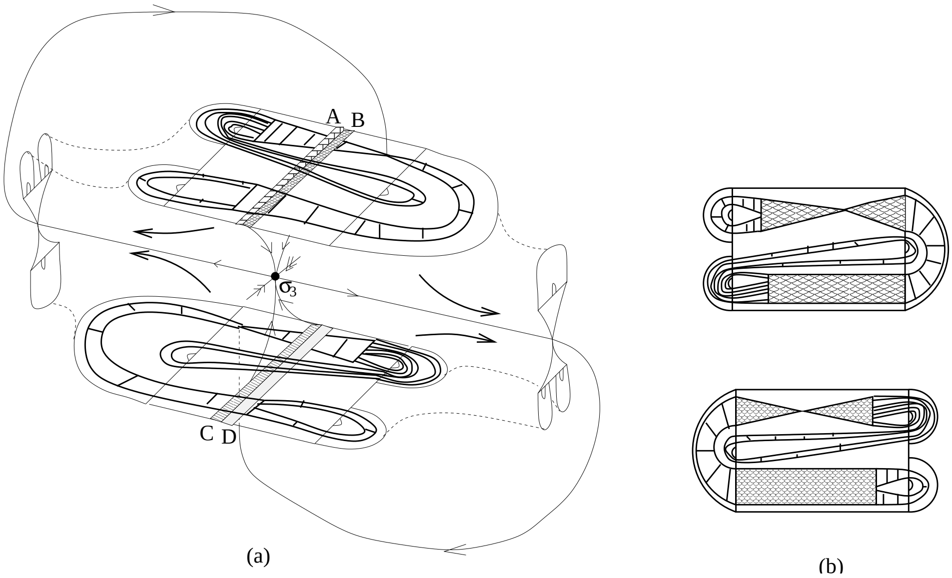


Fig. 3.35 The flow X^3

$$\mathscr{M}(X) \subset \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}, \tag{3.17}$$

a contradiction by Theorem 3.31 since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not belong to the right-hand side set. Nevertheless we shall see later that the main results in [110] are correct.

In order to state our last remark we observe that in all known examples of transitive sectional-Anosov flows on compact 3-manifolds the maximal invariant set is a homoclinic class (this also includes the ones in Theorem 3.31). This motivated the following conjecture whose three-dimensional version appeared first in [101]:

Conjecture 3.1 (Homoclinic class conjecture). The maximal invariant set of a transitive sectional-Anosov flow on a compact manifold is a homoclinic class.

It follows from the proof of Theorem 3.22 that this conjecture is true in the case of the geometric Lorenz attractor (see also [18] for the original proof).

A proof of the Homoclinic class conjecture on 3-manifolds was announced in Theorem C of [11]. It relies on transversal sections Σ_j, like $A \cup B$ or $C \cup D$ in Figure 3.34, with j running over twice the number of singularities of the attractor. The ones intersecting the attractor form a system of transversal sections

$\Sigma(E) := \{\Sigma_i(E) : 1 \leq i \leq k\}$ of size E. On each component $\Sigma_i(E)$ there is an interval $Q_i := \Sigma_i(E) \cap W^s_{loc}(\sigma)$, where σ is the corresponding singularity, such that the difference set $\hat{\Sigma}_i(E) := \Sigma_i(E) \setminus Q_i$ consists of two connected components like A and B or C and D in Figure 3.34. It would follow from Lemma 4 p.12 and the proof of the Main Theorem p.18 in [11] that for any small $\varepsilon > 0$ the attractor would intersect both connected components of $\hat{\Sigma}_i(E)$ for any $i \in \{1, \cdots, k\}$ and any size $E < \varepsilon$. In other words, the attractor would intersect each element of either $\{A, B\}$ or $\{C, D\}$ or $\{A, B, C, D\}$. This would imply that *every* sectional-Anosov flow X satisfies (3.17) which is not true due to Theorem 3.31.

Although there is a revised version [10] of [11] we have observed that the proof of Lemma 3.5 p. 80 in [10] is incorrect (althought the first part of such a lemma is true because of Corollary 6.29).

3.4.7 Recurrence far from closed orbits

The objective is to exhibit a sectional-Anosov flows on a compact manifold with recurrent points which cannot be accumulated by closed orbits. The construction goes through three steps.

First we describe the so-called *Cherry flow* ([27], [87]) which was used before. Let N^0 the two-dimensional torus and X^0 be a C^∞ vector field in N^0 satisfying the following properties:

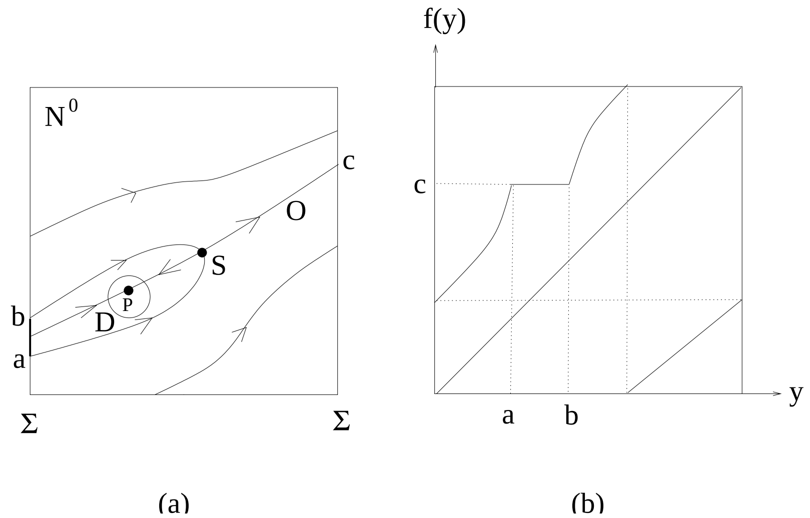


Fig. 3.36 : Cherry flow

(A) X^0 has two singularities, a hyperbolic saddle S and a hyperbolic sink P.

(B) X^0 is transverse to a meridian circle Σ in N^0.
(C) One of the two orbits in $W^u(S)\setminus\{S\}$ belongs to $W^s(P)$ and so it does not intersect Σ. The remaining orbit denoted by O intersects Σ in a first point c.
(D) The eigenvalues of the saddle S are such that the following holds: There is an open interval $(a,b)\subset\Sigma$ such that the positive orbit of $y\in(a,b)$ goes directly to P without re-intersecting Σ. The positive orbits of a or b do not re-intersect Σ too but they go to S instead. In particular, $a,b\in W^s(S)$. Finally, the positive orbit of $y\in\Sigma\setminus[a,b]$ re-intersects Σ in a first point $f(y)$. This yields a Poincaré map $f:\Sigma\setminus[a,b]\to\Sigma$ which we require to be expanding, i.e., there is $\lambda>1$ such that $f'(y)>\lambda$. Moreover, $f'(y)\to\infty$ as $y\to a^-$ or $y\to b^+$.

The map f in (D) can be extended to the whole Σ by setting $f(y)=c$ for every $y\in[a,b]$. The resulting map $f:\Sigma\to\Sigma$ is then a continuous endomorphism of degree 1 in Σ. Therefore f has a well defined rotation number. The vector field X^0 is called *Cherry flow* if its associated f has irrational rotation number. Figures 3.36-(a) and 3.36-(b) describes X^0 and f respectively.

The following lemma summarizes the main properties of the Cherry flow to be used here. Its proof can be found in [87] p. 187. Call a point $x\in\Sigma$ regular for X^0 if $X^0(x)\neq 0$.

Lemma 3.2. *If X^0 is a Cherry flow, then*

1. *X^0 has no periodic orbits.*
2. *$\Lambda^0=N^0\setminus W^s(P)$ is a transitive set of X^0. Consequently X^0 has a regular recurrent point $p\in\Lambda^0$.*

Next we describe a connected sum and construct an attracting set. Consider the vector field Y in closed disk T described in Figure 3.37-(b). Note that Y has a hyperbolic repelling equilibrium at P'. Choose another closed disk $D'\subset Int(T)$ with interior $Int(D')$ containing P' such that Y is transverse to $l'=\partial D'$ pointing outward. Choose one more closed disk $D\subset N^0$ containing P in its interior such that X^0 is transverse to the boundary $l=\partial D$ of D pointing outward. These disks are indicated in figures 3.37-(b) and 3.36-(a) respectively.

Remove $Int(D)$ from N^0 to obtain the manifold with boundary N^1 diffeomorphic to the punctured torus in Figure 3.37-(a). Remove $Int(D')$ from T glue the resulting manifold to N^1 by identifying l' and l. In this way we obtain the manifold in Figure 3.38-(a) which is diffeomorphic to a punctured torus. The vector fields X^1 and Y (which are transverse to l and l' respectively) induce a vector field X^2 in N^2 whose flow is depicted in Figure 3.38-(a). The point p in Figure 3.38-(a) represents the regular recurrent point p in Lemma 3.2. We fix at once a compact interval I as in Figure 3.38-(b).

Now we define an attracting set Λ to be used in Theorem A. Consider X^2 as in the previous subsection. Choose $\lambda_2<0$ and consider the vector field $F(x)=\lambda_2\cdot x$ in $[-1,1]$. Define X^3 as the vector field in $N^3=N^2\times[-1,1]$ whose flow is given by

$$X^3_t(q,x)=(X^2_t(q),F_t(x)),\quad \forall(q,x)\in N^3.$$

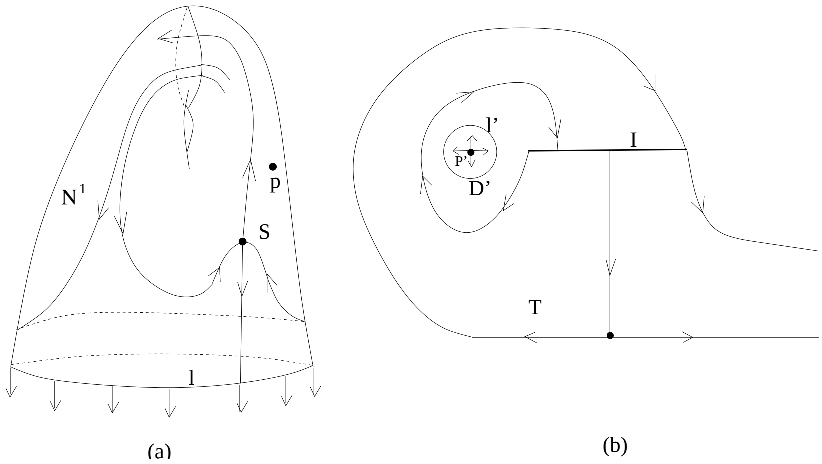


Fig. 3.37 : Deformed Cherry flow

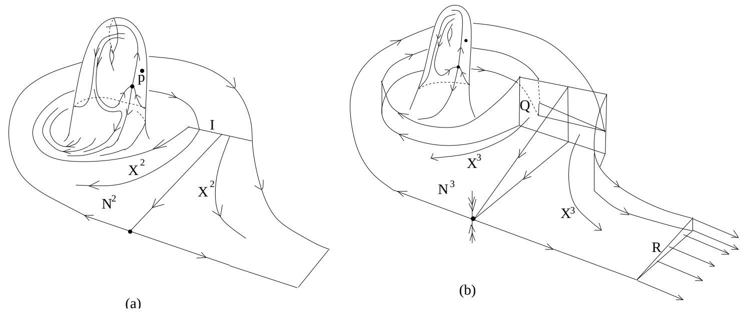


Fig. 3.38 : Connected sum

The portrait face of X^3 is depicted in Figure 3.38-(b). Notice that X^3 is transverse to both the square $Q = I \times [-1,1]$ and the cusp region R in the right-hand branch of N^3 indicated in Figure 3.38-(b).

Next we define a manifold U by flowing R back to N^3 as indicated in Figure 3.39. Notice that the resulting U is equipped with a vector field X induced by X^3 which is now transverse to the square $Q = I \times [-1,1]$ depicted in Figure 3.39. Moreover, U has a fibration of the form $\{*\} \times [-1,1]$.

The construction is done in a way that the positive orbits through Q goes to a geometric Lorenz attractor L contained in U. As already explained in Theorem 3.22, L is a sectional-hyperbolic set with sectional-hyperbolic splitting

$$T_L U = F_L^s \oplus F_L^c,$$

where the subbundle F_L^s is tangent to the fibers $\{*\} \times [-1,1]$ in U.

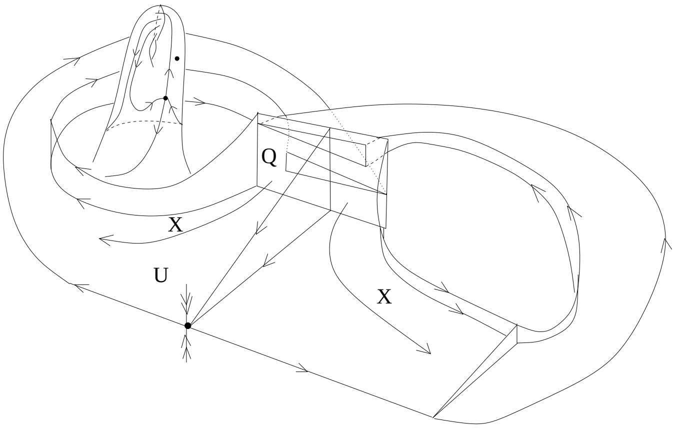


Fig. 3.39 : The attracting set

Finally, we define

$$\Lambda = \bigcap_{t \geq 0} X_t(U). \tag{3.18}$$

As $X_t(U) \subset U$ for all $t \geq 0$ we have that Λ is attracting set of X. We shall prove in the next section that Λ is the attracting set required in Theorem A.

Now we put together the previous results in order to prove

Theorem A. *There are sectional-Anosov flows on certain compact manifolds having recurrent points which cannot be accumulated by closed orbits.*

Proof. Let Λ the attracting set in (3.18), p be the recurrent point in Lemma 3.2 and $p \times 0 \in \Lambda$ be the corresponding regular recurrent point in Λ. As p is regullar and not accumulated by periodic orbits we have that $p \times 0 \in \Lambda$ is a recurrent point which cannot be accumulated by closed orbits. It remains to prove that Λ is sectional-hyperbolic. For this we proceed as follows.

Notice that $N^1 \times 0$ embeds into U and

$$X_t(N^1 \times 0) \subset N^1 \times 0, \qquad t \leq 0. \tag{3.19}$$

Therefore, the set

$$S = \bigcup_{t \geq 0} X_t(N^1 \times 0)$$

is a submanifold of U. Let TS be the tangent space of S. Define the splitting

$$T_S U = G_S^s \oplus G_S^c$$

over S by setting G^s_S as the line bundle tangent to the fibers $\{*\}\times[-1,1]$ and $G^c_S = TS$.

Now, one can easily prove that

$$\Lambda = S \cup L, \tag{3.20}$$

where L is the geometric Lorenz attractor. Recall that L has a sectional-hyperbolic splitting $T_L U = F^s_L \oplus F^c_L$ with F^s_L being tangent to the fibers $\{*\}\times[-1,1]$ in U. Then, (3.20) allows us to define a splitting

$$T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$$

over Λ by setting

$$E^i_z = \begin{cases} F^i_z, & \text{if } z \in L \\ G^i_z & \text{if } z \in S \end{cases}$$

for $i = s$ or c. This splitting is clearly invariant. Moreover, E^s_Λ is contracting (and dominates E^c_Λ) if we choose λ_2 with modulus large enough.

We claim that E^c_Λ is volume expanding. Indeed, the volume expansiveness is clear in L so we only have to prove it in S. Now $E^c_S = TS$ by the definition of G^c_S. Let Λ^0 be the transitive set in Lemma 3.2. Then, $\Lambda^0 \times 0 \subset S$ and so we have the decomposition

$$S = (\Lambda^0 \times 0) \cup (S \setminus (\Lambda^0 \times 0)).$$

It follows from the expansivity of f in (D) above that $T_{\Lambda^0\times 0}S$ is volume expanding. On the other hand, the points in $S \setminus (\Lambda^0 \times 0)$ are precisely the points in S whose positive orbit eventually fall into $T \times 0$.

Since the circle $l \times 0 \approx l' \times 0 \subset N^1 \times 0$ is transverse to the contracting subbundle of L (i.e. the fibers $\{*\}\times[-1,1]$) one has that $T_{S\setminus(\Lambda^0\times 0)}S$ is volume expanding too. From this it follows that E^c_Λ is volume expanding concluding the proof. $\square$

Chapter 4
Some properties of Anosov and sectional-Anosov flows

In this chapter we present properites of Anosov and sectional-Anosov flows

4.1 Properties of Anosov flows

We start with the Anosov flows.

4.1.1 Anosov closing and connecting lemmas. Structural stability

To begin with we state the following remark.

Remark 4.1. *In the case X is Anosov we have the existence of the invariant manifolds $W^s(x)$, $W^u(x)$, $W^{ss}(x)$ and $W^{uu}(x)$ for all $x \in M$. These manifolds form partitions of M and in fact they are leaves of corresponding foliations $\mathscr{F}^s$, $\mathscr{F}^u$, $\mathscr{F}^{ss}$ and $\mathscr{F}^{uu}$. Although these foliations are of class C^0 in general we obtain in the codimension one case $dim(E^s \oplus E^X) = 1$ that $\mathscr{F}^s$ is also C^1 (if X is C^2).*

Next we present some consequences of the shadowing lemma for flows. Recall that given a C^1 vector field X in a manifold M and $p, q \in M$ we write $p \prec q$ if for all $\varepsilon > 0$ there is $t > 0$ such that $X_t(B_\varepsilon(p)) \cap B_\varepsilon(q) \neq \emptyset$, where $B_\varepsilon(x)$ denotes the open ε-ball around x.

Theorem 4.2 (Anosov connecting lemma)**.** *If X is an Anosov flow on a closed manifold M, $p,q \in M$ and $p \prec q$, then there is $x \in M$ such that $\alpha(p) = \alpha(x)$ and $\omega(x) = \omega(q)$.*

Proof. Let $\varepsilon > 0$ small and $\delta > 0$ be as in the shadowing lemma for $\Lambda = M$. We can assume that q is not in the positive orbit of q (otherwise $x = p$ works). Then, we can use the negative orbit of p, the positive orbit of q and a trajectory from a point close to p to a point closed q as in Figure 4.1 in order to construct a δ-orbit $c(t)$ of X in U which, by the shadowing lemma, can be ε-shadowed by the orbit of some $x \in M$.

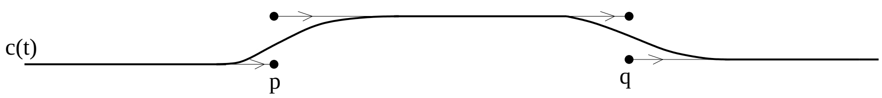


Fig. 4.1 Proof of the Anosov connecting lemma.

In particular, the negative (resp. positive) orbit of x stays ε-close to that of p (resp. q) hence $x \in W^u(p)$ (resp. $x \in W^s(q)$) from which we get $\alpha(p) = \alpha(x)$ (resp. $\omega(x) = \omega(q)$). □

Analogously but using that the closed δ-orbits in the shadowing lemma are shadowed by periodic orbits we get the following classical result.

Corollary 4.3 (Anosov closing lemma)**.** *The nonwandering set of every Anosov flow on a closed manifold is the closure of the periodic points.*

From the Anosov closing lemma we derive the following property.

Theorem 4.4. *Every Anosov flow is Axiom A.*

Proof. Let X be an Anosov vector field on a manifold M. Clearly $\Omega(X)$ is hyperbolic since M is. By using the hyperbolic structure in M we can see that every $p \in \Omega(X)$ can be approximated by closed orbits by the Anosov Closing Lemma. Hence the closed orbits are dense in $\Omega(X)$ and we are done. □

In particular, we have the following corollary.

Corollary 4.5. *Every Anosov flow on a closed manifold has an attractor, a repeller and so a periodic orbit.*

Proof. The existence of a periodic orbit follows at once from the Anosov closing lemma since the nonwandering set is not empty (we can also appeal to the Spectral Theorem). Since the time-reversed flow of an Anosov flow is Anosov, and repellers became attractors under time reversing, we only need to prove that every Anosov flow X on a closed manifold M has an attractor. For this purpose let $\Omega(X) = H_1 \cup \cdot \cup H_n$ be the decomposition of $\Omega(X)$ given in the Spectral Theorem. It turns out that $M = \bigcup i = 1^n W^s(H_i)$ and then there is some index $1 \le i \le n$ for which $Int(W^s(H_i)) \neq \emptyset$. By using the density of the periodic orbits in the homoclinic class H_i we have that there is an open set $U \subset W^s(H_i)$ and a periodic orbit $O \subset H_i$ such that $W^s(O) \pitchfork U \neq \emptyset$. Applying the Inclination lemma to such a transverse intersection and using $U \subset W^s(H_i)$ we get $W^u(O) \subset H_i$. Since H_i is a homoclinic class this implies $W^u(x) \subset H_i$ for all $x \in H_i$ and then H_i is an attractor. □

Now we present two direct consequences of the above corollary.

Theorem 4.6. *An Anosov flow on a closed manifold is transitive if and only if it has dense periodic orbits.*

Proof. The direct implication follows at one from the Anosov closing lemma. For the converse we see that every Anosov flow has a hyperbolic attractor by Corollary 4.5. The attractor must be the entire manifold since the periodic orbits are dense. Then, the result follows since attractors are always transitive sets. □

Theorem 4.7. *An Anosov flow X on a closed manifold M is transitive if and only if $W^s(p)$ is dense in M for all $p \in M$. Analogously replacing $W^s(p)$ by $W^u(p)$.*

Proof. It is clear that for a transitive Anosov flow we have that both $W^s(p)$ and $W^u(p)$ are dense for all $p \in M$ (simply use the dense orbit to spread both invariant manifolds). If each $W^s(p)$ (resp. $W^u(p)$) is dense we have that X is transitive since by Corollary 4.5 X has a repeller (resp. attractor). □

In the sequel we study the perturbation and topology of the Anosov flows. The most important result about the perturbation of Anosov flows is the following theorem proved by Anosov [5]. Hereafter r will denote a positive integer. Recall that a C^r vector field X on a manifold M is C^r *structural stable* if for every C^r vector field Y that is C^r close to X there is there is a homomorphism $h : M \to M$ (called *conjugation*) which sends orbits of X into orbits of Y.

Definition 4.8. *Let X and Y be two flows on a manifold M. We say that X and Y are* topologicaly equivalent *if there is a homomorphism $h : M \to M$ sending the orbits of X into orbits of Y. A C^r flow X in M is C^r* structural stable *if any other flow Y that is C^r close to X is topologicaly equivalent to X.*

Theorem 4.9. *Every C^r Anosov flow on a closed manifold is C^r structural stable.*

Proof. We present an outline of the proof using the shadowing lemma for flows (details in [132]). Pick $\varepsilon > 0$ and let $\delta > 0$ be as in the shadowing lemma for the hyperbolic set $\Lambda = M$ of X. Take $x \in M$ and fix a rectangle S_x of small diameter around x. Let Y be a C^r vector field that is C^r close to X. It is clear that the Y-orbit $O_Y(x)$ of x is a δ-orbit of X and, then, there is a unique X-orbit which ε-shadow $O_Y(x)$. It follows that such an orbit intersects S_x at some point $h(x)$ which gives the desired homomorphism. □

Corollary 4.10. *Every C^r transitive Anosov flow on a closed manifold is both C^r robustly transitive and C^r robustly periodic.*

Proof. Since Anosov flows are structural stable, and transitivity is invariant by conjugations, we have that every C^r transitive Anosov flow is C^r robustly transitive. Since every C^r transitive Anosov flow has dense periodic orbits, and denseness of periodic orbits is invariant by conjugations, we see that every C^r transitive Anosov flow is also C^r robustly periodic. □

4.1.2 Strong foliations and transitivity

Now we study the relationship between transitivity and denseness of strong stable foliations. Recall that a flow X on a manifold M is *transitive* if it has a dense orbit. A point $p \in M$ is a *periodic point* (of X) if it lies in a periodic orbit of X. The *period of* p is the smallest $T = T_p > 0$ such that $X_T(p) = p$. The main result is the one below due to J. Plante [127].

Theorem 4.11. *Let X be a transitive Anosov flow on a closed manifold M. Assume that there is a periodic point $p \in M$ such that $W^{uu}(p)$ is not dense in M. Then, there is $T > 0$ and a closed codimension one submanifold S transverse to X such that:*

1. *S intersects all flowlines of X and so X is suspended (Definition 1.25).*
2. $X_T(S) = S$ *and* $S \cap X_t(S) = \emptyset$ *for all* $0 < t < T$.
3. *If q is a periodic point of X with period* T_q*, then there is* $n \in \mathbb{N}$ *such that* $T_q = nT$.

The proof will use the following notation. Let X be a flow on a manifold M. If $C \subset M$ and $B \subset \mathbb{R}$ we denote

$$X_B(C) = \{X_t(x) : (t,x) \in B \times C\}.$$

It is clear that if $T > 0$ and C is compact in M, then $X_{[0,T]}(C)$ is a compact subset of M. When X is an Anosov flow on a manifold M we have that M is a hyperbolic set of M. In such a case the manifolds $W^s(x), W^u(x), W^{ss}(x), W^{uu}(x)$ are defined for all $x \in M$. These manifolds form continuous foliations denoted respectively by $\mathscr{F}^s, \mathscr{F}^u, \mathscr{F}^{ss}, \mathscr{F}^{uu}$. In particular, if $\mathscr{F}^*_x$ denotes the leaf of $\mathscr{F}$ containing x, then $\mathscr{F}^*_x = W^*(x)$ for $* = s, u, ss, uu$. In general, if $\mathscr{F}$ is a foliation on a manifold M, a subset $B \subset M$ is called $\mathscr{F}$-saturated in B is union of leaves of $\mathscr{F}$. If B is a saturated set of $\mathscr{F}$ then so is its closure $\overline{B}$.

Lemma 4.1. *If X is a transitive Anosov flow on a closed manifold M then* $W^s(x)$ *and* $W^u(x)$ *are dense in M,* $\forall x \in M$.

Proof. We only prove the result for s since the proof for u is obtained with the reversed flow $-X$. Pick $x \in M$. Then the closure $\overline{W^s(x)}$ is $\mathscr{F}^s$-invariant since $W^s(x)$ is a leaf of $\mathscr{F}^s$. Since M is connected by definition we only have to prove that $\overline{W^s(x)}$ is open in M. For this end pick $y \in \overline{W^s(x)}$ and a neighborhood U of y. Let $z \in U$ be periodic. If U is chosen small we have that $W^s(x) \cap W^u(z) \neq \emptyset$ by the Local Product Structure. Then, $z \in \overline{W^s(x)}$ by the Inclination Lemma. As X is transitive we have that the periodic points z are dense in U. Consequently $U \subset \overline{W^s(x)}$ and so $\overline{W^s(x)}$ is open. The result follows. □

Lemma 4.2. *Let X be a transitive Anosov flow on a closed manifold M. Let p be a periodic point of X such that* $W^{uu}(p)$ *is not dense in M. If* T_p *is the period of p, then*

$$M = X_{[0,T_p]}\left(\overline{W^{uu}(p)}\right).$$

Proof. Observe that $X_{[0,T_p]}\left(\overline{W^{uu}(p)}\right)$ is closed in M since $\overline{W^{uu}(p)}$ is. On the other hand

$$X_{[0,T_p]}\left(\overline{W^{uu}(p)}\right) \supset X_{[0,T_p]}\left(W^{uu}(p)\right) = W^u(p)$$

since T_p is the period of p. In addition, $W^u(p)$ is dense in M by Lemma 4.1 since X is transitive. As $X_{[0,T_p]}\left(\overline{W^{uu}(p)}\right)$ is closed we obtain the result. □

Lemma 4.3. *Let X be a transitive Anosov flow on a closed manifold M. If p is a periodic point of X and $W^{uu}(p)$ is not dense in M, then there is $T>0$ such that $X_T(\overline{W^{uu}(p)})=\overline{W^{uu}(p)}$ and*

$$M=\bigcup_{t\in[0,T)} X_t\left(\overline{W^{uu}(p)}\right).$$

In addition, the union is disjoint and then $\overline{W^{uu}(p)}\cap X_t(\overline{W^{uu}(p)})=\emptyset$ for all $0<t<T$. If q is a periodic point of X with period T_q, then there is $n\in\mathbb{N}$ such that $T_q=nT$.

Proof. By Lemma 4.2 we have that

$$M=X_{[0,T_p]}\left(\overline{W^{uu}(p)}\right),$$

where T_p is the period of p. The Zorn's Lemma implies that there is a non-empty set $K\subset\overline{W^{uu}(p)}$ which is minimal respect to the following properties:

1. K is closed in M.
2. K is $\mathscr{F}^{uu}$-saturated.
3. $X_T(K)=K$.

Define $K^*=X_{[0,T_p]}(K)$. By (1) we have that K^* is closed in M and by (2) we have that K^* is $\mathscr{F}^u$-saturated. These fact together with Lemma 4.1 imply $K^*=M$ and then

$$M=X_{[0,T_p]}(K).$$

First we claim that if $0<t<T_p$ and $K\cap X_t(K)\neq\emptyset$, then $K=X_t(K)$. Indeed, $K\cap X_t(K)$ satisfies the properties (1)-(3) above and so $K\cap X_t(K)=K$ by minimality. It follows that $K\subset X_t(K)$ and then $X_{-t}(K)\subset K$. But $X_{-t}(K)$ also satisfies (1)-(3). Then, $X_{-t}(K)=K$ proving $K=X_t(K)$. This proves our first claim.

Second we claim that there is $0<\delta<T_p$ such that $K\cap X_t(K)=\emptyset$ for all $0<t<\delta$. Indeed, suppose by contradiction that there is no such a δ. Then, there is $t>0$ arbitrarily close to 0 such that $K\cap X_t(K)\neq\emptyset$. Hence $K=X_t(K)$ by the first claim. It follows that $K=X_{nt}(K)$ for all $n\in\mathbb{N}$. From this, and the fact that t is close to 0, we would obtain that the set $\{r\in\mathbb{R}:K=X_r(K)\}$ is dense in $\mathbb{R}$. It would follow that K is dense in $X_{[0,T_p]}(K)=M$ implying that $W^{uu}(p)$ is dense in M since $K\subset\overline{W^{uu}(p)}$. This is a contradiction which proves our second claim.

Now we define

$$T=\sup\{0<\delta\leq T_p:K\cap X_t(K)=\emptyset,\forall 0<t<\delta\}.$$

The second claim implies $T>0$. Also $K\cap X_T(K)\neq\emptyset$ for, otherwise, $K\cap X_T(K)=\emptyset$ and then it would exist $\delta\in(T,T_p)$ such that $K\cap X_t(K)=\emptyset$ for $0<t<\delta$ contradicting the definition of T. Then, $K=X_T(K)$ by the first claim. Henceforth $X_{[0,T]}(K)$ is $\mathscr{F}^u$-saturated. As such a set is also compact (and nonempty) we obtain from Lemma 4.1 that $M=X_{[0,T]}(K)$. As $K=X_T(K)$ we obtain the union

$$M = \bigcup_{t \in [0,T)} X_t(K). \tag{4.1}$$

Suppose $r,t \in [0,T)$ satisfy $X_r(K) \cap X_t(K) \neq \emptyset$. Assume $r < t$. Hence $t - r \in (0,T)$ and $K \cap X_{t-r}(K) \neq \emptyset$. Then we get a contradiction by the definition of T. This proves $r \geq t$. Interchanging the roles of r and t in the above argument we obtain $r \leq t$. Hence $r = t$ and so the union (4.1) is disjoint.

Third we claim that if $q \in M$ is a periodic point of X with period T_q, then there is $n \in \mathbb{N}$ such that $T_q = nT$. Indeed, we have $T \leq T_q$ since the union (4.1) is disjoint. Then there is an integer $n \geq 1$ and $r \in [0, \delta_0)$ such that $T_q = nT + r$. By equality (4.1) we can assume $q \in K$. In addition,

$$X_{T_q}(K) = X_{nT+r}(K) = X_r(X_{nT}(K)) \overset{(3)}{=} X_r(K).$$

Hence $X_{T_q}(K) = X_r(K)$ and so $q \in X_r(K)$ because $q = X_{T_q}(q) \in X_{T_q}(K)$. As $q \in K$ we conclude that $q \in K \cap X_r(K)$. Consequently $K \cap X_r(K) \neq \emptyset$ and then $r = 0$ because of the definition of T and $0 \leq r \leq T$. Hence $T_q = nT$ which proves our third claim.

Finally we prove $K = \overline{W^{uu}(p)}$. Indeed, (4.1) implies that

$$p \in X_t(K) \tag{4.2}$$

for some $0 \leq t < T$. Let us prove $t = 0$. By contradiction suppose that $t > 0$. As $p \in X_t(K)$ we obtain $X_{-t}(p) \in K$ and then $\overline{W^{uu}(X_{-t}(p))} \subset K$ by properties (1) and (2). Consequently $X_{-t}(\overline{W^{uu}(p)}) \subset K$ yielding

$$\overline{W^{uu}(p)} \subset X_t(K).$$

This inclusion implies $K \subset X_t(K)$ and then $K \cap X_t(K) = K \neq \emptyset$. The last contradicts the definition of T since $0 < t < T$. This contradiction proves $t = 0$. Replacing in (4.2) we obtain $p \in K$. So $\overline{W^{uu}(p)} \subset K$ by properties (1) and (2). As $K \subset \overline{W^{uu}(p)}$ by definition we conclude that $K = \overline{W^{uu}(p)}$. Replacing in (4.1) we obtain the equality of the lemma. The proof follows. □

Lemma 4.4. *Let X be a transitive Anosov flow on a closed manifold M. If p is a periodic point of X such that $W^{uu}(p)$ is not dense in M, then $\overline{W^{uu}(p)}$ is $\mathscr{F}^{ss}$-saturated.*

Proof. Suppose by contradiction that there is $x \in \overline{W^{uu}(p)}$ such that $W^{ss}(x) \not\subset \overline{W^{uu}(p)}$. Then, there is $y \in W^{ss}(x) \setminus \overline{W^{uu}(p)}$. Let T be as in Lemma 4.3. By Lemma 4.3 we have that $y \in X_t(\overline{W^{uu}(p)})$ for some $t \in (0,T)$. As $y \in W^{ss}(x)$ one has $dist(X_{nT}(y), X_{nT}(x)) \to 0$ as $n \to \infty$. On the other hand, $X_{nT}(y) \in X_t(\overline{W^{uu}(p)})$ and $X_{nT}(x) \in \overline{W^{uu}(p)}$ because $X_T(\overline{W^{uu}(p)}) = \overline{W^{uu}(p)}$ by Lemma 4.3. It then follows that $dist(X_t(\overline{W^{uu}(p)}), \overline{W^{uu}(p)}) = 0$ and so $\overline{W^{uu}(p)} \cap X_t(\overline{W^{uu}(p)}) \neq 0$ because $X_t(\overline{W^{uu}(p)})$ and $\overline{W^{uu}(p)}$ are compact. This contradicts $0 < t < T$ by Lemma 4.3 and the proof follows. □

Lemma 4.5. *Let X be a transitive Anosov flow on a closed manifold M. If p is a periodic point of X such that $W^{uu}(p)$ is not dense in M, then $\overline{W^{uu}(p)}$ is a closed codimension one submanifold transverse to X.*

Proof. To simplify we denote $S = \overline{W^{uu}(p)}$. Clearly S is $\mathscr{F}^{uu}$-saturated and S is $\mathscr{F}^{ss}$-saturated by Lemma 4.4. These facts can be used to construct a manifold structure whose local coordinate systems on $x \in S$ have the form

$$\bigcup_{y \in I^{ss}(x)} I^{uu}(y)$$

where $I^{ss}(x)$ is an open interval which contains x and is contained in $W^{ss}(x) \subset K$; and $I^{uu}(y)$ is an open interval which contains y and is contained in $W^{uu}(y)$. With this manifold structure we obtain that the tangent space $T_xS = E^s_x \oplus E^u_x$ for all $x \in S$. As the sum $T_xM = E^s_x \oplus E^X_x \oplus E^u_x$ is direct we obtain that X is transverse to S. The proof follows. □

Now we prove Theorem 4.11. Let X be a transitive Anosov flow on a closed manifold M. Assume that there is a periodic point p of X such that $W^{uu}(p)$ is not dense in M. By Lemma 4.5 we have that $S = \overline{W^{uu}(p)}$ is a closed codimension one submanifold transverse to X. By Lemma 4.3 we have that S intersects every flowline of X. Moreover, there is $T > 0$ such that if T_p is the period of p, then $T_p = nT$ for some $n \in \mathbb{N}$, $X_T(S) = S$ and $S \cap X_t(S) = \emptyset\ \forall 0 < t < T$. This proves the result. □

Corollary 4.12. *Let X be a transitive Anosov flow on a closed manifold M. Suppose that there are a pair of periodic orbits of X whose periods T_1, T_2 satisfy $T_1/T_2 \notin \mathbb{Q}$. If p is a periodic point of X, then $W^{uu}(p)$ and $W^{ss}(p)$ are both dense in M.*

Proof. By contradiction suppose that there is some periodic point p of X such that $W^{uu}(p)$ is not dense in M. As X is transitive we conclude that there is a closed submanifold S transverse to X and $T > 0$ satisfying the conclusions of Theorem 4.11. In particular, the conclusion (3) implies that there are positive integers n, m such that $T_1 = nT$ and $T_2 = mT$. Hence $T_1/T_2 = nT/mT = n/m \in \mathbb{Q}$ which is absurd. This contradiction proves the result. The proof for $W^{ss}(p)$ is analogous. □

Theorem 4.13. *Let X be an Anosov flow on a closed manifold M such that the subbundle $E^s_M \oplus E^u_M$ in the hyperbolic decomposition $TM = E^s_M \oplus E^X_M \oplus E^u_M$ of X is tangent to a C^1 foliation of M. Then, X is suspended.*

Proof. Let $\mathscr{F}$ be the foliation tangent to the subbundle $E^s_M \oplus E^u_M$. If L is a leaf of $\mathscr{F}$ then L is tangent to $E^s_M \oplus E^u_M$. So, if $t \in \mathbb{R}$ we have that $X_t(L)$ is also tangent to $E^s_M \oplus E^u_M$. Consequently $X_t(L)$ is also a leaf of $\mathscr{F}$. Applying Lemma 1.6 we conclude that $\mathscr{F}$ is tangent to a closed one form ω in M. Clearly X is transverse to $E^s_M \oplus E^u_M$ and so X is transverse to $ker(\omega)$. We then conclude by Theorem 1.32 that X has a global cross section and so X is suspended. The proof follows. □

4.1.3 Existence of global cross sections

In this subsection we prove a classical criterium for the existence of cross sections due to Verjovsky [148].

Proposition 4.14. *Let M be a transitive Anosov flow on a closed n-manifold M. Then, X is a suspension if and only if $Rank(H_1(M,\mathbb{Z})) = 1$ and every closed orbit of X represents a non-zero element of the free part of $H_1(M,\mathbb{Z})$.*

Proof. Necessity. Assume that X is suspended. Then there is a transverse torus T intertsecting all flowlines of X. By cutting open M along T and regluing one sees by the Seifert-Van Kampen Theorem that $\pi_1(M) = \mathbb{Z}^{n-1} \times_A \mathbb{Z}$ for some $A \in Aut(\mathbb{Z}^n)$ hyperbolic (in particular, $A - Id$ is invertible in $\mathbb{Z}^{n-1}$). More precisely

$$\pi_1(M) = \{(a,m) \in \mathbb{Z}^{n-1} \times \mathbb{Z}\},$$

with the product

$$(a,m) \cdot (b,n) = (a + A^m(b), m+n). \tag{4.3}$$

Let $G \leq \pi_1(M)$ be the subgroup $G = \{a,0) : a \in \mathbb{Z}^{n-1}\}$. One sees using (4.3) that G is normal. We claim that G is the commutador $[\pi_1(M), \pi_1(M)]$ of $\pi_1(M)$. In fact, observe that an easily computation using (4.3) yields

$$[(a,m),(b,n)] = (a + A^m(b) - A^n(a) - b, 0).$$

Hence $[\pi_1(M), \pi_1(M)] \leq G$. Conversely if $(a,0) \in G$ we have that $b = (A - Id)^{-1}(a) \in \mathbb{Z}^{n-1}$ is well defined since $A - Id$ is invertible in $\mathbb{Z}^{n-1}$. Hence

$$[(0,1),(b,0)] = (0 + A(b) - b, 0) = ((A - Id)(b), 0) = (a,0)$$

proving $G \leq [\pi_1(M), \pi_1(M)]$. The claim follows. On the other hand, the quotient $\pi_1(M)/G$ is infinite cyclic generated by $(0,1) \cdot G$. By the Hurewicz Theorem we have

$$H_1(M,\mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)] = \pi_1(M)/G$$

is infinite cyclic. This proves $Rank(H_1(M,\mathbb{Z})) = 1$. To finish we observe that all closed orbits of X intersect T in the positive direction. This implies that all closed orbit of X is non-zero in $H_1(M,\mathbb{Z})$ and the result follows.

Sufficiency. Assume that $Rank(H_1(M,\mathbb{Z})) = 1$ and that all closed orbits of X are non-zero in the free part of $H_1(M,\mathbb{Z})$. By projecting onto the free part $\mathbb{Z}$ of $H_1(M,\mathbb{Z})$ we obtain an onto homomorphism $c : H_1(M,\mathbb{Z}) \to \mathbb{Z}$ whose kernel is the torsion group of $H_1(M,\mathbb{Z})$. Let $H : \pi_1(M) \to H_1(M,\mathbb{Z})$ be the Hurewizt homomorphism. Then the composition $h = c \circ H$ yields an onto homomorphism $h : \pi_1(M) \to \mathbb{Z}$. The kernel of h is formed precisely by those elements in $\pi_1(M)$ representing a torsion element of $H_1(M,\mathbb{Z})$. Since all closed orbits of X are non-zero in the free part of $H_1(M,\mathbb{Z})$ we conclude that $h(\gamma) \neq 0$ for every $\gamma \in \pi_1(M)$ represented by a closed orbit of X. Let $\hat{M} \to M$ be the Galois covering associated to such a kernel. Such a covering is regular and the group of covering maps is the free part of $H_1(M,\mathbb{Z})$

which is infinite cyclic. Hence $\hat{M} \to M$ is a $\mathbb{Z}$-cover. Let $\hat{X}$ the lift of X to $\hat{M}$. $\hat{X}$ is Anosov. Suppose that $\hat{X}$ has a closed orbit $\hat{O}$. Then $\hat{O}$ projects to a closed orbit O of X. The element γ of $\pi_1(M)$ represented by such an orbit is clearly contained in the kernel of h. Hence γ represents a torsion element of $H_1(M,\mathbb{Z})$ contradicting the assumption. We conclude that $\hat{X}$ has no closed orbits. By the Anosov closing Lemma we conclude that $\Omega(\hat{X}) = \emptyset$. Hence $\lim_{t\to\infty}(\hat{X}_t(\hat{x})) = \pm\infty$ for every $\hat{x} \in \hat{M}$. Since X is transitive we can assume that $\lim_{t\to\infty}\hat{X}_t(\hat{x}) = \infty$ for every $\hat{x} \in \hat{M}$. This implies that X is suspended by Theorem 1.26 and the proof follows. □

The arguments above are used to prove the following.

Corollary 4.15. *Let X be a transitive Anosov flow on a closed 3-manifold M. If there is a cohomology class $c \in H^1(M,\mathbb{Z})$ such that $c(\gamma) \neq 0$ for every homology class $\gamma \in H_1(M,\mathbb{Z})$ represented by a closed orbit of X, then X is suspended.*

Proof. Let $H : \pi_1(M) \to H_1(M,\mathbb{Z})$ be the Hurewicz homomorphism and define $h = c \circ H$. We have that $h \neq 0$ since c does not vanish at the closed orbits of X. Note that the kernel of h is formed precisely by those elements of $\pi_1(M)$ representing an element in the kernel of c. We conclude that $c(\gamma) \neq 0$ for every element of $\pi_1(M)$ represented by a closed orbit of $\pi_1(M)$. Let $\hat{M} \to M$ be the Galois's covering associated to such a kernel. As before we have that such a covering is a $\mathbb{Z}$-cover of M. Let $\hat{X}$ be the lift of X to $\hat{M}$. Hence $\hat{X}$ is Anosov. If $\Omega(\hat{X}) \neq \emptyset$ then it would exists a closed orbit $\hat{O}$ of $\hat{X}$ in $\hat{M}$. This closed orbit projects to a closed orbit O of X whose representant γ in $\pi_1(M)$ satisfies $h(\gamma) = 0$. This contradicts the fact that $c \neq 0$ at the closed orbits. This proves $\Omega(\hat{X}) = \emptyset$. The rest of the proof is similar to the previous proof. □

4.2 Properties of sectional-Anosov flows

In this section we present some basic properties of sectional-Anosov flows.

4.2.1 Basic properties

In the case of sectional-Anosov flows X on M the strong stable manifold's family $\{W^{ss}(p) : p \in M(X)\}$ extends to a continuous foliation W^{ss} in M with the property that every pair of points in the same leaf are asymptotic one to another. By adding the flow direction to W^{ss} we obtain a foliation W^s now tangent to $E^s_M \oplus E^X_M$. Unlike the Anosov case W^s may have singularities, all of which being the leaves through the singularities of X. Note that W^s is transverse to ∂M because it contains the flow direction.

Next we state two direct consequences of the hyperbolic lemma.

Corollary 4.16. *All sectional-Anosov flows on closed manifolds are Anosov.*

Proof. Note that in a boundaryless manifold the maximal invariant set is the manifold itself. It then follows from Corollary 2.7 that the flows in the statement are singular free. Now apply the hyperbolic lemma. □

Observe that we can replace Corollary 2.7 by [148] in the above proof to obtain the same conclusion.

Corollary 4.17. *Every periodic orbit of a sectional-Anosov flow on a compact manifold is hyperbolic. In particular, all such flows have countably many closed orbits.*

The two alternatives in Lemma 2.8 can occur in the same sectional-Anosov flow. The following theorem generalizes an observation in [114].

Theorem 4.18. *If X is sectional-Anosov, $\sigma \in \Lambda \cap Sing(X)$ and $\Omega(X) \setminus \{\sigma\}$ is not closed, then σ is Lorenz-like. In particular, every singularity of a sectional-Anosov flow which either is transitive or has dense periodic orbits on a compact 3-manifold is Lorenz-like.*

Proof. Since $\Omega(X) \setminus \{\sigma\}$ is not closed there is a sequence $x_n \to \sigma$ such that $x_n \in \Omega(X) \setminus \{\sigma\}$ for all n. Since $x_n \neq \sigma$ we can assume that $x_n \in W^u(\sigma) \cup W^s(\sigma)$.

Suppose that there is $p = x_n \in W^u(\sigma) \setminus \{\sigma\}$ close to σ. Then, there are sequences $y_m \to p$ and $t_m \to \infty$ such that $X_{t_m}(y_m) \to p$. Clearly there is a sequence $t'_m < t_m$ such that $X_{t'_m}(y_m) \to z$ for some $z \in W^s(\sigma) \setminus \{\sigma\}$. Notice that such a z also exists if there is $x_n \in W^s(\sigma)$.

Now suppose by contradiction that σ is not Lorenz-like. It follows from Lemma 2.8 that σ has two positive eigenvalues thus $W^{ss}(\sigma) = W^s(\sigma)$. Now we observe that the existence of z contradicts Lemma 2.7. □

The following theorem gives a sufficient condition for a singularity of a sectional-Amosov flow to be Lorenz-like.

Theorem 4.19. *Every singularity in the closure of the unstable manifold of a periodic orbit of a sectional-Anosov flow on a compact 3-manifold is Lorenz-like.*

Proof. Let X a sectional-Anosov flow on a compact 3-manifold M and O be a periodic orbit of X. If $\sigma \in Cl(W^u(O))$ were not Lorenz-like, then $W^s(\sigma) = W^{ss}(\sigma)$ would be one-dimensional. As $\sigma \in Cl(W^u(O))$, and obviously $\sigma \notin O$, we would have that $Cl(W^u(O))$ contains also one of the two-connected components of $W^s(\sigma) \setminus \{\sigma\}$. However, such connected components exit M by Lemma 2.8 contradiction. □

Proposition 4.20. *Every sectional-Anosov flow without Lorenz-like singularities on a compact* 3*-manifold has hyperbolic nonwandering set. In particular, all such flows have homoclinic orbits and infinitely many periodic orbits.*

Proof. Let X be a sectional-Anosov flow on a compact 3-manifold M as in the statement. It is easy to see from Lemma 2.8 applied to $\Lambda = M(X)$ that $\Omega(X) \setminus Sing(X)$ is closed in M since X has no Lorenz-like singularities. Therefore $\Omega(X) \setminus Sing(X)$ is hyperbolic by the hyperbolic lemma. Then the result follows since $\Omega(X) = (\Omega(X) \setminus Sing(X)) \cup Sing(X)$. □

On the other hand, it is easy to see that all Anosov flows on compact 3-manifolds are codimension one. The analogous fact holds easily for sectional-Anosov flows.

Proposition 4.21. *Every sectional-Anosov flow on a compact* 3*-manifold is codimension one.*

It follows from corollaries 2.11 and 2.12 that the maximal invariant set of a transitive sectional-Anosov flow with singularities X on a compact n-manifold M has empty interior and topological dimension $\leq n-1$. In that case we have the inequalities

$$dim(E^c_{M(X)}) - 1 \leq dim(M(X)) \leq n-1.$$

The lower bound $dim(E^c_{M(X)}) - 1$ is improved to $dim(E^c_{M(X)})$ when X has a periodic orbit. More precisely we have the following corollary.

Corollary 4.22. *If X is a transitive sectional-Anosov flow with singularities and periodic orbits on a compact n-manifold M, then*

$$dim(E^c_{M(X)}) \leq dim(M(X)) \leq n-1.$$

Proof. If O is a periodic orbit, then $dim(W^u(O)) = dim(E^c_{M(X)})$. On the other hand, $W^u(O) \subset M(X)$ and then $dim(E^c_{M(X)}) \leq dim(M(X))$. The upper bound follows from Corollary 2.12 since maximal invariant sets are proper. □

The following definition is a minor modification of the classical definition of expanding attractor [150]. A partially hyperbolic attractor Λ is *expanding* if its topological dimension coincides with the dimension of its central subbundle E^c_Λ. As noted in [150] not every partially hyperbolic attractor is expanding (e.g. the ambient manifold of a transitive Anosov flow). The following gives a sufficient condition for the maximal invariant set of a vector field to be an expanding attractor. Recall that a sectional-Anosov flow is *codimension one* if its stable subbundle is one-dimensional.

Corollary 4.23. *The maximal invariant set of a transitive codimension one sectional-Anosov flow with singularities and periodic orbits on a compact manifold is an expanding attractor.*

Proof. If X is a sectional-Anosov flow on a compact n-manifold as in the statement, then $dim(E^c_{M(X)}) = n-1$. Now the result follows from Corollary 4.22. □

4.2.2 Existence of singular partitions

We present results about existence of singular partitions for sectional-Anosov flows. The first one is the following.

Theorem 4.24. *If X is a sectional-Anosov flow on a compact 3-manifold M and $q \in M$, then every compact invariant subset $H \subset \omega(q)$ of X with $H \neq \omega(q)$ has singular partitions of arbitrarily small size.*

Proof. To prove the result it suffices to prove by Proposition 1.50 that for every $z \in H$ regular there is a cross section of small diameter Σ_z such that $z \in Int(\Sigma_z)$ and $H \cap \partial \Sigma_z = \emptyset$. Fix $z \in H$ regular.

We claim that $H \cap W^{ss}(z)$ has empty interior in $W^{ss}(z)$. Indeed, if $\omega(q)$ has a singularity then $\omega(q)$ cannot contain a local strong stable manifold by Theorem 2.9. From this and $H \subset \omega(q)$ we conclude that $H \cap W^{ss}(z)$ has empty interior in $W^{ss}(z)$. If $\omega(q)$ has no singularities then $\omega(q)$ (and so H) are hyperbolic sets by the hyperbolic lemma. It would follow from Lemma 2.9 that $H = \omega(q)$ which is against the assumption $H \neq \omega(q)$. This proves the claim.

Using it we can fix for all $z \in H$ a foliated rectangle of small diameter R_z^0 such that $z \in Int(R_z^0)$ and $H \cap \partial^h R_z^0 = \emptyset$. Observe that $\omega(q)$ is not a periodic orbit for it has a singularity. Hence the positive orbit of q cannot intersects the stable leaf $\mathscr{F}^s(z, R_z^0)$ infinitely many times (c.f. Lemma 5.6 p. 369 in [110]).

Therefore, the positive orbit of q intersects either only one or the two connected components of $R_z^0 \setminus \mathscr{F}^s(z, R_z^0)$. In the former case we select some point q' of the positive orbit inside that component, a point z' in the other component and define Σ_z as the subrectangle of R_z^0 bounded by $\mathscr{F}^s(q', R_z^0)$ and $\mathscr{F}^s(z', R_z^0)$. Notice that $H \cap \mathscr{F}^s(q', R_z^0) = \emptyset$ for if not then it would exist $h \in W^{ss}(q) \cap H$ which implies $\omega(q) = \omega(h) \subset H$ and then $H = \omega(q)$ which is against the hypothesis once more.

On the other hand, we clearly have $\omega(q) \cap \mathscr{F}^s(z', R_z^0) = \emptyset$ and so $H \cap \mathscr{F}^s(z', R_z^0) = \emptyset$ since the positive orbit of q do not intersect the component of $R_z^0 \setminus \mathscr{F}^s(z, R_z^0)$ containing z'. Since $H \cap \partial^h R_z^0 = \emptyset$ and $\partial^h \Sigma_z \subset \partial^h R_z^0 = \emptyset$ we get that Σ_z satisfies the required properties.

If the positive orbit of q intersects both components of $R_z^0 \setminus \mathscr{F}^s(z, R_z^0)$ we choose two points q', q'' of the positive orbit in each connected component and define Σ_z as the rectangle of R_z^0 bounded by $\mathscr{F}^s(q', R_Z^0)$ and $\mathscr{F}^s(q'', R_Z^0)$. In a similar way we can prove that this Σ_z satisfies the required properties. □

For the second result we shall use the following definition.

Definition 4.25. *We say that a C^1 vector field X with hyperbolic closed orbits has the* Property (P) *if for every periodic orbit O there is a singularity σ such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$.*

The advantage of this property is given by the elementary fact below.

Lemma 4.6. *Every point in the closure of the periodic orbits of a vector field with the Property (P) is accumulated by points for which the omega-limit set is a singularity.*

With this definition we have the following.

Theorem 4.26. *Let X be a sectional-Anosov flow with the Property (P) on a compact* 3*-manifold M. If $x \in M$ is not approximated by points for which the omega-limit set* has *a singularity, then $\omega(x)$ has a singular partition of arbitrarily small size.*

Proof. By Proposition 1.50 we have to prove that for all $z \in \omega(x)$ there is a cross section of small diameter Σ_z such that $z \in Int(\Sigma_z)$ and $\omega(x) \cap \partial\Sigma_z = \emptyset$. For this we shall proceed as in the proof of Theorem 4.24, but taking care with the fact that $\omega(x)$ is not necessarily contained in a singular omega-limit set. Property (P) will supply the alternative argument.

Suppose for a while that $\omega(x) \cap Sing(X) \neq \emptyset$. Then, the constant sequence $x_n = x$ yields a sequence of points whose omega-limit set has a singularity converging to x. Since this contradicts the assumption we obtain $\omega(x) \cap Sing(X) = \emptyset$, and so, $\omega(x)$ is hyperbolic by the hyperbolic lemma.

We claim that $\omega(x) \cap W^{ss}(z)$ has empty interior in $W^{ss}(z)$ for all $z \in \omega(x)$. Indeed, suppose by contradiction that it is not so. Then, $\omega(x)$ contains a local strong stable manifold through some of its points. Since $\omega(x)$ is hyperbolic we could apply Lemma 2.9 to $H = \omega(x)$ in order to conclude that x belongs to a hyperbolic repeller. It follows that there is a periodic orbit sequence accumulating on x. By Lemma 4.6 we would have that x is accumulated by points whose omega-limit set is a singularity contradicting the assumption. Therefore, the claim is true.

By the claim we can fix for all $z \in \omega(x)$ a foliated rectangle of small diameter R^0_z such that $z \in Int(R^0_z)$ and $\omega(x) \cap \partial^h R^0_z = \emptyset$. If the positive orbit of x intersects $\mathscr{F}^s(z, R^0_z)$ infinitelty many times we would have as in Theorem 4.24 that $\omega(x)$ is a periodic orbit in whose case the result is trivial. Therefore, we can assume that the positive orbit of x does not intersect $\mathscr{F}^s(z, R^0_z)$. Then, it intersects either only one or the two connected components of $R^0_z \setminus \mathscr{F}^s(z, R^0_z)$.

If the positive orbit intersects only one component we select some point x' of the positive orbit inside that component, a point z' in the other component and define Σ_z as the subrectangle of R^0_z bounded by $\mathscr{F}^s(x', R^0_z)$ and $\mathscr{F}^s(z', R^0_z)$. Since the positive orbit does not passes through the connected component of $R^0_z \setminus \mathscr{F}^s(z, R^0_z)$ containing z' we have that $\omega(x) \cap \mathscr{F}^s(z', R^0_z) = \emptyset$. Now suppose for a while that there is $h \in \omega(x) \cap \mathscr{F}^s(x', R^0_z)$. Since $\omega(x)$ is hyperbolic we can apply the shadowing lemma for flows to the positive orbit of x in order to find a periodic orbit sequence accumulating on h. Such periodic orbits are in turn accumulated by points for which the omega-limit set is a singularity by Lemma 4.6. Since the stable manifolds in $M(X)$ have uniformly large size we have that x' is also accumulated by points for which the omega-limit set is a singularity. This would imply that x is accumulated by points whose omega-limit set is a singularity, a contradiction. This contradiction proves that $\omega(x) \cap \mathscr{F}^s(x', R^0_z) = \emptyset$. Since $\partial^h \Sigma_z \subset \partial^h R^0_z$ and $\partial^v \Sigma_z = \mathscr{F}^s(z', R^0_z) \cup \mathscr{F}^s(x', R^0_z)$ we have that Σ_z has the required properties.

Now we consider the case when the positive orbit intersects both components of $R^0_z \setminus \mathscr{F}^s(z, R^0_z)$. In such a case we choose two points x', x'' of that orbit, in each connected component, and define Σ_z as the rectangle of R^0_z bounded by $\mathscr{F}^s(x', R^0_Z)$ and $\mathscr{F}^s(x'', R^0_Z)$. Again we have that $\omega(x) \cap (\mathscr{F}^s(x', R^0_Z) \cup \mathscr{F}^s(x'', R^0_Z)) = \emptyset$ so Σ_z satisfies the required properties. This completes the proof. □

To state our last result about existence of singular partitions we need the following short definition. Let M be a compact 3-manifold, $X \in \mathscr{X}^1(M)$ and σ be a Lorenz-like singularity of X. It follows that $W^{ss}(\sigma)$ splits $W^s(\sigma)$ in two connected components denoted by $W^{s,+}(\sigma)$ and $W^{s,-}(\sigma)$ respectively (see Figure 4.2).

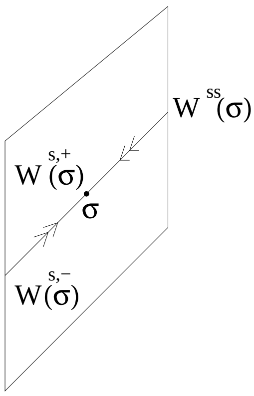


Fig. 4.2 $W^{s,+}(\sigma)$ and $W^{s,-}(\sigma)$

We use these sets in the following result.

Theorem 4.27. *Let X a sectional-Anosov flow on a compact 3-manifold, $\sigma \in Sing(X)$ Lorenz-like and $q \in W^u(\sigma)$ be non-recurrent. If there are $p, r \in Per(X)$ such that*

$$W^u(p)\cap W^s(\sigma)\subset W^{s,+}(\sigma),\qquad W^u(r)\cap W^s(\sigma)\subset W^{s,-}(\sigma), \tag{4.4}$$

$W^u(p)\cap W^s(\sigma)$ is dense in $W^{s,+}(\sigma)$ and $W^u(r)\cap W^s(\sigma)$ is dense in $W^{s,-}(\sigma)$, then $\omega(q)$ has singular partitions of arbitrarily small size.

Proof. We can assume that $\omega(q)$ is not a periodic orbit for, otherwise, the result is obvious. By Proposition 1.50 we only have to prove that for all $z\in\omega(q)$ there is cross section Σ_z close to z such that $z\in Int(\Sigma_z)$ and $\omega(q)\cap\partial\Sigma_z=\emptyset$.

First observe that $W^{ss}(z)\cap\omega(q)$ has empty interior in $W^{ss}(z)$ for, otherwise, q would be recurrent which is against the hypothesis. From this we can fix a foliated rectangle of small diameter R^0_z such that $z\in Int(R^0_z)$ and $\omega(q)\cap\partial^h R^0_z$.

We clearly have that $W^u(p)\cap W^{s,+}(\sigma)\neq\emptyset$ and $W^u(r)\cap W^{s,-}(\sigma)\neq\emptyset$ since they are dense in $W^u(p)$ and $W^u(r)$ respectively. Using it and linear coordinates around σ we can construct an open interval $I=I_a$, contained in a suitable cross section throught a, such that $I\setminus\{a\}$ is formed by two intervals $I^+\subset W^u(p)$ and $I^-\subset W^u(q)$. Since $W^u(p)\cap W^s(\sigma)$ is dense in $W^u(p)$ and $W^u(r)\cap W^s(\sigma)$ is dense in $W^u(r)$ we have that $W^{s,+}(\sigma)\cap I^+$ is dense in I^+ and $W^{s,-}(\sigma)\cap I^-$ is dense in I^-.

Since $z\in\omega(q)$ we have that the positive orbit of q contains a sequence in $Int(R^0_z)$ converging to z. If infinitely many elements of such a sequence belongs to $\mathscr{F}(z,R^0_z)$, then Lemma 5.6 in [110] implies that $\omega(q)$ is a periodic orbit which contradicts our initial assumption. Therefore, the positive orbit of q intersects either only one or the two connected components of $R^0_z\setminus\mathscr{F}^s(z,R^0_z)$.

First assume these intersections occurs in one component only. Since the positive orbit of q carries the positive orbit of I into such a component we can assume that the whole I is contained in that component. Hence, the stable manifolds throught I form a subrectangle R_I in that component. On the other hand, $R_I\setminus\mathscr{F}^s(q,R^0_z)$ is formed by two components R_I^+ and R_I^- formed by the stable manifolds through I^+ and I^- respectively (see Figure 4.3). Since $W^{s,+}(\sigma)\cap I^+$ is dense in I^+ and $W^{s,-}(\sigma)\cap I^-$ is dense in I^- we have that $W^{s,+}(\sigma)\cap R_I^+$ is dense in R_I^+ and $W^{s,-}(\sigma)\cap I^-$ is dense in R_I^-.

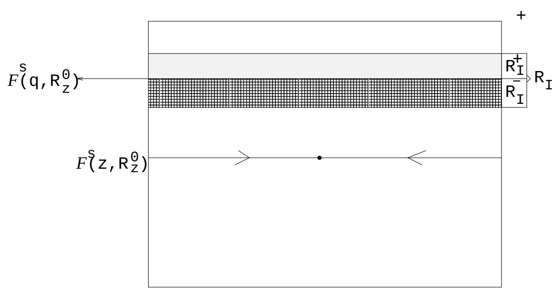


Fig. 4.3 $W^{s,+}(\sigma)$ and $W^{s,-}(\sigma)$

Let us prove that $\omega(q) \cap Int(R_I) = \emptyset$. Indeed suppose by contradiction that this is not the case. Then, the positive orbits of q intersects R_I infinitely many times. If it does in $\mathscr{F}^s(q, R_z^0)$ then $\omega(q)$ would be again a periodic orbit which is a contradiction. We conclude that the positive orbit intersects either R_I^+ or R_I^-. If it intersects R_I^+, then the positive orbit of I^- would intersects R_I^+. Since $W^{s,+}(\sigma) \cap R_I^+$ is dense in R_I^+ we conclude that I^- intersects $W^{s,+}(\sigma)$. But this contradicts (4.4) since $I^- \subset W^u(r)$. Therefore the positive orbit of q does not intersect R_I^+. In an analogous way we can prove that such an orbit does not intersect R_I^- so we get a contradiction. From this we conclude that $\omega(q) \cap Int(R_I) = \emptyset$.

Now select any point $z' \in Int(R_I)$, a point z'' in the component of $R_z^0 \setminus \mathscr{F}^s(z, R_z^0)$ which is not intersected by the positive orbit of q and define Σ_z as the subrectangle of R_z^0 bounded by $\mathscr{F}^s(q', R_z^0)$ and $\mathscr{F}^s(z', R_z^0)$. This rectangle satisfies the desired properties.

If the positive orbit intersect both components of $R_z^0 \setminus \mathscr{F}^s(z, R_z^0)$ we carrie I with the positive orbit as before to obtain two subrectangles R_I^t and R_I^b, like R_I, in each component. We then we select two points $z' \in Int(R_I^t)$ and $z'' \in Int(R_I^b)$ and define Σ_z as the rectangle in R_z^0 bounded by $\mathscr{F}^s(z', R_z^0)$ and $\mathscr{F}^s(z'', R_z^0)$. Again we have that this Σ_z works. The proof follows. □

Chapter 5
Codimension one Anosov flows

The results of this chapter will be based on the following definition.

An Anosov flow on a manifold M is *codimension one* if $dim(E^s_M) = 1$ or $dim(E^u_M) = 1$.

We start describing a relationship between the existence of codimension one Anosov flows and the fundamental group of the underlying manifold. Afterward we state some basic properties of these flows. Finally we prove the Verjovsky Theorem of transitivity of codimension one Anosov flows on closed n-manifolds $n \geq 4$.

5.1 Some basic properties

Let X be a codimension one Anosov flow on a closed manifold M. We can assume that $dim(E^s) = 1$ for otherwise we consider the time-reversed flow. It follows that $\mathscr{F}^u$ and $\mathscr{F}^s$ are foliations of M of codimension one and dimension two respectively. Let $\pi : \widehat{M} \to M$ be the universal cover of M. Hence there is a freely properly discontinuous action

$$\pi_1(M) \times \widehat{M} \to \widehat{M}$$

so that $M = \pi_1(M)\backslash\widehat{M}$. Let $\widehat{X}$ be the lift of X to $\widehat{M}$. Hence the diagram below

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{X}_t} & \widehat{M} \\ \pi \big\downarrow & & \big\downarrow \pi \\ M & \xrightarrow{X_t} & M \end{array}$$

commutes, i.e.

$$X_t(\pi(\hat{x})) = \pi(\widehat{X}_t(\hat{x})), \quad \forall t \in \mathbb{R}\, \forall \hat{x} \in \widehat{M}.$$

The following two lemmas are straightforward.

Lemma 5.1. *$\widehat{X}_t$ is Anosov.*

Now, let $\widehat{\mathscr{F}^{uu}}$, $\widehat{\mathscr{F}^{ss}}$, $\widehat{\mathscr{F}^{u}}$, $\widehat{\mathscr{F}^{s}}$ be the strong unstable, strong stable, unstable and stable foliations of $\widehat{X}_t$.

Lemma 5.2. *If $\hat{x} \in \widehat{M}$ and $x = \pi(\hat{x})$, then $\pi/\widehat{\mathscr{F}^u}\colon \widehat{\mathscr{F}}^u_{\hat{x}} \to \mathscr{F}^u_x$ is a covering map. Analogously replacing u by s, uu, ss.*

Lemma 5.3. *The foliations $\widehat{\mathscr{F}^*}$ $(* = u, s, uu, ss)$ are all invariant by the action $\pi_1(M) \times M \to M$.*

Proof. By definition we have the following:

Leaf of $\widehat{\mathscr{F}^u}$ = connected components of π^{-1}(leaf of $\mathscr{F}^u$).

Pick a leaf $\widehat{L}$ of $\widehat{\mathscr{F}^u}$ and $\gamma \in \pi_1(M)$. Hence $\pi(\widehat{L}) = L$ for same leaf L of $\mathscr{F}^u$. Clearly $\gamma(\widehat{L})$ is connected and $\pi(\gamma(\widehat{L})) = \pi(\widehat{L}) = L$ (as $\pi \circ \gamma = \pi \quad \forall \gamma \in \pi_1(M)$) $\therefore \gamma(\widehat{L})$ is a leaf of $\widehat{\mathscr{F}^u}$. Analogous proof works for *u* by *s*, *uu*, *ss*. □

Lemma 5.4. *Let $\gamma \in \pi_1(M) - 1$ and $\widehat{L}$ be a leaf of $\widehat{\mathscr{F}^u}$.*

(a) It $\gamma(\widehat{L}) = \widehat{L}$, then $\pi(\widehat{L})$ is a leaf of $\mathscr{F}^u$ containing a closed orbit θ which satisfies $\theta^n \in \gamma$ for some $n \in \mathbb{Z}$.

(b) If $L = \pi(\widehat{L})$ is a leaf of $\mathscr{F}^u$ containing a closed orbit θ, then $\gamma = [\theta] \in \pi_1(M)$ is a nontrivial $(\neq 1)$ element of $\pi_1(M)$ fixing $\widehat{L}$ (i.e. $\gamma(\widehat{L}) = \widehat{L}$).

Proof. (a). Suppose that $\gamma(\widehat{L}) = \widehat{L}$. Pick a base point $x_0 \in L := \pi(\widehat{L})$ and let $\hat{x} \in \pi^{-1}(x_0) \cap \widehat{L}$. By the assumption one has $\gamma(\hat{x}) \in \widehat{L}$. Since $\widehat{L}$ is path connected we have that there is a curve $\hat{c} \subseteq \widehat{L}$ joint $\hat{x}$ with $\gamma(\hat{x})$, i.e. $\hat{c}(0) = \hat{x}$, $\hat{c}(1) = \gamma(\hat{x})$. Define $c = \pi \circ \hat{c}$. Clearly $c \in \gamma$ by the definition of fundamental group. On the other hand, c is cannot be null-homotopic in L for, otherwise, $\gamma(\hat{x}) = \hat{x} \therefore \gamma = 1$ since the action $\pi_1(M) \times M \to M$ is free. This contradicts $\gamma \neq 1$. Hence c is not null-homotopic in L. It follow that L is *not* simply connected and then L is periodic (i.e. L contains a closed orbit θ). Since $\pi_1(L) = \mathbb{Z}$ with generator $[\theta]$ it follows that $\exists n \in \mathbb{Z}$ s.t. $\theta^n \in \gamma$. This proves (a).

(b). Suppose that $L = \pi(\widehat{L})$ is periodic, i.e. it contains a closed orbit θ. Let $x \in \theta$ and $T > 0$ be the period of θ, i.e. $X_T(x) = x$. We can assume that $T = 1$ without loss of generality. Consider the closed curves $c\colon [0,1] \to L$ given by $c(t) = X_t(x)$. Set $\gamma = [c] \in \pi_1(M)$. Observe that $\gamma \neq 1$ for, otherwise, θ would be null homotopic in M which is absurd. The lift of c to $\widehat{M}$ is precisely the curve $\hat{c}$ with $\hat{c}(0) = \hat{x}$, $\hat{c}(1) = \gamma(\hat{x})$. Since $\hat{c} \subseteq \widehat{L}$ we have $\gamma(\hat{x}) \in \widehat{L}$ for $\hat{x} \in \widehat{L}$. Hence

$$\gamma(\widehat{L}) \cap \widehat{L} \neq \emptyset$$

$\therefore \gamma(\widehat{L}) = \widehat{L}$ (recall that $\widehat{L}$, $\gamma(\widehat{L})$ are leaves of $\widehat{\mathscr{F}^u}$). This proves (b). □

Proposition 5.1. *The leaves of $\mathscr{F}^u$ are either planes $\mathbb{R}^{n-2} \times \mathbb{R}$ or cilinder $\mathbb{R}^{n-2} \times S^1$ or Moebious bands $\mathbb{R}^{n-2} \hat{\times} S^1$. In particular, the leaves of $\widehat{\mathscr{F}^u}$ are planes $\mathbb{R}^{n-1}$ in $\widehat{M}$ and so $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$.*

Proof. The first assertion is a direct consequence of the hyperbolic theory. The second assertion follows from the fact that the leaves of $\mathscr{F}^u$ are all simply connected (otherwise the leaves of $\mathscr{F}^u$ would not π_1-inject into M). Since $\widehat{M}$ is foliated by planes we have that $\widetilde{M}$ is *homeomorphic* to $\mathbb{R}^n$ by the Palmeira's thesis. □

Remark 5.2. *(a) A leaf of $\mathscr{F}^u$ is periodic $\Leftrightarrow$ it is either a cylinder or a Moebious Band.*

(b) If M is orientable, and $\mathscr{F}^u$ is transversely orientable, then $\mathscr{F}^u$ have no Moebious band leaves.

Lemma 5.5. *If M is a closed manifold supporting codimension one Anosov flows, then M is aspherical.*

Proof. Let $\pi : M \to \widehat{M}$ be the universal cover of M. By the proposition we have that $\widehat{M}$ is $\mathbb{R}^m$ where $m = \dim M$. If $n \geq 2$ then any continuous map $f\colon S^n \to M$ lifts to a map $\hat{f}\colon S^n \to \widehat{M}$ since $\pi_1(S^n) = 1 \quad \forall n \geq 2$. By Proposition 1.19 the map $\hat{f}\colon S^n \to \widehat{M}$ extends to a map $\bar{\hat{f}}\colon E^{n+1} \to \widehat{M}$. Setting $\bar{f} = \pi_0\ \bar{\hat{f}}$ we obtain an extension $\bar{f}\colon E^{n+1} \to M$ of $f\colon S^n \to M$ to E^{n+1} proving the result. □

Lemma 5.6. *If M is a closed manifold supporting codimension one Anosov flows, then $\pi_1(M)$ is torsion free.*

Proof. This follows from Lemma 5.5 since, by Proposition 1.22, the fundamental group of an aspherical manifold of finite dimension is torsion-free. □

Proposition 5.3. *A closed manifold supporting codimension one Anosov flows is irreducible.*

Proof. Let M be a closed n-manifold supporting codimension one Anosov flows. Then, the universal cover of M is $\mathbb{R}^n$. Let S be an embedded $(n-1)$-sphere in M. Since $\pi_1(S^n) = 1$ for $n \neq 1$ we can lift S to a sphere $\widehat{S}$ in the universal covering space $\mathbb{R}^n$. This sphere bounds a n-ball $\widehat{B}$ in $\mathbb{R}^n$ since $\mathbb{R}^n$ is irreducible (Generalized Schoenflies's Theorem). Such a ball is contained in the interior of a fundamental domain of the covering $\mathbb{R}^n \to M$. Hence $\widehat{B}$ project to a n-ball with boundary S. □

5.2 Exponential growth of fundamental group

In this section we give an important application of Theorem 1.24 concerning the topology of closed manifolds supporting Anosov flows. The three-dimensional case of the theorem below was proved by Margulis while the general case is due to Plante and Thurston.

Theorem 5.4 (Plante-Thurston-Margulis)**.** *If M is a closed manifold supporting codimension one Anosov flows, then $\pi_1(M)$ has exponential growth.*

Proof. Let ϕ_t be an codimension one Anovov flow on M. Assume that $\dim E^s = 1 \therefore \dim(E^u \oplus E^0) = n-1$. By structural stability we can assume that ϕ_t is C^∞ (C^2 suffices). Hence $E^u \oplus E^0$ is C^1 $\therefore$ the weak unstable foliation $\mathscr{F}^u$ (tangent to $E^u \oplus E^0$) is C^1. We have that $\mathscr{F}^u$ has no null-homotopic closed transversal (this follows from the C^1 Haefliger Theorem since $\mathscr{F}^u$ has no one-side holomony leaf). Hence *any* leaf of $\mathscr{F}^u$ satisfy the hypothesis (1) of the Plante's Theorem. Let us check the hypothesis (2) for every leaf $W^u(x)$ of $\mathscr{F}^u$ which *does not* contain periodic orbits.

Fix φ 1-ball $D_1^w(x)$ in $W^{uu}(x)$ (the strong unstable manifold) centered at x. Fix $R > 0$. For every $y \in D_1^{uu}(x)$ we set

$$I_R(y) = \varphi_{[0,R]}(y).$$

As M is compact we have that

$$\text{length}(I_R(y)) \approx C \cdot R$$

for some continuous $V > 0$ independing on R. So

$$\bigcup_{y \in D_1^{uu}(x)} I_R(y) \subseteq \overbrace{D_{c \cdot R+1}(x)}^{\text{the } (c \cdot R+1)\text{-ball in } W^u(x)}$$

$\therefore$

$$G(x, c \cdot R+1) \geq \text{ Volume } \left(\bigcup_{y \in D_1^{uu}(x)} I_R(y) \right). \qquad (\oplus)$$

But

$$\bigcup_{y \in D_1^{uu}(x)} I_R(y) = \bigcup_{0 \leq t \leq R} \varphi_t(D_1^{uu}(x))$$

Thus, by Fubini's Theorem, we get

$$\text{Vol}(\left(\bigcup_{y \in D_1^{uu}(x)} I_R(y) \right) = \int_0^R \text{area}(D_1^{uu}(x)) \cdot dt$$

(Here we use that $W^u(x)$ *does not* contain periodic orbits.) As $D_1^{uu}(x) \subseteq W^{uu}(x)$ we get

$$\text{Area}(\varphi_t(D_1^{uu}(x)) \geq \delta \cdot K \cdot e^{\lambda t}.$$

Where K, λ are the hyperbolicity constant and $\delta = \dim(D_1^{uu}(x)) > 0$. Hence

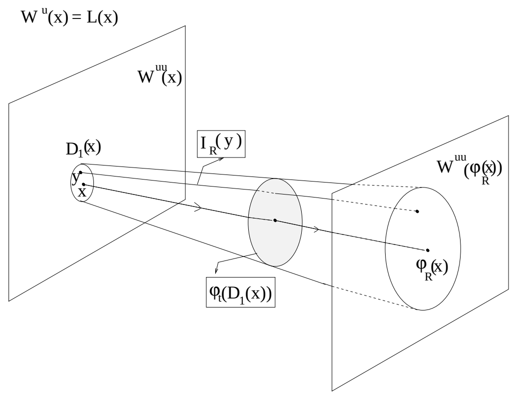


Fig. 5.1

$$
\begin{aligned}
\mathrm{Vol}\left(\bigcup_{y\in D_1^{uu}(x)} I_R(y)\right) &\geq \int_0^R \delta\cdot K\cdot e^{\lambda t}\,dt\\
&= \frac{\delta\cdot K}{\lambda}\,[e^{\lambda t}]_0^R = \frac{\delta\cdot K}{\lambda}\cdot[e^{\lambda R}-1]\\
&\geq \frac{\delta\cdot K}{\lambda}\cdot e^{\lambda R}.
\end{aligned}
$$

Applying $(\oplus)$ we get

$$G(x, c\cdot R+1) \geq \frac{\delta\cdot K}{\lambda}\cdot e^{\lambda R}.$$

Hence

$$G(x,R) \geq \frac{\delta\cdot K}{\lambda}\cdot e^{\lambda\left(\frac{R-1}{c}\right)} = B\cdot e^{cR}$$

where

$$\begin{cases} B = \frac{\delta\cdot K}{\lambda}\cdot e^{\lambda/c}\\ C = \frac{\lambda}{c}\end{cases}$$

This prove (2) i.e. every non periodic leaf $L(x)$ of $\mathscr{F}^u$ has exponential growth. Now the result follows from Theorem 1.24. □

Corollary 5.5. *The manifolds T^3, S^3, $S^2\times S^1$ cannot support Anosov flows.*

Proof. These manifolds are closed and the groups $\pi_1(T^3)$, $\pi_1(S^3)$, $\pi_1(S^2\times S^1)$ have no exponential growth. □

5.3 Verjovsky Theorem

In this section we shall prove the following theorem due to Verjovsky [145].

Theorem 5.6 (Verjovsky)**.** *A codimension one Anosov flow on a closed n-manifold $n \geq 4$ is transitive.*

The proof we present here is the one in the Barbot's thesis [16]. Some previous lemmas are needed. To begin with we consider a codimension one Anosov flows X on closed n-manifolds M. In the lemmas below we shall assume that $dim(E^s) = 1$. Hence $\mathscr{F}^u$ of X is codimension one and $\mathscr{F}^s$ is two-dimensional. Let $\widetilde{M} \to M$ be the universal cover of M. Let $\widetilde{X}$ be the lift of X to $\widetilde{M}$. Let $\widetilde{\mathscr{F}^*}$ be the lift of $\mathscr{F}^*$ to $\widetilde{M}$ $(* = S, u, ss, uu)$.

Lemma 5.7. *A leaf of $\widetilde{\mathscr{F}^u}$ intersects a leaf of $\widetilde{\mathscr{F}^{ss}}$ at most once.*

Proof. If there were a lift $\widetilde{L}$ of $\widetilde{\mathscr{F}^u}$ intersecting a lift $\tilde{\ell}$ of $\widetilde{\mathscr{F}^{ss}}$ in two points, then we would have an picture as below (note that $\widetilde{\mathscr{F}^{ss}}$ is orientable because $\widetilde{M}$ is simply connected):

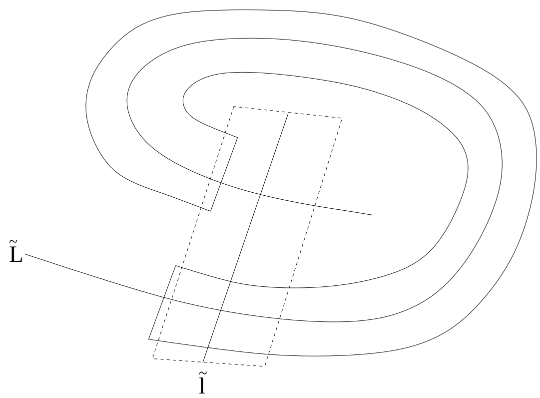


Fig. 5.2

With this it would be easy to construct a null homotopy closed curve $(C = \pi \circ \tilde{c})$ of $\mathscr{F}^u$ in M a contradiciton. □

Lemma 5.8. *If $\widetilde{X}, \tilde{y} \in \widetilde{M}$ satisfy the property below*

$$\exists \widetilde{X}_n \to \widetilde{X} \text{ and } T_n \geq 0 \text{ such that } y_n := \widetilde{X}_{T_n}(x_n) \to \tilde{y} \text{ as } n \to \infty \qquad (**)$$

then $\widetilde{X}$ and $\tilde{y}$ belongs to the same orbit of $\widetilde{X}$.

Proof. Let $\widetilde{L}_{\widetilde{X}}$, $\widetilde{L}_{\tilde{y}}$ be leaves of $\widetilde{\mathscr{F}^u}$ containing $\tilde{x}$, $\tilde{y}$, resp. By using local product structure and the stable foliation $\widetilde{\mathscr{F}^s}$ we can assume in (**) that $\widetilde{X}_n \in \widetilde{L}_{\tilde{x}}$ $\forall n$ (unless T_n is bounded).

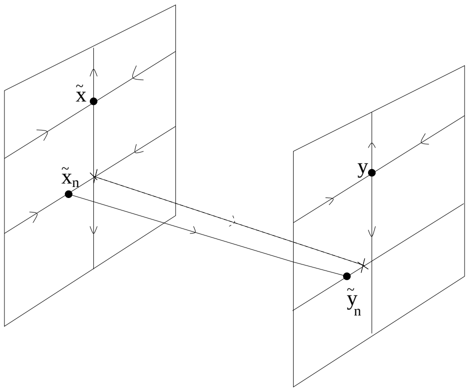


Fig. 5.3

By using local product once more we see that $\tilde{y} \in \widetilde{L}_{\tilde{x}}$ unless $\widetilde{L}_{\tilde{x}}$ intersects $\ell_{\tilde{y}}$ twice ($\ell_{\tilde{y}}$ is the leaf of $\widetilde{\mathscr{F}^{ss}}$ containing $\tilde{y}$). The last is absurd hence $\tilde{y} \in \widetilde{L}_{\tilde{x}}$. We conclude that the negative $\widetilde{X}$-orbit of $\tilde{y}$ contains $\widetilde{X}$. □

Theorem 5.7. *The orbit space $\widetilde{M}/\tilde{X}$ is diffeomorphic to $\mathbb{R}^n$.*

Proof. First we observe that $\widetilde{M}/\widetilde{X}$ is a $(n-1)$-manifold. (The chards are obtained from the ones of $\widetilde{M}$ by just projecting). Let $\pi_\theta : \widetilde{M} \to \widetilde{M}/\tilde{X}$ be the natural projection. We observe that $\forall \widetilde{O} \in \widetilde{M}/\tilde{X}$ there is an open neighborhood $\tilde{U}$ of $\widetilde{O}$ such that $\pi_\theta^{-1}(\widetilde{U}) \cong \widetilde{U} \times O_{\widetilde{X}}(\tilde{x})$ for some $\tilde{x} \in \widetilde{M} \therefore \pi_\theta : \widetilde{M} \to \widetilde{M}/\widetilde{X}$ is a fibration with fiber $\mathbb{R}$ (to obtain the last property one uses the fact that $\Omega(\widetilde{X}) = \emptyset$). By the exact sequence of the fibration we have

$$1 \to \underset{\substack{\shortparallel \\ 1}}{\pi_1(\text{Fibre})} \hookrightarrow \underset{\substack{\shortparallel \\ 1}}{\pi_1(\widetilde{M})} \to \pi_1(\widetilde{M}/\widetilde{X}) \to 1 \therefore \pi_1(\widetilde{M}/\widetilde{X}) = 1/1 = 1$$

This proves that $\widetilde{M}/\widetilde{X}$ is simply connected. By Lemma 2 we have that $\widetilde{M}/z\widetilde{X}$ is *Hausdorff*. Moreover, the projection $\pi_\theta(\widetilde{\mathscr{F}^u})$ of $\widetilde{\mathscr{F}^u}$ yields a foliation by $(n-2)$ planes of

$$\widetilde{M}/\widetilde{X} \therefore \boxed{\widetilde{M}/\widetilde{X} = \mathbb{R}^{n-1}}$$

by Palmeira's Theorem [65]. □

Notation:

- $O_X =$ orbit space $\widetilde{M}/\widetilde{X}$ of $\widetilde{X}$
- $\pi_\theta : \widetilde{M} \to O_X$ the natural projection.

- $\mathscr{Y}^\sigma = \pi_\theta(\widetilde{\mathscr{F}^\sigma})$ $(\sigma = s, u)$ are the projection of $\widetilde{\mathscr{F}^\sigma}$ to θ_X. Therefore, Y^u is a foliation by $(n-2)$ planes of $O_X \simeq \mathbb{R}^{n-1}$ and $\mathscr{Y}^s$ is a foliation by lines of O_X.

  Recall that there is an action

$$\pi_1(M) \times \widetilde{M} \to \widetilde{M}$$

preserving $\widetilde{\mathscr{F}^\sigma}$ $(\sigma - s, u, ss, uu)$. This actions induces an action

$$\pi_1(M) \times \theta_X \to \theta_X$$

which preserves $\mathscr{Y}^\sigma$ $(\sigma = s, u)$.

Remark 5.8. *Not every leaf of $\mathscr{Y}^u$ intersects all leaves of $\mathscr{Y}^s$ (one can have a picture as below).*

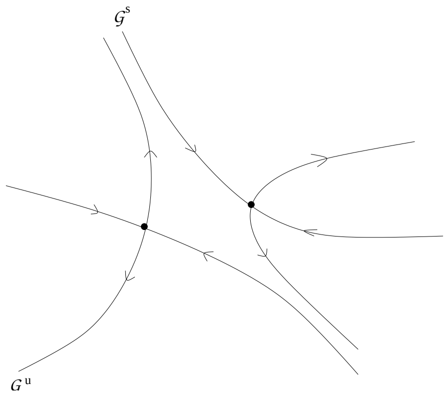


Fig. 5.4

Proof of Theorem 5.6: We rever the flow to assume cod $\mathscr{F}^s = 1$. By passing to a double cover if necessary we can assume that $\mathscr{F}^u$ is orientable. We can define (via orientability of $\mathscr{F}^u$) $\forall x \in M$ the sets

$$[x, \infty), \quad (x, \infty), \quad (-\infty, x], \quad (-\infty, x)$$

as the oriented half-intervals in $\mathscr{F}^u_x$. To prove that X is transitive we shall prove that every repeller Ω_0 of X is an attractor of X. To prove it we use the following lemma:

Lemma 5.9. *If $x \in \Omega_0 \Rightarrow (x, \infty) \cap \Omega_0 \neq \emptyset$ and $(-\infty, x) \cap \Omega_0 \neq \emptyset$.*

Proof. We only prove $(x, \infty) \cap \Omega_0 \neq \emptyset$ $\forall x \in \Omega_0$ (the other proof is similar). By contradiction we assume that

$$A = \{x \in \Omega_0 : (x,\infty) \cap \Omega_0 = \emptyset\} \neq \emptyset.$$

We claim that A is a finite union of closed orbits. In fact, pick $x_0 \in A$ and consider $y \in \alpha_Y(x_0)$

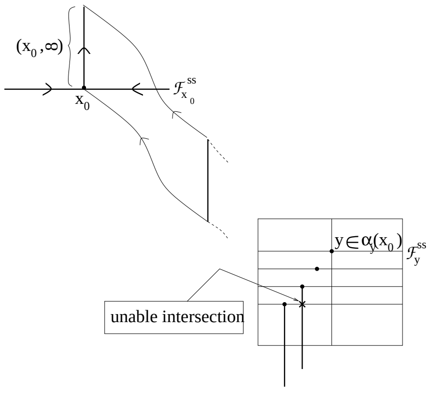


Fig. 5.5

Obvioiusly $y \in \Omega_0$ (because Ω_0 is a repellor) $\therefore \mathscr{F}^s_y \subseteq \Omega_0$. Now suppose that the orbit of x_0 hits a cross-section of y infinitely many times. By using local product structure we observe that $x_0 \in \mathscr{F}^s_y$. As $t \in \alpha_Y(x_0)$ we have that the X-orbit of y is periodic and that $x_0 \in$ orbit of y. This proves the claim.

By the claim we pick $x_0 \in A$ and let $\theta_0 = X_{\mathbb{R}}(x_0)$ be the orbit of x_0 (which is periodic). Let $\gamma_0 = [\theta_0]$ be the element θ_0 represents in $\pi_1(M)$. Then θ_0 is a fixed point of γ_0 (viewing γ_0 as a map $O_x \hookleftarrow$). Denote by F_0 the leaf of $\mathscr{Y}^s$ containing θ_0. Set $f_0' = F_0 - \theta_0$

By the claim once more we have that every orbit in A is closed. Since the ones represented by elements in F_0' are not closed we have that

$$x \in \Gamma \stackrel{\text{def}}{=} \mathscr{F}^s_{x_0} - \theta_0 \text{ implies } (x,\infty) \cap \Omega_0 \neq \emptyset.$$

By local product (and $\Omega_0 = \overline{\mathscr{F}^s_{x_0}}$) the last is equivalent to

$$(x,\infty) \cap \mathscr{F}^s_{x_0} \neq \emptyset,\ \forall x \in \Gamma.$$

Consequently, $\forall \theta \in F_0'$ one has

$$(\theta,\infty) \cap F_0 \neq \emptyset.$$

(Where $(*,\infty)$ is the half-intend induced by $\mathscr{Y}^{u^0}$ in O_X). This allow us to define the map

$$h\colon F_0' \to O_X$$

by the identity

$$\ell[\theta, h(\theta)] = \inf\{\ell[\theta, \theta'] : \theta' \in (\theta, \infty) \cap F_0\}.$$

Local product structure implies

$$h(\theta) > 0, \quad \forall \theta \in F_0'.$$

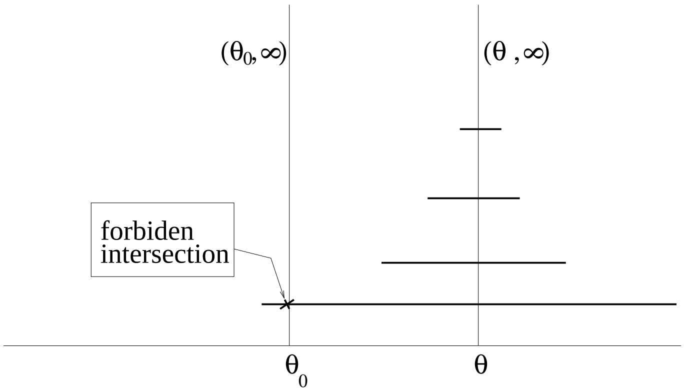


Fig. 5.6

Key fact: (*KF*)

Since $n = \dim M \geq 4 \Rightarrow n-2 > 2 \Rightarrow \dim(F_0) > 2 \quad (\dim \mathscr{Y}^s = n-2) \therefore F_0' = F_0 - \theta_0$ is *connected*.

Our second claim is that there is a leaf F_1 of $\mathscr{Y}^s$ such that $h(\theta) \in F_1, \quad \forall \theta \in F_0'$. In fact, we set $\forall z \in \theta_X$

$$V(z) = \{\theta \in F_0' : h(\theta) = \mathscr{Y}_z^s\}$$

(•) $F_0' = \bigcup_{z \in \theta_X} V(z)$ (obvious because $h(\theta) \in \mathscr{Y}^s_{h(\theta)}$)

(•) $V(z)$ is open in F_0'

(•) The $V(z)$'s are pairwise *disjoint*.

(•) $V(z)$ is closed in F_0' (because of the previous two statements).

By the **KF** we get $V(z) = F_0'$ for some z and the claim follows.

Now we observe that $F_1 \neq F_0$. Indeed, if $F_1 = F_0$ then we would have a picture as below which is impossible by local product structure. We conclude that $F_1 \neq F_0$.

Our third claim is that if F_1 is leaf as in the second claim, then γ_0 (viewed as a covering map) fixes F_1. Indeed, we first observe that γ_0 commutes with h, i.e.

$$\boxed{h(\gamma_0\theta) = \gamma_0(h\theta) \quad \forall \theta \in F_0'}$$

to see this we observe that γ_0 carries $[\theta, h\theta]$ to an interval $[\gamma_0\theta, \gamma_0(h\theta)]$ in $\mathscr{Y}^u_\theta$ because $\mathscr{Y}^u_\theta$ is γ_0-invariant. We have the figure below. Analogously F_0 does not intersect $(\gamma_0\theta, \gamma_0 h\theta)$ because it is fixed by $\gamma_0 \therefore \gamma_0(h\theta) = h(\gamma_0\theta)$. Now, $h(\theta) \in F_1$

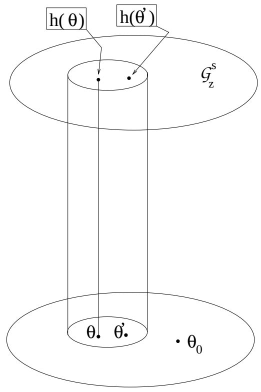


Fig. 5.7

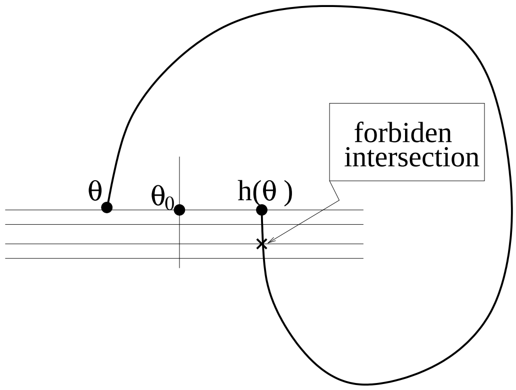


Fig. 5.8

$\forall \theta \in F_0$

$$\begin{aligned}&\Rightarrow h(\theta) \in F_1 \cap \gamma_0^{-1}(F_1)\\ &\therefore F_1 \cap \gamma_0^{-1}(F_1) \neq \emptyset\\ &\therefore \gamma_0(F_1) = F_1 \quad \text{proving our third claim.}\end{aligned}$$

Now we finish the proof of Lemma 5.9. Because $\gamma_0 \neq$ Id it follows from our third claim that γ_0 has a unique fixed point θ_1 in F_1. Moreover, γ_0/F_0 , γ_0/F_1 are contractions with global fixed points θ_0, θ_1 respectively. Take a $\overbrace{(n-3)-\text{sphere}}^{\dim(O_X)=n-1\ \dim F_0=n-2}$ S_0 in F_0 centered at θ_0 $\therefore S_1 = h(S_0)$ is also a $(n-3)$-sphere in F_1. Clearly S_0 is the boundary of a $(n-2)$-ball $B_0^{n-2} \subseteq F_0$ centered at θ_0. By Jordan-Browder (or generalized Alexander's Theorem) we get that S_1 is also the boundary of a $(n-2)$ ball $B_1' \subseteq F_1$. As $F_1 \neq F_0$ the set

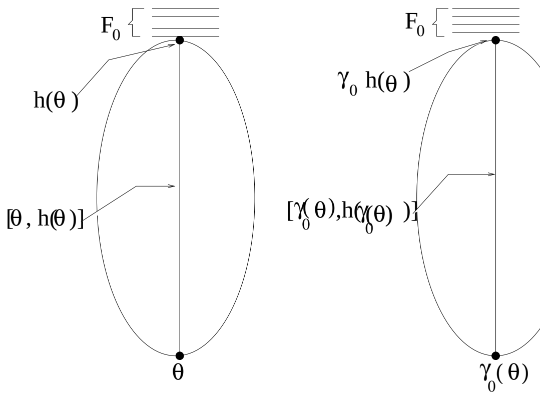


Fig. 5.9

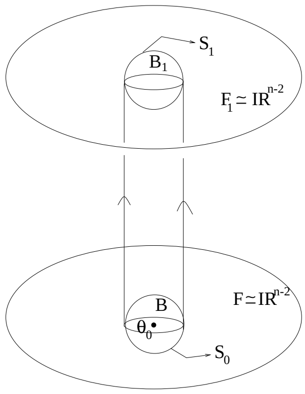


Fig. 5.10

$$S = B_0 \cup \left(\bigcup_{\theta \in S_0} [\theta, h(\theta)] \right) \cup B_1$$

is then a $(n-2)$ sphere in $O_X = \mathbb{R}^{n-1}$. It follows that S bounds a $(n-1)$-ball $\mathbb{B}$ in O_x Note that the curve $[\theta_0, \infty)$ enters in $\mathbb{B}$ through $\theta_0 \in S - \partial\mathbb{B}$. *Observe that* $[\theta_0, \infty)$ *has no accumulation points in* θ_X . In fact, any accumulation point of (θ_0, ∞) produces a closed curve in $\widetilde{M}$ transverse to $\widetilde{\mathscr{F}^s}$ $\therefore$ exists null-homotopic closed curve in $M \pitchfork \mathscr{F}^s$ a contradiction (*alternatively* any accumation point of (θ_0, ∞) produces a couple of intersection points between a leaf of $\widetilde{\mathscr{F}^u}$ and a leaf of $\widetilde{\mathscr{F}^s}$ contradicting Lemma 5.7).

It follows that (θ_0, ∞) must intersect $\partial\mathbb{B}$ in another point $\neq \theta_0$. This point must be in $B_1 \subseteq F_1 \therefore (\theta_0, \infty) \cap F_1 \neq \emptyset \therefore (\theta_0, \infty) \cap F_0 \neq \emptyset$ a contradiciton. This contradiction proves Lemma 5.9. □

Now we finish with the proof of Theorem 5.6.

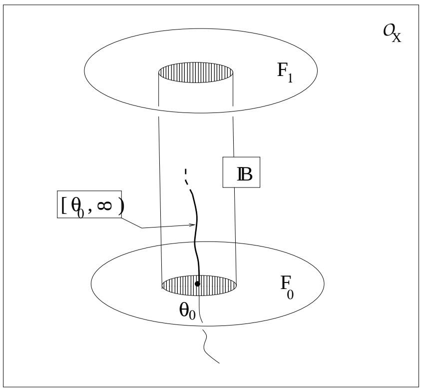


Fig. 5.11

As before we consider a repeller Ω_0 of X and we have to prove that Ω_0 is an attractor. By Lemma 5.9 one has

$$(x,\infty)\cap\Omega_0 \neq \emptyset,\ (-\infty,x)\cap\Omega_0 \neq \emptyset \quad \forall x \in \Omega_o$$

We claim that $\forall x \in \Omega_0$, $(x,\infty)\cap\Omega_0$ *is dense in* (x,∞). Indeed, suppose that exists $x \in \Omega_0$ such that $(x,\infty)\cap\Omega_0$ is *not* dense in (x,∞). Then, there is an open interval $I_X \subseteq (x,\infty)$ with $I_x \cap \Omega_0 = \emptyset$. By taking a bigger interval if necessary we can assume that $x \in \partial I_x$. Let $y \in \omega_X(x)$ be fixed. By using I_x as in the proof of Remark 1 (see proof of Lemma 5.9) we obtain that y is a periodic point (and $x \in W^s(y)$). By the λ-Lemma we have that the positive orbit of I_x accumulates on (y,∞). Then, as $I_x \cap \Omega_0 = \emptyset$, we get $(y,\infty)\cap\Omega_0 = \emptyset$. This contradicts Lemma 5.9 because $y \in \Omega_0$. This contradiction shows the claim.

Now, by the previous claim and the fact that Ω_0 is closed we get $[x,\infty) \subseteq \Omega_0$. Analogously we ge $(\infty,x] \subseteq \Omega_0$ and so $\mathscr{F}^u_x \subseteq \Omega_0 \ \ \forall x \in \Omega_0$. With this information it is easy to prove that Ω_0 is an attractor. Since Ω_0 is also a repeller we conclude that $\Omega_0 = M$. As Ω_0 is transitive for X we have that X is transitive flow. This completes the proof of Theorem 5.6. □

5.4 Anosov flows on 3-manifolds

In this section we study Anosov flows on closed 3-manifolds.

5.4.1 Transverse torus

We shall prove the following result due to Fenley and Brunella [44], [34] (see [93]).

Theorem 5.9. *If T is a torus transverse to an Anosov flow on a closed 3-manifold, then T is incompressible.*

Proof. Let M be a closed 3-manifold and let T be a torus transverse to an Anosov flow X in M. It follows that T is 2-sided. As M supports Anosov flows and $\dim M = 3$ we have that M is irreducible. It follows from Corollary 1.13 that one of the following properties hold:

(1) T is incompressible,
(2) T bounds a solid torus ST,
(3) T belongs to the interior of a 3-ball B.

If (1) holds then we are done. If (3) holds then T separates M and one of the connected components of $M \setminus T$ (M' say) belongs to the interior $\mathrm{Int}(B)$ of B (prove it as exercise). As T separates M we have that $\Omega(X) \cap T = \emptyset$. In particular, X is not transitive and so there is a spectral decomposition

$$\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k$$

formed by hyperbolic basis sets. One of then (Λ_1, say) is an attractor which can be assumed to belong to M' by reversing the flow it necessary. Hence $\Lambda_1 \subseteq M' \subseteq \mathrm{Int}(B) \therefore \Lambda_1$ is contained in the 3-ball B. Since B is simply connected we conclude that X has a null-homotopic periodic orbit (e.g. one in Λ_1). This is a contradiction which proves that (3) cannot occur. Now we assume (2). As before we can assume that exists attractor Λ_1 of X contained in the interion $\mathrm{Int}(ST)$ of ST. Let U be a smooth compact isolating block of Γ_1 contained in $\mathrm{Int}(ST)$. We can choose $\partial U \pitchfork X$ (Lyapunov function) $\therefore \partial U$ is a finite union of tori $T_1, \ldots, T_k$. Now, since T_i is contained in ST (which is a solid torus) we have that none of the T_i's is incompressible because $\pi_1(ST) = \mathbb{Z}$ and $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$. Cleary such tori are 2-sided (as they are $\pitchfork X$). As M is irreducible, Corollary 1.13 implies that either T_i is contained in a 3-ball or T_i bounds a solid torus $\forall i$. In the first case we get a contradiciton as before (case (3)). Hence T_i bounds a solid torus in M, $\forall i$. With this argument, replacing T by T_i if necessary, we can assume that $T = T_i$ and that ST is a isolating block of Λ_1.

Next the apply an argument due to Brunella. Define $\mathscr{F}_{ST}$ as the foliation induced by $\mathscr{F}^u$ (the wave unstable foliation) in ST. Clearl $\mathscr{F}_{ST}$ is transverse to $\partial(ST) = T$ (recall $X \pitchfork T$). Moreover, $\mathscr{F}_{ST}$ has no Reeb components (X Anosov $\therefore \mathscr{F}^u$ has no closed leaves). Let $2\mathscr{F}_{ST}$ be the double foliation defined in the double manifold $2ST$. ($2ST$ is obtained by gluing two copies of ST with the identity map $\mathrm{Id}\colon T = \partial(ST) \to T$). Hence $2ST$ is a genus one closed 3-manifold as ST is a solid torus [He]. It follow from Seifet-Van Kampen that

$$\pi_1(2ST) = \mathbb{Z} \therefore \pi_1(2ST) \text{ has no exponential growth.} \qquad (*)$$

Actually $2ST = S^2 \times S^1$. On the other hand, we can see that $2\mathscr{F}_{ST}$ has no Reeb component. Indeed, suppose that there is a Reeb component R of $2\mathscr{F}_{ST}$. Since $\mathscr{F}_{ST}$

has no Reeb component we would have that R intersects $2(ST)$ in a way that $R \cap ST$ is a *Half-Reeb component* as below:

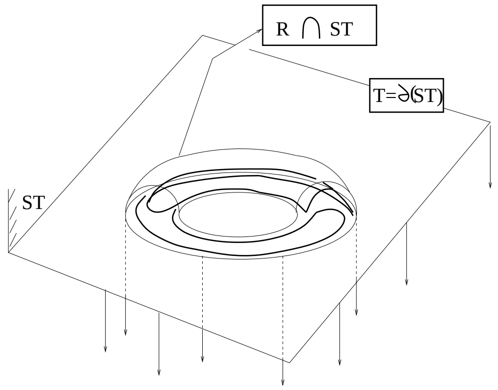


Fig. 5.12

By observing that X points outward to ST in $T = \partial ST$ (say) we observe that $\partial(R \cap ST)$ has a closed orbit θ. Such a closed orbit must be contained in Λ_1 (ST is an solated block) $\therefore$ its full unstable manifold also belongs to Λ_1 (Λ_1 is an attractor). This is a contradiction since $\partial(R \cap ST)$ is part of the unstable manifold of θ. Hence $2\mathscr{F}_{ST}$ has no Reeb component as desired. To finish the proof one see that $2\mathscr{F}_{ST}$ has a leaf of exponential growth (a non periodic leaf in Λ_1). Since $\mathscr{F}_{ST}$ has no Reeb component we get that no leaf of $2\mathscr{F}_{ST}$ intersects a null-homotopic closed transversal of $2\mathscr{F}_{ST}$ (this is the Novikov theorem). It would follow from the Plante's Theorem that $\pi_1(2ST)$ has exponential growth contradicting (*). This contradiciton proves the result. □

Next we present some applications of Theorem 5.9. A 3-manifold is called *atoroidal* if it does not have incompressible torus.

Corollary 5.10 ([44]). *Anosov flows on closed atoroidal orientable 3-manifolds are transitive.*

Proof. Every non-transitive Anosov flow X on a closed 3-manifold M exhibits an attractor $A \neq M$. By using Lyapunov functions we can arrange an isolating block U of A whose boundary ∂U is a disjoint union of closed surfaces $\pitchfork X$. Since X is defined on an orientable manifold all such surfaces are tori. By Theorem 5.9 all such tori are incompressible which contradicts that M is atoroidal. This proves the result. □

Recall that a manifold is *hyperbolic* if it admits a metric with negative curvature.

Corollary 5.11 ([44]). *An Anosov flow on a closed hyperbolic 3-manifold is transitive.*

Proof. By passing to a double covering we can assume that the manifold is orientable. It follows from the Preissman's Theorem [129] that every hyperbolic 3-manifold is atoroidal. Then Corollary 5.10 applies. □

Remark 5.12. *These are examples of closed hyperbolic 3-manifolds supporting Anosov flows (this was proved by S. Goodman [55]).*

Another application is given below.

Theorem 5.13. *An Anosov flow on a torus bundle over S^1 is transitive.*

Proof. By passing to a double covering we can assume that the torus bundle M is orientable. Let X be an Anosov flow in M. If X were not transitive then there were a transverse torus T of X which in the boundary of an isolating block U of an attractor A of X (the argument is the same as in the proof of Corollary 1.9. By Theorem 5.9 we have that T is incompressible. As M is a torus bundle over S^1 and M supports Anosov flows we have that M has Anosov monodromy. Then T is isotopic to the torus fiber of M (see Lemma 5.2 p. 54 in [63]). Hence the manifold M^1 obtained by containing open M along T is diffeomorphic to $[0,1]\times T^2$. (Note that $T \pitchfork X \Rightarrow T$ is two-sided). Let $2M^1$ be the double of M^1 and let $2\mathscr{F}_{M^1}$ be the double foliation $\mathscr{F}_{M^1} = \mathscr{F}^u/M^1$. One can see that $2M^1 = T^3$ (the 3-torus). As in the Brunella's argument we observe that $2\mathscr{F}_{M^1}$ does not exhibit closed null-homotopic transversals $\therefore$ no leaf of $2\mathscr{F}_{M^1}$ intersect such transversals. Again $2\mathscr{F}_{M^1}$ has a leaf with exponential growth. Hence $\pi_1(2M^1)$ would have exponential growth, a contradiction because $\pi_1(2M^1) = \pi_1(T^3) = \mathbb{Z}^3$. This contradiction proves the result. □

5.4.2 Product Anosov Flows

In this chapter we describe some techniques for the study of Anosov flows. To start with we consider an Anosov flow on a closed 3-manifold M.

We denote by $\mathscr{F}^s, \mathscr{F}^u$ the weak stable and weak unstable foliations of X respectively. We can assume that X is C^∞ and then $\mathscr{F}^u$ is C^1. Denote by $\pi : \hat{M} \to M$ the universal cover of M.

Note that $\pi_1(M)$ is identified with the set of fiber-preserving diffeomorphisms in $\hat{M}$ and $M = \hat{M}/\pi_1(M)$. In addition, since π is a local diffeomorphism, there is a vector field $\hat{X}$ in $\hat{M}$ such that for all $\hat{x} \in \hat{M}$ one has $\hat{X}(\hat{x}) = D\pi^{-1}(X)(X(X))$ where $x = \pi(\hat{x})$.

The vector field $\hat{X}$ is called the lift of X to $\hat{M}$. It follows that $\hat{X}$ is Anosov since X is. In particular $\hat{X}$ is equipped with the weak stable and the weak unstabe foliations $\hat{\mathscr{F}}^s, \hat{\mathscr{F}}^u$ respectively. Such foliations are precisely the ones obtained by lifting

$\mathscr{F}^s, \mathscr{F}^u$ to $\hat{M}$ via π, namely a leaf of $\hat{\mathscr{F}}^*$ is a connected component of $\pi^{-1}(L)$ for some leaf L of $\mathscr{F}^*$ $(* = s, u)$.

Note that $\pi \circ \alpha = \pi$ for all $\alpha \in \pi_1(M)$ since M is precisely the space of orbits of the action of $\pi_1(M)$ over $\hat{M}$. Hence if $\hat{L}$ is a leaf of $\hat{\mathscr{F}}^*$ and $\alpha \in \pi_1(M)$ then the commom value $\pi(\alpha(\hat{L})) = \pi(\hat{L})$ is a leaf of $\mathscr{F}^*$ $(* = s, u)$. In particular, if $\hat{L}$ is a leaf of $\hat{\mathscr{F}}^*$ and $\alpha \in \pi_1(M)$ then $\alpha(\hat{L})$ is also a leaf of $\hat{\mathscr{F}}^*$.

Now suppose that $\hat{L}$ is a leaf of $\hat{\mathscr{F}}^u$ which is fixed by some $\alpha \in \pi_1(M) \setminus \{1\}$. If $\hat{x} \in \hat{L}$ then $\alpha(\hat{x}) \in \hat{L}$ is different from $\hat{x}$ since $\alpha \neq 1$. There is a curve $\hat{c}$ in $\hat{L}$ joining $\hat{x}$ with $\alpha(\hat{x})$ since $\hat{L}$ is path-connected. Such a curve is not closed because $\hat{x} \neq \alpha(\hat{x})$.

The curve $c = \pi \circ \hat{c}$ is a closed curve contained in the leaf $L = \pi(\hat{L})$ of $\mathscr{F}^u$. Clearly c is not null homotopic in L as c lift to the non-closed curve $\hat{c}$ in $\hat{L}$.

We conclude that L is not a plane leaf of $\mathscr{F}^u$ and so it contains a periodic orbit of X. Conversely suppose that L is a leaf of $\mathscr{F}^u$ containing a periodic orbit O of X.

Let α be the element of $\pi_1(M)$ represented by the closed curve $O \subset M$. Note that $\alpha \neq 1$ since O is not null homotopic.

Let us prove that $\alpha(\hat{L}) = \hat{L}$ for all leaf $\hat{L}$ of $\hat{\mathscr{F}}^u$ with $\pi(\hat{L}) = L$. It suffices to show that $\alpha(\hat{L}) \cap \hat{L} \neq \emptyset$ since both $\alpha(\hat{L})$ and $\hat{L}$ are leaves of $\hat{\mathscr{F}}^u$. Choose $\hat{x} \in \hat{L}$ such that $\pi(\hat{x}) \in O$.

Since $\hat{L}$ is simply connected we have that $\pi : \hat{L} \to L$ is precisely the universal cover of L. Then we can lift O to a curve $\hat{O}$ in $\hat{L}$ such that $\hat{O}(0) = \hat{x}$. By definition $\alpha(\hat{x}) = \hat{O}(1) \in \hat{L}$ which proves $\hat{x} \in \hat{L} \cap \alpha^{-1}(\hat{L})$. Hence $\hat{L} \cap \alpha^{-1}(\hat{L}) \neq \emptyset$ and then $\alpha(\hat{L}) \cap \hat{L} \neq \emptyset$ as desired.

We have then proved that a leaf $\hat{L}$ of $\mathscr{F}^u$ is fixed by some $\alpha \in \pi_1(M) \setminus \{1\}$ precisely when the leaf $\pi(\hat{L})$ of $\mathscr{F}^u$ contains a periodic orbit of X. Similarly for the leaves of $\hat{\mathscr{F}}^s$.

Next we consider the leave spaces V^s, V^u of $\hat{\mathscr{F}}^s, \hat{\mathscr{F}}^u$ respectively. As $\hat{\mathscr{F}}^u$ is a foliation by planes we have that $\hat{M}$ is homeomorphic to $\mathbb{R}^3$ by the Palmeira's thesis.

On the other hand, we can endow V^u with an structure of 1-manifold by considering intervals transverse to $\hat{\mathscr{F}}^u$ in $\hat{M}$. Such an structure is well defined since the leaves of $\hat{\mathscr{F}}^u$ do not have self-accumulation (otherwise we could construct a closed transversal to $\mathscr{F}$ via π).

It is clear that every leaf of $\hat{\mathscr{F}}^u$ disconnects $\hat{M}$ (again by the abcense of closed transversal to $\mathscr{F}^u$). Moreover V^u is simply connected. It then follows that V^u is a simply connected 1-manifold possibly non-Hausdorff. Similarly for V^s.

Since the action of $\pi_1(M)$ in $\hat{M}$ leads invariant $\hat{\mathscr{F}}^s, \hat{\mathscr{F}}^u$ we have that such an action induces an action $\pi_1(M) \times V^* \to V^*$ $(* = s, u)$. Such an action plays a fundamental role in some results in the theory of Anosov flows.

The result below summarizes some important properties of these actions. The statement is done for V^u and a similar statement holds for V^s. See the previous Chapter 5 for the proof.

Theorem 5.14. *The following properties hold:*

1. $\alpha \in \pi_1(M) \setminus \{1\}$ *has a fixed point in* V^u *if and only if* α *is represented by a periodic orbit of* X.
2. X *is transitive if and only if the action* $\pi_1(M) \times V^u \to V^u$ *is* minimal, *i.e. the orbit* $O(f) = \{\alpha(f) : \alpha \in \pi_1(M)\}$ *is dense in* V^u, $\forall f \in V^u$.
3. *The isotropy group* $Stab(f) = \{\alpha \in \pi_1(M) : \alpha(f) = f\}$ *of* f *is either trivial or infinite cyclic,* $\forall f \in V^u$. *Such a group is not trivial precisely when* f *represents a leaf* $\hat{L}$ *of* $\hat{\mathscr{F}}^u$ *whose projection* $L = \pi(\hat{L})$ *contains a periodic orbit of* X. *In that case* $Stab(f)$ *is the infinite cyclic group generated by the representant of the unique periodic orbit of* X *in* L.
4. *The set* $\{f \in V^u : \alpha(f) = f$ *for some* $\alpha \in \pi_1(M) \setminus \{1\}\}$ *is dense in* V^u.
5. *If* f *is a fixed point of* $\alpha \in \pi_1(M) \setminus \{1\}$ *then* α *is contracting or expanding in a neghborhood of* f. *In particular, the set* $Fix(\alpha)$ *of fixed points of* α *is discrete.*

The structure of the 1-manifold V^u is also interesting. As mentioned before we have that V^u may be non-Hausdorff. This motivates the following definition.

Definition 5.15. *An Anosov flow* X *on a closed* 3*-manifold is* product *if* V^u *is Hausdorff (*$\Leftrightarrow V^u$ *is homeomorphic to* $\mathbb{R}$*).*

Not every Anosov flow is product. Many examples of this type can be obtained by the following observation due to Barbot [15].

Theorem 5.16. *Product Anosov flows are transitive.*

Proof. By hypothesis we have $V^u = \mathbb{R}$. Since X is not transitive the action $\pi_1(M) \times \mathbb{R} \to \mathbb{R}$ is not minimal and so there is an orbit of $\pi_1(M)$ whose closure Γ is not $\mathbb{R}$. Let I a maximal open interval in $\mathbb{R} \setminus \Gamma$ and f be a boundary point of I. We claim that $Stab(f) \neq 1$. In fact, there is $\alpha \in \pi_1(M) \setminus \{1\}$ fixing some element of I by part (4) of Theorem 5.14. Since I is maximal we have that $\alpha(I) = I$ and so α^2 fixes f. This proves $\alpha^2 \in Stab(f) \setminus \{1\}$ since $\pi_1(M)$ is torsion free. By part (3) of Theorem 5.14 we have that $Stab(f)$ is infinite cyclic. Then we can fix $\delta \in Stab(f) \setminus \{1\}$ a generator of $Stab(f)$. On one hand, by Theorem 5.14 there is an open interval J containing f such that δ/J is contracting. In particular, $\delta^k(g) \neq g$ for all $g \in J$ and $k \in \mathbb{N}^*$. On the other hand, we have that there is $g \in J$ fixed by some $\alpha \in \pi_1(M) \setminus \{1\}$. Necessarily one has $\alpha(I) = I$ and then $\alpha^2(f) = f$. It follows that $\alpha^2 \in Stab(f)$ and then there is $k \in \mathbb{Z}^*$ such that $\alpha^2 = \delta^k$ yielding $\delta^k(g) = g$. Obviously we can assume that $k > 0$ and so we obtain a contradiction since $\delta^k(g) \neq g$ $\forall g \in J$. The proof follows. □

It follows that the Anomalous Anosov flow in Chapter 1 is not product. We say that $f_1, f_2 \in V^u$ are separated if they exhibit disjoint neighborhoods in V^u. Otherwise f_1, f_2 are non-separated (denoted by $f_1 \approx f_2$). Given $\alpha \in \pi_1(M)$ we denote $Fix^{\approx}(\alpha) = \{f \in V^u : \alpha(f) \approx f\}$. The following gives a sufficient condition for X to be product.

Theorem 5.17. *Anosov flows on closed* 3*-manifolds whose fundamental group have non-trivial center are product.*

Proof. Recall the definition of the center $Z = Z(\pi_1(M))$ in Chapter 6. By assumption there is $\delta \in Z \setminus \{1\}$ which can be assumed to be orientation-preserving. Fix $n \in \mathbb{N}^*$. Every $f \in Fix(\delta^n)$ represents a periodic orbit of X. Since the set of periodic orbits of X is countable we have that $Fix(\delta^n)$ is countable. But $Fix^{\approx}(\delta^n) \setminus Fix(\delta^n)$ is formed by non-separated points of V^u. As the set of non-separating points of V^u is countable we conclude that $Fix^{\approx}(\delta^n) \setminus Fix(\delta^n)$ is countable for all $n \in \mathbb{N}^*$ too. It follows that $Fix^{\approx}(\delta^n) = Fix(\delta^n) \cup (Fix^{\approx}(\delta^n) \setminus Fix(\delta^n))$ is countable. It is clear that $Fix^{\approx}(\delta^n)$ is also closed in V^u. As $\delta \in Z$ we have that $Fix(\delta^n)$ is $\pi_1(M)$-invariant. In fact, consider $\alpha \in \pi_1(M)$ and $f \in Fix^{\approx}(\delta^n)$. Hence $\delta^n(f) \approx f$ and then $\alpha\delta^n(f) \approx \alpha(f)$. Because $\delta \in Z(\pi_1(M))$ we have $\delta^n(\alpha(x)) = \alpha\delta^n(x) \approx \alpha(f)$ proving $\alpha(f) \in Fix^{\approx}(\delta^n)$. Hence $\alpha(Fix^{\approx}(\delta^n)) = Fix^{\approx}(\delta^n)$ for all $\alpha \in \pi_1(M)$ as desired. The last invariance implies that the union of the leaves in $\hat{M}$ representing some element in $Fix^{\approx}(\delta^n)$ is projected into M to a compact $\mathscr{F}^u$-invariant subset which is transversely countable. The existence of closed tranversely countable sets implies necessarily the existence of a compact leaf of $\mathscr{F}^u$, a contradiction. We conclude that $Fix^{\approx}(\delta^n) = \emptyset$ and so $\delta^n(f)$ and f are separated $\forall f \in V^u$.

For all $n \in \mathbb{N}^*$ we consider the family $\mathscr{G}_n = \{J \subset V^u : J$ is an open δ^n-invariant interval$\}$. This family is not empty for some $n \in \mathbb{Z}^*$. In fact, let $\hat{V} = V^u/\delta$ be the quotient space induced by the action of δ in V^u. Since δ has no fixed points and V^u is simply connected we have that the natural quotient map $V^u \to \hat{V}$ is the universal covering of $\hat{V}$. Note that $\pi_1(\hat{V}) =< \delta >$ is non-trivial. Then there is a path γ in $\hat{V}$ which is not null-homotopic. The lift of γ to V^u yields a closed interval $[f, \delta^n(f)]$ in V^u joining f to $\delta^n(f)$ for some $(f,n) \in V^u \times \mathbb{Z}^*$. Hence $J = \cup_{k\in\mathbb{Z}}\delta^{kn}([f, \delta^n(f)])$ is an open δ^n-invariant interval. This proves $\mathscr{G}_n \neq \emptyset$. Without loss of generality we can assume that $n = 1$. If $J, J' \in \mathscr{G}$ then $J \cap J'$ is either $\emptyset$ or an open interval. In the last case the boundary points of $J \cap J'$ are fixed by δ since δ is orientation-preserving. Because δ has no fixed points we conclude that the family $\mathscr{G}$ is disjoint. If $J \in \mathscr{G}$ and $\alpha \in \pi_1(M)$ then $\alpha(J) \in \mathscr{F}$ because δ commutes with α. Hence $\mathscr{H} = \cup_{J\in\mathscr{G}} J$ is $\pi(M)$-invariant. One sees that $\mathscr{H}$ is open and closed in V^u. In fact, note that $K = \overline{\mathscr{H}} \setminus \mathscr{H}$ is formed by non-separating points of V^u and so K is countable. On the other hand, K is closed because $\mathscr{H}$ is open. And K is $\pi_1(M)$-invariant because K is δ-invariant and δ is central. As before we can use K to construct a transversely countable compact $\mathscr{F}^u$-invariant set. Since $\mathscr{F}$ has no closed leaves we conclude that $K = \emptyset$ and so $\mathscr{H}$ is closed in V^u. To finish we let $\hat{N}$ be the union of the leaves of

$\hat{\mathscr{F}}^u$ represented by points in $\mathscr{H}$ and let N be the projection of $\hat{N}$ to M. As $\mathscr{H}$ is open-closed in V^u we have that N is open-closed in M. Since M is connected we conclude that $N = M$. This proves that $\mathscr{H} = V^u$ and so X is product. The result follows. □

One can easily see that a torsion free group with non-trivial center has an infinite cyclic normal subgroup (e.g. the cyclic group generated by a central element). A sort of converse holds for by the following lemma.

Corollary 5.18. *Anosov flows on closed* 3*-manifolds whose fundamental group contains an infinite cyclic normal subgroup are product.*

Proof. Let M be a closed 3-manifold whose fundamental group $\pi_1(M)$ contains an infinite cyclic normal subgroup. By the previous corollary we have that $\pi_1(M)$ exhibits a finite index normal subgroup H with non-trivial center. Let $\tilde{M} \to M$ be the Galois covering associated to H. Such a covering is finite since H has finite index. Hence $\hat{M}$ is closed. Every Anosov flow in M lifts to an Anosov flow in $\hat{M}$ which is product by Theorem 5.17. Since $\hat{M}$ and M have the same universal covering we conclude that all Anosov flows in M are product. The proof follows. □

By definition, a 3-manifold M is *Seifert* if it exhibits a foliation by circles with some exceptional leaves [66]. Since all closed Seifert 3-manifolds have infinite cyclic normal subgroup we obtain the following corollary.

Corollary 5.19. *Anosov flows on closed Seifert* 3*-manifolds are product. In particular Anosov flows on circle bundles over closed surfaces are product.*

Theorem 5.20. *Circle bundles over a closed orientable surface cannot support suspended Anosov flows.*

Proof. Let M be a circle bundle over a closed orientable surface Σ. Suppose by contradiction that M supports suspended Anosov flows. It follows that M is a torus bundle over S^1. The exact sequence of the fibrering yields,

$$1 \longrightarrow \pi(T^2) \xrightarrow{\beta} \pi_1(M) \xrightarrow{\Phi} \pi_1(S^1) \longrightarrow 1$$

and

$$1 \longrightarrow \pi_1(S^1) \xrightarrow{\phi} \pi_1(M) \xrightarrow{\varphi} \pi_1(\Sigma) \longrightarrow 1$$

Let us suppose for a while that $Ker(\varphi) \cap Im(\beta) = 1$. Let δ the generator of $Ker(\varphi) = Im(\phi) = \mathbb{Z}$, and a, b be the generators of $Im(\beta) = Ker(\Phi) = \mathbb{Z}^2$. Since $Ker(\varphi)$ is normal we have $a\delta a^{-1}\delta^{-1} \in Ker(\varphi)$ as $\delta \in Ker(\varphi)$. Since $Im(\beta)$ is normal we have $a\delta a^{-1}\delta^{-1} \in Im(\beta)$ as $a^{-1} \in Im(\beta)$. Thus $a\delta a^{-1}\delta^{-1} \in \mathrm{Ker}(\varphi) \cap Im(\beta) = 1 \therefore a\delta a^{-1}\delta^{-1} = 1$, i.e. a and δ commute in $\pi_1(M)$. Analogously b and δ commute in $\pi_1(M)$. On the other hand, if $a^n b^m \delta^k = 1$ then $a^n b^m = \delta^{-k}$ and so

$a^n b^m \in Im(\beta) \cap Ker(\varphi) = 1$. Thus $a^n b^m = 1$ and so $m = n = 0$ since $Im(\beta) = \mathbb{Z}^2$ is torsion free. If follows that $k = 0$ since $\pi_1(M)$ is torsion free (as M supports Anosov flows). We conclude that the subgroup in $\pi_1(M)$ generated by a, b, δ is $\mathbb{Z}^3$. In other words $\pi_1(M)$ contains $\mathbb{Z}^3$ as a subgroup. Then $\pi_1(M)$ could be finitely covered by T^3 a contradiction since T^3 cannot support Anosov flows. Therefore $Ker(\varphi) \cap Im(\beta) \neq 1$. Then there is $n \in \mathbb{Z}^*$ such that $\delta^n \in Im(\beta)$, where δ is the generator of $Ker(\varphi)$. As $Im(\beta) = Ker(\Phi)$ we have $0 = \Phi(\delta^n) = n\Phi(\delta) \therefore \Phi(\delta) = 0$. We conclude that $Ker(\varphi) \subset Im(\beta)$. In other words we have the normal sequence $Ker(\varphi) \lhd Im(\beta) \lhd \pi_1(M)$. By the Group Isomorphism Theorem we obtain

$$\frac{\pi_1(M)}{Im(\beta)} = \frac{\pi_1(M)/Ker(\varphi)}{Im(\beta)/Ker(\varphi)}.$$

Thus one has

$$\pi_1(S^1) = \frac{\pi_1(\Sigma)}{Im(\beta)/Ker(\varphi)}.$$

This proves that the quotient group $\frac{Im(\beta)}{Ker(\varphi)}$ has infinite index in $\pi_1(\Sigma)$. By Poincaré duality the cohomological dimension $c.d\frac{Im(\beta)}{Ker(\varphi)} \leq 1$ and so $\frac{Im(\beta)}{Ker(\varphi)}$ is free. Since such a group is abelian we conclude that $\frac{Im(\beta)}{Ker(\varphi)} = \mathbb{Z}$. Hence $\pi_1(\Sigma)$ has two generators, and so, $\Sigma = T^2$. The above shows that M is a circle bundle over the torus. Hence $\pi_1(M)$ exhibits a infinite cyclic normal subgroup with abelian quotient. By Lemma 1.1 we conclude that $\pi_1(M)$ is almost nilpotent and then $\pi_1(M)$ has polynomial growth by Theorem 1.4. Since M supports Anosov flows we obtain a contradiction by Theorem 5.4. This contradiction proves the result. □

One can see that a suspended Anosov flow is product. The converse is false in general by the following corollary.

Corollary 5.21. *There are product Anosov flows which are not suspended.*

Proof. Let X be the geodesic flow on the unitary tangent bundle $M = T_1\Sigma$ over a negatively curved closed surface Σ. Since Σ is negatively curved we have that X is Anosov by Theorem 3.7. Moreover, X is product by Corollary 5.19 since M is a circle bundle over Σ. Finally X is not suspended by Theorem 5.20. The proof follows. □

The following lemma will be used in Section 5.4.3.

Lemma 5.10. *Let X be a transitive Anosov flow on a closed 3-manifold M. Let $\pi_1(M) \times V^u \to V^u$ be the action of the fundamental group $\pi_1(M)$ on the unstable leave space V^u of X in the universal cover. If A is a non-trivial abelian normal subgroup of $\pi_1(M)$, then $Fix(\beta) = \emptyset\ \forall \beta \in A - 1$.*

Proof. By Lemma 1.3 applied to the restricted action $A \times V^u \to V^u$ one has that $\cup_{\beta \in A-1} Fix(\beta)$ is discrete. Suppose by contradiction that $Fix(\beta) \neq \emptyset$ for some $\beta \in A - 1$ and choose $f \in Fix(\beta)$. Since X is transitive we have that the action $\pi_1(M) \times$

$V^u \to V^u$ is minimal. Hence we can find $\gamma \in \pi_1(M)$ such that $\gamma(f) \neq f$ is arbitrarily close to f. Define $\beta' = \gamma\beta\gamma^{-1}$. Then $f' = \gamma(f) \in Fix(\beta')$ and $\beta' \in A$ since A is normal. Because $\cup_{\beta \in A-1} Fix(\beta)$ is discrete and $f' \neq f$ is arbitrarily close to f we have that $\beta' = 1$. Hence $\beta = 1$ by the definition of β' which is impossible. This contradiction proves that $Fix(\beta) = \emptyset \; \forall \beta \in A-1$ and the theorem follows. □

5.4.3 Armendariz Theorem

Here we shall prove a result about classification of Anosov flows on closed 3-manifolds with solvable fundamental group due to Armendariz [9], [146].

Plante announced that the conclusion of this result holds for all codimension one Anosov flows [124] on solvable manifolds. However, the three-dimensional part of this result however has incorrect proof since it uses the (false) statement that all transitive Anosov flows on closed 3-manifolds are product. A correct proof of the Plante's extension of the Armendariz's Theorem is due to T. Barbot [16]. S. Matsumoto [83] also proved the Plante's extension but in dimension ≥ 4. The proofs by Barbot and Matsumoto are based on proving that the stable/unstable manifolds of Anosov flows on solvable 3-manifolds are transversely afine. The proof giving here is only three-dimensional and does not uses transversely afine structure.

We start with the following lemma.

Lemma 5.11. *Let X be a transitive Anosov flow with a transverse torus T on a closed* 3*-manifold M. Let V^u be the unstable leave space in the universal cover associated to X. If the fundamental group $\pi_1(T^2)$ of T^2 is fixed point free, then X is suspended.*

Proof. It suffices to prove that all closed orbits of X intersect T. To prove it we assume by contradiction that there is a closed orbit which does not intersect T. Then, there is a closed curve in T which is freely homotopic to a closed orbit of X (this is proved using the transitivity of X and the boundary closed orbit trick). Hence some element of $\pi_1(T^2) - 1$ has a fixed point in V^u a contradiction. This contradiction proves the result. □

From this we obtain the corollary below.

Corollary 5.22. *Let X be a transitive Anosov flow on a closed* 3*-manifold M. Then, X is a suspension if and only if X exhibits a* normal torus*, i.e. a transverse torus whose fundamental group is normal in* $\pi_1(M)$.

Proof. First suppose that X is suspended. Then M is a torus bundle over S^1 and X exhibits a transverse torus T which is isotopic to the fibre of M. Hence the fundamental group of T is normal in $\pi_1(M)$. Conversely suppose that X exhibits a transverse torus whose fundamental group is normal in $\pi_1(M)$. Such a subgroup is isomorphic to $\mathbb{Z}^2$ and then it is non-trivial abelian normal subgroup of $\pi_1(M)$. By Lemma 5.10 we have that $A = \pi_1(T^2)$ is fixed point free. Hence X is suspended by Lemma5.11 and the proof follows. □

Next we classify Anosov flows on torus bundles over S^1.

Theorem 5.23. *Anosov flows on torus bundles over S^1 are suspended.*

Proof. Let M be a torus bundle over S^1. Let X be an Anosov flow on M. X is transitive by Theorem 5.13. As M is a torus bundle over S^1 supporting Anosov flows we have that $H_1(M,\mathbb{Z})$ is infinite cyclic. Note that π_1(torus fibre of M) injects into $\pi_1(M)$. This yields a rank-two free abelian normal subgroup A of $\pi_1(M)$. By Lemma 5.10 we have that there is no element in $A-1$ having fixed points. The last implies that the closed orbits of X are non-zero elements of $H_1(M,\mathbb{Z})$. Then Theorem 4.14 applies. □

Now we state the main result of this section.

Theorem 5.24. *(Armendariz Theorem) An Anosov flow on a closed 3-manifold with solvable fundamental group is suspended.*

Proof. Let X be an Anosov flow on a closed 3-manifold M with solvable fundamental group $\pi_1(M)$. Let $G_0 = 1 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = \pi_1(M)$ be the normal series such that G_{i+1}/G_i is abelian for all $0 \leq i \leq n-1$. Define

$$i_0 = \sup\{0 \leq i \leq n : | G_j/G_{j-1} | < \infty, \forall i < j \leq n\}.$$

We have that G_{i_0} has finite index in $\pi_1(M)$ because G_j/G_{j-1} is finite $\forall j \geq i_0+1$ and

$$\left|\pi_1(M)/G_{i_0}\right| = |G_n/G_{n-1}| \cdot |G_{n-1}/G_{n-2}| \cdots \left|G_{i_0+1}/G_{i_0}\right|.$$

Let $\hat{M} \to M$ be the Galois covering associated to G_{i_0}. Hence $\hat{M}$ is compact and $\pi_1(\hat{M}) = G_{i_0}$. In particular $\pi_1(\hat{M})$ is solvable. We note that $Rank(H_1(\hat{M},\mathbb{Z})) \geq 1$. In fact, $H_1(\hat{M},\mathbb{Z}) = \pi_1(\hat{M})/[\pi_1(\hat{M}),\pi_1(\hat{M})]$ by the Hurewitz homomorphism. Hence $H_1(\hat{M},\mathbb{Z}) = G_{i_0}/G'_{i_0}$ where we denote $H' = [H,H]$ for simplicity. As G_{i_0}/G_{i_0-1} is abelian we have that $G'_{i_0} \subset G_{i_0-1}$. By the definition of i_0 we have that G_{i_0}/G_{i_0-1} is infinite. As

$$G_{i_0}/G_{i_0-1} = \frac{G_{i_0}/G'_{i_0}}{G_{i_0-1}/G'_{i_0}}$$

by the First Isomorphism Theorem one has that G_{i_0}/G'_{i_0} is infinite since G_{i_0}/G_{i_0-1} is. We conclude that $H_1(\hat{M},\mathbb{Z})$ is infinite and then $Rank(H_1(\hat{M},\mathbb{Z}) \geq 1$ as desired. The last implies that there is an onto homomorphism $h : H_1(\hat{M},\mathbb{Z}) \to \mathbb{Z}$. Componing with the Hurewizt homomorphism we obtain an onto homomorphism $\pi_1(\hat{M}) \to \mathbb{Z}$ whose kernel is normal and even finitely generated by Theorem 1.16. Hence $\hat{M}$ fibers over S^1 with fibre a closed surface F by Theorem 1.15. Since $\pi_1(\hat{M})$ is solvable we have that $\pi_1(F) = N$ also does. Hence F is either the torus or the Klein bottle. In any case $\hat{M}$ is finitely covered by a torus bundle over S^1, and so, we can assume that $\hat{M}$ itself is a torus bundle over S^1. By Theorem 5.23 we have that $\hat{X}$ is suspended. Hence $\hat{X}$ has a cross section and then X also does by Proposition 1.33. This proves that X is suspended and the result follows. □

A nice corollary of the Armendariz's Theorem is due to E. Ghys [53].

Theorem 5.25. *Let X be an Anosov flow on a closed 3-manifold M. Then X is suspended if and only if X is product and every closed orbit of X represents a non-zero element of $H_1(M,\mathbb{Z})$.*

Proof. Suppose that X is suspended. Then X exhibits a transverse torus T intersecting all the orbits of X. By analyzing the trace of the unstable foliation in T we can prove that X is product. On the other hand, T defines an homomorphism $U_T : \pi_1(M) \to \mathbb{Z}$ in the usual way (this is the Poincaré dual of T). Clearly the commutador $[\pi_1(M),\pi_1(M)]$ is contained in the kernel $Ker(U_T)$ of U_T since $\mathbb{Z}$ is abelian. Now, if $\gamma \in \pi_1(M)$ is represented by a closed orbit of X then $U_T(\gamma) \neq 0$ since T intersects every flowline of X. Hence $\gamma \notin Ker(U_T)$ and then $\gamma \notin [\pi_1(M),\pi_1(M)]$. It follows from the Hurewitz homomorphism that γ is not zero in $H_1(M,\mathbb{Z})$. Conversely assume that X is product and that every closed orbit of X represents a non-zero element in $H_1(M,\mathbb{Z})$. Hence the commutador $[\pi_1(M),\pi_1(M)]$ admits a fixed point free action in $V^u = \mathbb{R}$. By Theorem 1.5 we conclude that $[\pi_1(M),\pi_1(M)]$ is abelian. Hence $\pi_1(M)$ is solvable and so X is suspended by the Armendariz's Theorem. The proof follows. □

An exercice for the reader is to prove that every Anosov flow with a transverse torus on a closed 3-manifold with zero first Betti number is not transitive.

5.4.4 Abelian normal subgroup

We shall study Anosov flows on closed 3-manifolds for which the fundamental group exhibits a non-cyclic abelian normal subgroup. This investigation is due to T. Barbot [16].

We start with the following observation due to Verjovsky [145].

Proposition 5.26. *If M is a closed 3-manifold supporting Anosov flows, then the center of $\pi_1(M)$ is either trivial or infinite cyclic.*

Proof. By contradiction suppose that there is a closed 3-manifold M supporting Anosov flows X such that $Z(\pi_1(M))$ is neither trivial nor infinite cyclic. In particular, $\pi_1(M)$ has non-trivial center and then X is product by Theorem 5.17. Hence there is a minimal hyperbolic action $\pi_1(M) \times \mathbb{R} \to \mathbb{R}$. Since X is Anosov we have that there is $\alpha \in \pi_1(M) - 1$ such that $Fix(\alpha) \neq \emptyset$. Of course $Fix(\alpha)$ is discrete. It has neither upper nor lower bound since such a bound must be fixed by $Z(\pi_1(M))$ which is not cyclic, a contradiction. Hence we can assume that $Fix(\alpha) = \mathbb{Z}$. Now, $\gamma(Fix(\alpha)) = Fix(\alpha)$ for every $\gamma \in Z(\pi_1(M))$ since γ and α commute. This yields an homomorphism $\theta : Z(\pi_1(M)) \to \mathbb{Z}$ given by translations. θ must have nontrivial kernel since $Z(\pi_1(M))$ is not cyclic. Then, there is $\beta \in Z(\pi_1(M)) - 1$ fixing all the elements of $Fix(\alpha)$. In particular, $Fix(\beta) \neq \emptyset$ contradicting Lemma 5.10. The proof follows. □

Recall that if A is a subgroup of a group G then the set $Z_G(A)$ defined by

$$Z_G(A) = \{g \in G : ga = ag, \forall a \in A\}$$

is called the centralizer of A in G.

Corollary 5.27. *If M is a closed 3-manifold supporting Anosov flows and A is a non-cyclic abelian normal subgroup of $\pi_1(M)$, then $\pi_1(M)/Z_{\pi_1(M)}(A)$ is infinite.*

Proof. Let $\hat{M} \to M$ be the Galois covering associated to $Z = Z_{\pi_1(M)}(A)$, i.e. $\pi_1(\hat{M}) = Z$. Hence $\hat{M}$ is a closed 3-manifold as $\pi_1(M)/Z_{\pi_1(M)}(A)$ is finite. Clearly $A \leq Z(\pi_1(\hat{M}))$ and $\hat{M}$ supports Anosov flows since M does. As A is not cyclic and $A \leq Z(\pi_1(\hat{M}))$ we conclude that $Z(\pi_1 M))$ is not cyclic contradicting Proposition 5.26. The proof follows. □

Now we state the following result due to Barbot.

Theorem 5.28. *Let X be an Anosov flow on a closed 3-manifold M. If $\pi_1(M)$ contains a non-cyclic abelian normal subgroup, then X is suspended.*

Proof. Let $A \triangleleft \pi_1(M)$ be a non-cyclic abelian normal subgroup. Since $\pi_1(M)$ is torsion free we have that either A is a rank-two free abelian group (i.e. $\mathbb{Z}^2$) or $Rank(A) = 1$ ([66]). On the other hand, denote by $\mathscr{G}$ the set of groups G with $cdG < \infty$ such that $cdN < cdG - 1$ for all subgroup N of infinite index in G. As M suppors Anosov flows a theorem of Strebel [140] implies $\pi_1(M) \in \mathscr{G}$. Then, Theorem 1.7 implies that either $\pi_1(M)$ is virtually solvable or $\pi_1(M)/Z_{\pi_1(M)}(A)$ is finite. However, the last possibility cannot occur by Corollary 5.27. Hence $\pi_1(M)$ is virtually solvable and then X is suspended by the Armendariz's Theorem. The proof follows. □

The celebrated Seifert Fibered Conjecture ([38] or [57]) asserts that, among closed orientable 3-manifolds with infinite fundamental group, the Seifert ones are precisely the ones whose fundamental group exhibit infinite cyclic normal subgroups. As a corollary of Theorem 5.28 and the Seifert Fibered Conjecture we obtain the following result.

Corollary 5.29. *Let M be a closed* 3*-manifold supporting Anosov flows. If* $\pi_1(M)$ *contains a non-trivial abelian normal subgroup, then M is either Seifert or a torus bundle over S^1.*

A different proof of this corollary still using Poincaré duality can be found in [102].

Chapter 6
Sectional-Anosov flows on 3-manifolds

In this chapter we present some properties of sectional-Anosov flows on compact 3-manifolds.

6.1 Singular partition

In this section we introduce the concept of *singular cross section* which is very useful to study sectional-Anosov flows.

Hereafter M will denote a compact 3-manifold and X will denote a sectional-Anosov flow in M.

Given a disjoint collection of rectangles $\mathscr{S}$ we define $\partial^*\mathscr{S} = \bigcup_{S\in\mathscr{S}} \partial^* S$ for $* \in \{h, v, o\}$ where $S^o = S \setminus \partial^h S$.

Definition 6.1. *A* singular cross section *of X is a finite disjoint collection $\mathscr{S}$ of foliated rectangles with $M(X) \cap \partial^h \mathscr{S} = \emptyset$ such that for every $S \in \mathscr{S}$ there is a leaf l_S of $\mathscr{F}^s$ in S^o such that the return time $t_{\mathscr{S}}(x)$ for $x \in S \cap Dom(\Pi_{\mathscr{S}})$ goes uniformly to infinity as x approaches l_S. In other words,*

$$\lim_{\delta\to 0^+} \inf\{t_{\mathscr{S}}(x) : x \in S \cap Dom(\Pi_{\mathscr{S}}), dist(x, l_S) \leq \delta\} = \infty. \tag{6.1}$$

We define the singular curve *of $\mathscr{S}$ as the union,*

$$l_{\mathscr{S}} = \bigcup_{S\in\mathscr{S}} l_S.$$

It follows from the definition that if $\mathscr{S}$ is a singular cross section, then the leaf space $\mathscr{S}/\mathscr{F}^s$ is a disjoint union of copies of $[0, 1]$. So, there is a natural order "$<$" on each connected component. A leaf L will also denote the corresponding element of the leaf space. A *band* of $\mathscr{S}$ is a set V in some component of $\mathscr{S}$ which is union of leaves of $\mathscr{F}^s$ (or $\mathscr{F}^s$-*invariant* for short). It follows that the boundary ∂V is formed by two leaves of $\mathscr{F}^s$, V^- and V^+, and two curves in $\partial^h S$ transverse to $\mathscr{F}^s$. The union of these curves will be denoted by $\partial^v V$ and $\partial^h V$ respectively. Note that $V^- < V^+$ in the natural order. If $L < L'$ are leaves of $\mathscr{F}^s$ in the same connected component of $\mathscr{S}$ we denote by $[L,L']$ (resp. (L,L')) the unique band satisfying $\partial^v[L,L'] = L \cup L'$ (resp. $\partial^v(L,L') = L \cup L'$) A band V will be *open* or *closed* depending on whether $V = [V^-,V^+]$ or $V = (V^-,V^+)$. By a *band around a leaf* L of $\mathscr{F}^s$ we mean a band V with $L \subset V^o$, where $V^o = (V^-,V^+)$. We still call band a finite disjoint collection of bands $\mathscr{V}$. In such a case we denote

$$\mathscr{V}^o = \bigcup_{V \in \mathscr{V}} V^o.$$

6.1.1 Properties

Now we present some properties of singular cross sections. Hereafter X will be a sectional-Anosov flow of a compact 3-manifold M. Denote by $B_\delta(K)$ the open δ-ball around K.

Lemma 6.1. *For every singular cross section* $\mathscr{S}$ *there is* $\delta > 0$ *such that the following properties hold for every band* $\mathscr{V} \subset B_\delta(l_{\mathscr{S}})$*:*

1. *Dom*$(\Pi_{\mathscr{V}})$ *is* $\mathscr{F}^s$*-invariant.*
2. *If* L *is a leaf of* $\mathscr{F}^s$ *and* $L \subset Dom(\Pi_{\mathscr{V}})$*, then there is a leaf* $f(L)$ *of* $\mathscr{F}^s$ *such that* $\Pi_{\mathscr{V}}(L) \subset Int(f(L))$ *and the restriction* $\Pi_{\mathscr{V}}/L : L \to f(L)$ *is continuous. In particular,* $\Pi_{\mathscr{V}}(L) \cap \partial^h \mathscr{V} = \emptyset$*.*

Proof. By (6.1) we can select δ such that if $\mathscr{V}$ is a band $\mathscr{V} \subset B_\delta(l_{\mathscr{S}})$, then $t_{\mathscr{V}}(x)$ is uniformly large for all $x \in Dom(\Pi_{\mathscr{V}})$.

Now pick a leaf L intersecting $Dom(\Pi_{\mathscr{V}})$ at some point x. We can assume that $L \subset W^{ss}_x$ by just projecting along the flow. Since $t_{\mathscr{V}}(x)$ is uniformly large we have that $X_{t_{\mathscr{V}}(x)}(L)$ stays close to $\Pi_{\mathscr{V}}(x)$. From this we get (1) by projecting onto $\mathscr{V}$. Setting $f(L) = \mathscr{F}^s_{\Pi_{\mathscr{V}}(x)}$ we get (2) by the Tubular Flow Box Theorem [87]. □

For the next property we recoment the reader to see Definition 1.41 of triangular maps and Definition 1.44 of properties **(H1)-(H2)** in Appendix 1.8.

Proposition 6.2. *Let* $\mathscr{S}$ *be a singular cross section such that* $l_{\mathscr{S}}$ *is not accumulated by periodic orbits. There is* $\delta > 0$ *such that if* $\mathscr{V} \subset B_\delta(l_{\mathscr{S}})$ *is a band, then* $\Pi_{\mathscr{V}}$ *is a triangular map satisfying* **(H1)-(H2)**.

Proof. By Lemma 6.1 there is $\delta > 0$ such that if $\mathscr{V} \subset B_\delta(\mathscr{V})$ is a band, then $\Pi_{\mathscr{V}}$ is a triangular map with associated foliation $\mathscr{F}^s$. Since $l_{\mathscr{S}}$ is not accumulated by periodic orbits we can further assume that $\mathscr{V}$ does not intersect the periodic orbits. We shall use this last property to prove that $\Pi_{\mathscr{V}}$ satisfies **(H1)-(H2)**. To simplify the notation we write F, Σ and $\mathscr{F}$ instead of $\Pi_{\mathscr{V}}$, $\mathscr{V}$ and $\mathscr{F}^s$ respectively. We also write $L \in \mathscr{F}$ to say that L is a leaf of $\mathscr{F}$. Recall Definition 1.43.

Proof of **(H1)**: Let $L \in \mathscr{F}$ be such that $L \subset Dom(F)$ and $n(L) = 0$. It follows from the definition of $n(L)$ that $F(L) \subset \Sigma \setminus (L_- \cup L_+)$ and so $F(L) \subset Int(\Sigma)$. Then, F is C^1 in a connected neighborhood of L by the Tubular Flow Box Theorem.

Proof of **(H2)**: Let $L_* \in \mathscr{F}$ be such that $L_* \subset Dom(F)$, $1 \leq n(L_*) < \infty$ and

$$F^{n(L_*)}(L_*) \subset Dom(F).$$

It follows from the definition of $n(L_*)$ that

$$F^{n(L_*)+1}(L_*) \subset \Sigma \setminus (L_- \cup L_+).$$

Recalling the definition of $n(L_*)$ one has that $F^i(L_*) \subset L_- \cup L_+$ for all $0 \leq i \leq n(L_*) - 1$. Denote by L_i the leaf of $\mathscr{F}$ containing $F^i(L_*)$.

Now, for all $1 \leq i \leq n(L_*)$ we choose bands V_i centered at L_i. Although V_i is *not* contained in Σ we have that $V_i \setminus L_i$ consists of two connected components V_i^1, V_i^2 such that V_i^1 (say) is contained in $\Sigma \setminus (L_- \cup L_+)$ and V_i^2 is part of a small extension of Σ (as a cross section).

By the Tubular Flow-Box Theorem we can choose these bands in a way that the positive trajectories starting at V_i go directly to V_{i+1} and the positive orbits in the last band $V_{n(L_*)}$ goes directly to $\Sigma \setminus (L_- \cup L_+)$.

There are two cases to consider, namely

$$L_* \subset \Sigma \setminus (L_- \cup L_+) \quad \text{or} \quad L_* \subset L_- \cup L_+.$$

We shall consider the case $L_* \subset \Sigma \setminus (L_- \cup L_+)$ first.

As $L_* \subset \Sigma \setminus (L_- \cup L_+)$ there are leaves L, L' in the component of Σ containing L_* such that $L < L_* < L'$ (in the natural order).

Define $S = [L, L']$. We make S close to L_* by just taking L and L' close to L_*. Clearly S is a saturated neighborhood of L_* in Σ which is also connected. On the other hand, $S \setminus L_*$ has two connected components, i.e., the saturated sets $S_1 = [L, L_*)$ and $S_2 = (L_*, L]$. We shall prove that if S is close to L_* then S satisfies **(H2)**.

If S is close to L_* then the positive trajectories starting at S go directly to V_1 since $F(L_*) \subset L_1$. The positive trajectories in one component of $S \setminus L_*$ (say S_1) go directly to V_1^1 while trajectories in the other component S_2 go to V_1^2.

Then, for the first component S_1, one has $F(S_1) \subset V_1^1$ and so

$$F(S_1) \subset \Sigma \setminus (L_- \cup L_+).$$

For this component we define

$$n^1(L_*) = 1.$$

This definition and the previous inclusion imply

$$F^{n^1(L_*)}(S_1) \subset \Sigma \setminus (L_- \cup L_+).$$

Now we take care of the component S_2. The positive trajectories through S_2 meet successively the bands $V_1, \cdots, V_{n(L_*)}$ before meet $\Sigma \setminus (L_- \cup L_+)$.

If no such trajectory intersects $\cup_{1 \le i \le n(L_*)} V_i^1$ before it meets $\Sigma \setminus (L_- \cup L_+)$, then we define

$$n^2(L_*) = n(L_*) + 1.$$

Otherwise the trajectories intersect $\cup_{1 \le i \le n(L_*)} V_i^1$ in a first element $V_{i_0}^1$ $(1 \le i_0 \le n(L_*))$ and then we define

$$n^2(L_*) = i_0.$$

Observe that $i_0 \ne 1$ by the Tubular Flow Box Theorem because we have assume that the positive trajectories through S_1 goes directly to V_1^1. Consequently

$$n^2(L^*) \ne 1.$$

On the other hand,

$$F^{n^2(L_*)}(S_2) \subset \Sigma \setminus (L_- \cup L_+)$$

because $F(V_{n(L^*)}) \subset \Sigma \setminus (L_- \cup L_+)$ (if $n^2(L_*) = n(L_*) + 1$) and $V_{i_0}^1 \subset \Sigma \setminus (L_- \cup L_+)$ (otherwise).

All together imply (1) and (3) of **(H2)**. We obtain (2) of **(H2)** as a consequence of the Tubular Flow Box Theorem. This finishes the proof when $L_* \subset \Sigma \setminus (L_- \cup L_+)$.

The proof when $L_* \subset L_- \cup L_+$ follows from similar arguments with the sole exception that we have that $S \setminus L_*$ has one component instead of two. This completes the proof. □

Lemma 6.2. *There is a neighborhood U of $M(X)$ with the following property: For every $\lambda > 0$ and every singular cross section $\mathscr{S} \subset U$ there is $\delta > 0$ such that if $\mathscr{V} \subset B_\delta(l_{\mathscr{S}})$ is a band, then there is a cone field C_α in $\mathscr{V}$ transverse to $\mathscr{F}^s$ such that the following properties hold for all $x \in Dom(\Pi_{\mathscr{V}})$ where $\Pi_{\mathscr{V}}$ is differentiable,*

$$D\Pi_{\mathscr{V}}(x)(C(x)) \subset Int(C_{\alpha/2}(\Pi_{\mathscr{V}}(x))) \text{ and } \| D\Pi_{\mathscr{V}}(x) \cdot v_x \| \ge \lambda \cdot \| v_x \|,$$

for all $v_x \in C_\alpha(x)$.

Proof. Note that $M(X)$ is Lyapunov stable for it is an attracting set. Then, Lemma 6.5 p.1589 in [108] implies that for all $\alpha \in (0,1]$ there are a neighborhood U_α of $M(X)$ and constants $T_\alpha, K_\alpha, \lambda_\alpha > 0$ such that:

1. The sectional-hyperbolic splitting $T_{M(X)}M = E^s_{M(X)} \oplus E^c_{M(X)}$ extends to a continuous splitting $T_{U_\alpha}M = E^s_{U_\alpha} \oplus E^c_{U_\alpha}$ on U_α with $E^s_{U_\alpha}$ being invariant.
2. If $x \in U_\alpha$ and $t \ge T_\alpha$, then

$$DX_t(x)(C_\alpha(E^c_x)) \subset Int(C_{\alpha/2}(E^c_{X_t(x)})),$$

where

$$C_\alpha(E^c_x) = \{v_x \in T_xM : \angle(v_x, E^c_x) \le \alpha\}.$$

3. If $x \in U_\alpha$ is non-singular then $X(x) \in C_\alpha(E^c_x)$ and if $t \geq T_\alpha$ and $v_x \in C_\alpha(E^c_x) \cap N_x$, then
$$|| P^t_x(v_x) || \cdot || X(X_t(x)) || \geq K_\alpha e^{\lambda_\alpha t} || v_x || \cdot || X(x) ||,$$
where N_x is the orthogonal complement of $X(x)$ in T_xM; and P^t_x is the Linear Poincaré Flow induced by X (for the corresponding definition see Chapter 1).

   Choose $\alpha \in (0,1]$ such that
$$\inf\{\angle(v_x, E^s_x) : x \in U_\alpha, v_x \in C_\alpha(E^c_x) \setminus \{0\}\} > 0. \tag{6.2}$$

For such an α we let $U_\alpha, T_\alpha, K_\alpha, \lambda_\alpha > 0$ be satisfying (1)-(3) above.

Define $U = U_\alpha$.

Taking $\lambda > 0$ and a singular cross section $\mathscr{S} \subset U$ Fix $D > 0$ such that
$$\frac{|| X(x) ||}{|| X(y) ||} \geq D, \qquad \forall S \in \mathscr{S} \text{ and } x, y \in S.$$

Fix $T_\lambda > 0$ large enough such that
$$K_\alpha e^{\lambda_\alpha t} \cdot D \geq \lambda, \quad \forall t \geq T_\lambda. \tag{6.3}$$

Take $\delta > 0$ such that if $\mathscr{V} \subset B_\delta(l_{\mathscr{S}})$ is a band, then
$$t_{\mathscr{V}}(x) > T_\lambda, \quad \forall x \in Dom(\Pi_{\mathscr{V}}).$$

Define the cone field C_α in $\mathscr{V}$ by
$$C_\alpha(x) = C_\alpha(E^c_x) \cap T_x\mathscr{V}.$$

By (6.2) we have that C_α is transverse to $\mathscr{F}^s$. Moreover, recalling (16) p. 1596 in [108] one has
$$D\Pi_{\mathscr{V}}(x) = P^{t_{\mathscr{V}}(x)}_x.$$

Then, Property (2) yields
$$D\Pi_{\mathscr{V}}(x)(C_\alpha(x)) \subset Int(C_{\alpha/2}(\Pi_{\mathscr{V}}(x))$$

for all $x \in Dom(\Pi_{\mathscr{V}})$ where $D\Pi_{\mathscr{V}}(x)$ exists. Then, Property (3) implies
$$|| D\Pi_{\mathscr{V}}(x)(v_x) || \geq K_\alpha e^{\lambda_\alpha t_{\mathscr{V}}(x)} \cdot D \, || v_x || \, .$$

So (6.3) with $t = t_{\mathscr{V}}(x)$ implies
$$|| D\Pi_{\mathscr{V}}(x)(v_x) || \geq \lambda \cdot || v_x ||$$

proving the result. □

Combining lemmas 6.1, 6.2 and Proposition 6.2 we obtain the main result of this section.

Theorem 6.3. *Let X be a sectional-Anosov flow on a compact 3-manifold M. For all singular cross section $\mathscr{S}$ close to $M(X)$ and $\lambda > 0$ there is $\delta > 0$ such that if $\mathscr{V} \subset B_\delta(l_\mathscr{S})$ is a finite disjoint collection of bands, then $\Pi_\mathscr{V}$ is a λ-hyperbolic triangular map with associated foliation $\mathscr{F}^s$. If additionally $l_\mathscr{S}$ does not intersect the closure of the periodic orbits, then $\Pi_\mathscr{V}$ also satisfies* **(H1)-(H2)**.

6.1.2 Adapted bands

Let us present the definition of adapted band.

Definition 6.4. *Let $\mathscr{S}$ be a singular cross section of a sectional-Anosov flow on a compact 3-manifold. An* adapted band *of $\mathscr{S}$ is a closed band $\mathscr{V}$ satisfying the following identity*

$$\partial^v \mathscr{V} \cap \Pi_\mathscr{V}^{-1}(\mathscr{V}^o) = \emptyset. \tag{6.4}$$

In the sequel we present some properties of these bands.

6.1.2.1 Return map for adapted bands

We start with an study of the return map of an adapted band. Hereafter we consider a sectional-Anosov flow X on a compact 3-manifold M.

Lemma 6.3. *For all singular cross section $\mathscr{S}$ there is $\delta > 0$ such that if $\mathscr{V} \subset B_\delta(l_\mathscr{S})$ is an adapted band, then $Dom(\Pi_{\mathscr{V}^o})$ is open in $\mathscr{V}^o$ and $\Pi_{\mathscr{V}^o}$ is a C^1 local embedding.*

Proof. Take δ as in Lemma 6.1. By the Tubular Flow Box Theorem it suffices to prove that there is no $x \in Dom(\Pi_{\mathscr{V}^o})$ such that $X_t(x) \in \partial \mathscr{V}$ for some $t \in (0, t_{\mathscr{V}^o}(x)]$.

Suppose by contradiction that such an x exists. Then, $X_t(x) \in \partial^v \mathscr{V}$ by Lemma 6.1-(2). Hence the number

$$t_m = \max\{t \in (0, t_{\mathscr{V}^o}(x)] : X_t(x) \in \partial^v \mathscr{V}\}$$

is well defined and satisfies

$$t_m < t_{\mathscr{V}^o}(x)$$

since $\mathscr{V}^o$ is an open vertical band and $X_{t_m}(x) \in \partial^v \mathscr{V}$. Define

$$z = X_{t_m}(x).$$

Then, $z \in \partial^v \mathscr{V}$. Since $\mathscr{V}^o \subset \mathscr{V}$ and $t_m < t_{\mathscr{V}^o}(x)$ one has

$$z \in Dom(\Pi_{\mathscr{V}}).$$

But the definition of $t_{\mathscr{V}^o}(x)$ and t_m implies

$$X_s(x) \notin \mathscr{V}, \quad \forall s \in (t_m, t_{\mathscr{V}^o}(x)).$$

So,

$$t_{\mathscr{V}}(z) = t_{\mathscr{V}^o}(x) - t_m.$$

Then,

$$\Pi_{\mathscr{V}}(z) = \Pi_{\mathscr{V}^o}(x) \in \mathscr{V}^o$$

and so

$$z \in \Pi_{\mathscr{V}}^{-1}(\mathscr{V}^o).$$

It follows that

$$z \in \partial^v \mathscr{V} \cap \Pi_{\mathscr{V}}^{-1}(\mathscr{V}^o)$$

and then

$$\partial^v \mathscr{V} \cap \Pi_{\mathscr{V}}^{-1}(\mathscr{V}^o) \neq \emptyset$$

contradicting (6.4). The proof follows. □

Now we prove a Markov-type property for the return map of an adapted band.

Lemma 6.4. *If $\mathscr{S}$ is a singular cross section close to $M(X)$, then there is $\delta > 0$ such that if $V \subset B_\delta(l_{\mathscr{S}})$ is an adapted band which does not intersect the stable manifolds of the singularities, then for every closed band $\overline{V} \subset V^o$ and every $x \in Dom(\Pi_{V^o})$ with $\Pi_{V^o}(x) \subset \overline{V}^o$ there is a closed band $B_x \subset V^o$ around $\mathscr{F}_x^s$ satisfying the following properties:*

1. *$B_x \subset Dom(\Pi_{V^o})$ and $\Pi_{V^o}/_{B_x}$ is continuous.*
2. *$\Pi_{V^o}(B_x) \subset \overline{V}$.*
3. *$\Pi_{V^o}(\partial^v B_x) \subset \partial^v \overline{V}$.*

Proof. Fix δ as in Lemma 6.3 and an adapted band $V \subset B_\delta(l_{\mathscr{S}})$. Denote by $\mathscr{B}$ the set of open bands $B \subset V^o$ around $\mathscr{F}_x^s$ satisfying (1)-(2) of the lemma.

Since V is adapted we have from Lemma 6.3 applied to $\mathscr{V} = \{V\}$ that Π_{V^o} is a C^1 local embedding with open domain $Dom(\Pi_{V^o})$. In particular, $\mathscr{B} \neq \emptyset$ since $\Pi_{V^o}(x) \in \overline{V} \subset v^o$.

Endow $\mathscr{B}$ with the inclusion order, take its maximal element (B_x^-, B_x^+) and define

$$B_x = [B_x^-, B_x^+].$$

Let us prove that this B_x satisfies the conclusions of the lemma.

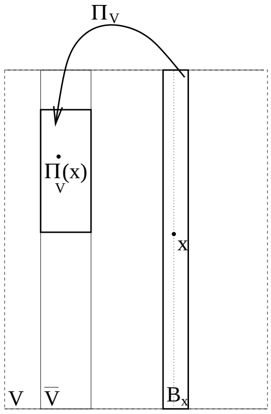


Fig. 6.1 The band B_x.

It follows from the definition of $\mathscr{B}$ that (B_x^-, B_x^+) is around $\mathscr{F}_x^s$, and so, B_x is around $\mathscr{F}_x^s$ too.

Now we claim that

$$B_x \subset V^o.$$

Indeed, it suffices to check

$$\partial^v B_x \subset V^o$$

since $(B_x^-, B_x^+) \subset V^o$ by definition. Suppose by contradiction that

$$\partial^v B_x \not\subset V^o.$$

Then we have either

$$B_x^- \not\subset V^o \quad \text{or} \quad B_x^+ \not\subset V^o.$$

We shall assume the first possibility since the second one can be handled analogously. In such a case we have $B_x^- = V^-$. Choose $a \in V^-$ and let $I \subset (B_x^-, B_x^+)$ be an open interval transverse to $\mathscr{F}^s$ such that $a \in \partial I$.

On the one hand, since V adapted, $a \in \partial^h V$ and $\overline{V} \subset V^o$ we have

$$Cl(O_X^+(a)) \cap \overline{V} = \emptyset$$

On the other hand, $I \subset (B_x^-, B_x^+) \in \mathscr{B}$ and so

$$O_X^+(x) \cap \overline{V} \neq \emptyset \quad \forall x \in I.$$

It follows that a satisfies $(\mathrm{P})_{\overline{V}}$ and so a is contained in the stable manifold of a periodic orbit or a singularity by Theorem B. However, the former case cannot occur because of the argument involving the Inclination Lemma [87] in the third paragraph of p. 367 in [110]. We conclude that a is in the stable manifold of a singularity, a contradiction since $a \in V$. This contradiction proves $B_x^- \subset V$. Therefore $\partial^v B_x \subset V$ and the claim follows.

Replacing V^* by B_x^* in the argument above (for $* = +, -$) one gets the inclusion

$$\partial^v B_x \subset Dom(\Pi_{V^o}).$$

But $(B_x^-, B_x^+) \subset Dom(\Pi_{V^o})$ by the definition of $\mathscr{B}$ since $(B_x^-, B_x^+) \in \mathscr{B}$. So,

$$B_x \subset Dom(\Pi_{V^o}).$$

As V is adapted we have that $\Pi_{V^o}/_{B_x}$ is continuous. Then B_x satisfies (1).

Now we check that B_x satisfies (2)-(3). Note that (2) is true in $Int^v(B_x)$ by the definition of $\mathscr{B}$ since $Int^v(B_x) = (B_x^-, B_x^+) \in \mathscr{B}$. Then (2) follows since $\Pi_{V^o}/_{B_x}$ continuous and $\overline{V}$ is a closed band. Because (B_x^-, B_x^+) is maximal we obtain (3). The proof follows. □

Now we state the main results of this subsection.

Theorem 6.5. *Let $\mathscr{S}$ be a singular cross section close to $M(X)$. There is $\delta > 0$ such that if $V \subset B_\delta(l_{\mathscr{S}})$ is an adapted band which does not intersect the stable manifold of the singularities, then V^o contains a non-wandering point if and only if V^o contains a periodic point.*

Proof. Choose δ as in Lemma 6.4 and an adapted band $V \subset B_\delta(l_{\mathscr{S}})$. Since every periodic point is non-wandering we only have to prove that if V^o contains a non-wandering point, then V^o contains a periodic point.

Assume that V^o contains a non-wandering point p. Then $Dom(\Pi_{V^o}) \neq \emptyset$, and so, there is $q \in V^o$ such that $\Pi_{V^o}(q)$ is defined.

Choose a sequence of closed bands $\overline{V}_n \subset V^o$ with the following properties for all n large:

(d)$q, \Pi_{V^o}(q) \in \overline{V}_n^o$;
(e)$\overline{V}_n \subset \overline{V}_{n+1}^o$;
(f).The vertical boundaries of $\overline{V}_n$ are $(1/n)$-close to those of V.

By hypothesis V does not intersect the stable manifolds of the singularities. Then, we can apply Lemma 6.4 to the bands $\overline{V}_n$ and $x = q$ in order to obtain a sequence $B_{q,n} \subset V^o$ of closed vertical bands satisfying (1)-(3) in that lemma. By (e) above one has $B_{q,n} \subset B_{q,n+1}^o$ for all n so the union

$$B_\infty = \bigcup_n B_{q,n}$$

is an open band in $Dom(\Pi_{V^o})$. In addition, $\Pi_{V^o}/_{B_\infty}$ is continuous by Lemma 6.4-(1).

On the other hand, consider the one-dimensional map $f : Dom(f) \subset V \to V$ induced by Π_{V^o}. which is continuous for V is adapted.

The property (f) above together with Lemma 6.4-(3) and the fact that imply that the lateral limits

$$\lim_{L\to(B_\infty^-)+} f(L) \quad \text{and} \quad \lim_{L\to(B_\infty^+)-} f(L)$$

exist and belong to different elements of $\{V^+, V^-\}$. If $[B_\infty^-, B_\infty^+] \subset V$ then by the above limits and the fact that $\Pi_{V^o}/_{B_\infty}$ is continuous we have that f has a fixed point in V^o. This fixed point corresponds to a leaf whose image under Π_V falls into itself. Henceforth Π_V has a fixed point which corresponds to a periodic orbit intersecting V. Then the result follows in this case.

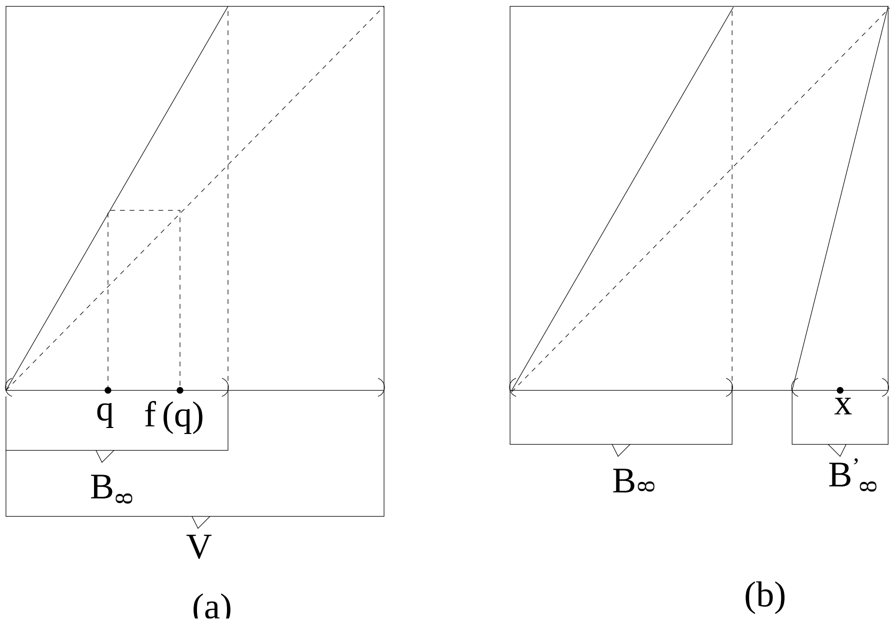


Fig. 6.2 Graphs of $f/_{B_\infty}$ and $f/_{B_\infty \cup B'_\infty}$.

Now assume

$$[B_\infty^-, B_\infty^+] \not\subset V.$$

As $(B_\infty^-, B_\infty^+) \subset V$ we conclude that $B_\infty^- = V^-$ or $B_\infty^+ = V^+$. We shall assume $B_\infty^- = V^-$ since the proof for the other case is similar.

As $B_\infty^- = V^-$ we have $B_\infty^+ \subset V$ for, otherwise, $B_\infty = V$ and then f has a fixed point since it has a non-wandering point and then we would be done too. We can further assume that $f/_{B_\infty}$ is orientation-preserving for, otherwise, $f/_{B_\infty}$ would have a fixed point and again we are done. It follows that the graph of $f/_{B_\infty}$ in V^o is like that in Figure 6.2-(a). But p is a non-wandering point so $Int(V \setminus B_\infty) \cap Dom(f) \neq \emptyset$ therefore we can choose $x \in Int^v(V \setminus B_\infty) \cap Dom(f)$.

Replacing q by x in the previous argument we obtain an open vertical band B'_∞ containing x such that the graph of $f/_{B_\infty \cup B'_\infty}$ is like that in Figure 6.2-(b). Then, the result follows since this map has infinitely many periodic points. □

We finish this subsection with a lemma about holonomy maps. Given a cross section Σ of X and $D \subset M$ disjoint from Σ we define the *holonomy map* $\Pi_{D,\Sigma}$ from D into Σ by

$$Dom(\Pi_{D,\Sigma}) = \{x \in D : X_t(x) \in \Sigma \;\text{ for some }\; t > 0\}$$

and

$$\Pi_{D,\Sigma}(x) = X_{t_{D,\Sigma}(x)}(x),$$

where $t_{D,\Sigma}(x)$ is the flight time

$$t_{D,\Sigma}(x) = \inf\{t > 0 : X_t(x) \in \Sigma\}.$$

Following the proof of Lemma 6.3 we can prove

Lemma 6.5. *If $\mathscr{V}$ is an adapted band of a singular cross section and D is a submanifold transverse to X and disjoint from $\mathscr{V}$, then $Dom(\Pi_{D,\mathscr{V}^o})$ is open in D and $\Pi_{D,\mathscr{V}^o}$ is a C^1 local embedding.*

6.1.2.2 Existence

Now we give two results about existence of adapted bands.

Theorem 6.6. *For every singular cross section $\mathscr{S}$ close to $M(X)$ there is $\delta > 0$ such that if $L_0 \subset B_\delta(l_{\mathscr{S}})$ is a leaf of $\mathscr{F}^s$ which does not accumulated by periodic orbits, then there is an adapted band around (and arbitrarily close to) L_0.*

Proof. Let δ as in Theorem 6.3 with $\lambda > 4$ and $L_0 \subset B_\delta(l_{\mathscr{S}})$ be a leaf which does not accumulated by periodic orbits. Then, we can choose a closed band $V \subset B_\delta(l_{\mathscr{S}})$ around (and arbitrarily close to) L_0 which does not intersect any periodic orbit. We also choose V in a way that

(B)The leaves V^- and V^+ are equidistant to L_0 in the leaf space.

We claim that there is a leaf $L^- \neq L_0$ in V such that

$$L^- \cap Dom(\Pi_V) = \emptyset.$$

Indeed, if L^- does not exist then every leaf $L \neq L_0$ belongs to $Dom(\Pi_V)$. By the choice of δ we have from Theorem 6.3 applied to $\mathscr{V} = \{V\}$ that the map

$$F = \Pi_V/(V \setminus L_0) \colon V \setminus L_0 \to V$$

is a λ-*hyperbolic triangular map* with $\lambda > 4$ and domain $V \setminus L_0$ satisfying **(H1)-(H2)**. Since such a map has a periodic point by Theorem 1.45 we have that V intersects a periodic orbit which is absurd.

Without loss of generality we can assume that $L^- < L_0$ and then

$$L^- < L_0 < V^+. \tag{6.5}$$

We choose the desired band $\mathscr{V} = \{Q\}$ with Q being a closed band in S around L_0. For this we define the leaves $Q^- < Q^+$ in V and set

$$Q = (Q^-, Q^+).$$

We define Q^- at once by

$$Q^- = L^-.$$

To define Q^+ we proceed as follows. First of all define the closed band

$$W = [Q^-, V^+].$$

Next we proceed according to the following cases.

Case 1: $V^+ \cap Dom(\Pi_W) = \emptyset$. In this case we define

$$Q^+ = V^+.$$

Therefore $[Q^-, Q^+] = W$. So, $\partial^v[Q^-, Q^+] \cap Dom(\Pi_{[Q^-,Q^+]}) = \emptyset$ and then Q satisfies (6.4). It follows that Q is adapted. By (6.5) we have that Q is around L_0 and then we are done.

Case 2: $V^+ \cap Dom(\Pi_W) \neq \emptyset$. Then $V^+ \subset Dom(\Pi_W)$ by $\mathscr{F}^s$-invariance. Obviously one has $\Pi_W(V^+) \subset L^- \cup V^+ \cup Int^v(W)$ and so we have three *possibilities*:

$$\Pi_W(V^+) \subset L^- \quad \text{or} \quad \Pi_W(V^+) \subset V^+ \quad \text{or} \quad \Pi_W(V^+) \subset Int^v(W).$$

In the first possibility we define

$$Q^+ = V^+.$$

Let us prove that $Q = [Q^-, Q^+]$ so defined satisfies (6.4). Indeed, as $\Pi_W(V^+) \subset L^-$ we have that

$$(Q^- \cup V^+) \cap \Pi_W^{-1}(Int(W)) = \emptyset.$$

But

$$\partial^v[Q^-, Q^+] = \partial^v W = Q^- \cup V^+,$$

$[Q^-, Q^+] = W$ and $Int([Q^-, Q^+]) = Int(W)$. Replacing above one gets

$$\partial^v[Q^-, Q^+] \cap \Pi_{[Q^-,Q^+]}^{-1}(Int([Q^-, Q^+])) = \emptyset$$

which is precisely (6.4). It follows that Q is adapted. By (6.5) we have that Q is around L_0 and then we are done.

In the second possibility we have that V^+ is an invariant leaf. Then it would exist a periodic orbit passing through $V^+ \subset V$. This contradicts the fact that V does not intersect such orbits. So, this possibility cannot occur.

Then we arrive to the third possibility

$$\Pi_W(V^+) \subset Int^v(W).$$

It follows that there is an intermediary leaf

$$L^- \leq \tilde{V}^+ < V^+$$

such that the vertical band $\tilde{W}$ defined by

$$\tilde{W} = (\tilde{V}^+, V^+]$$

satisfies

$$\tilde{W} \subset Dom(\Pi_W) \quad \text{and} \quad \Pi_W(\tilde{W}) \subset Int^v(W).$$

In particular, $\Pi_W/_{\tilde{W}}$ is continuous (actually C^1). Take the intermediary leaf $\tilde{V}^+$ so that $\tilde{W}$ is maximal with these properties.

Now, we know from Proposition 6.2 that the derivative of Π_W along the direction transverse to $\mathscr{F}^s$ is bigger than 4. Henceforth, the diameter of $\Pi_W(\tilde{W})$ in the direction transverse to $\mathscr{F}^s$ is at least twice the one of $\tilde{W}$. Then, (B) implies

$$L_0 < \tilde{V}^+. \tag{6.6}$$

If $\tilde{V}^+ \cap Dom(\Pi_W) = \emptyset$ then we define

$$Q^+ = \tilde{V}^+.$$

We can prove as before that the resulting band Q satisfies (6.4). It follows that Q is adapted. By (6.5) and (6.6) we have that Q is around L_0 and then we are done.

If $\tilde{V}^+ \cap Dom(\Pi_W) \neq \emptyset$ then $\tilde{V}^+ \subset Dom(\Pi_W)$ by invariance. Again we have

$$\Pi_W(\tilde{V}^+) \subset L^- \cup V^+ \cup Int^v(W)$$

and so we have three situations:

$$\Pi_W(\tilde{V}^+) \subset L^- \quad \text{or} \quad \Pi_W(\tilde{V}^+) \subset V^+ \quad \text{or} \quad \Pi_W(\tilde{V}^+) \subset Int^v(W).$$

In the first situation we define

$$Q^+ = \tilde{V}^+$$

and the resulting band Q satisfies (6.4) hence it is adapted. By (6.5) and (6.6) we have that Q is around L_0 and then we are done.

To finish we prove that the remainder situations *cannot occur*. For the last one we simply observe that if it does, then we could contradict the maximality of $\tilde{W}$ using the Tubular Flow Box Theorem. For the second one

$$\Pi_W(\tilde{V}^+) \subset V^+ \tag{6.7}$$

we proceed as follows: Let $f : Dom(f) \subset W \to W$ be the one-dimensional map induced by Π_W in the leaf space (e.g. Lemma 6.1). The inclusion (6.7) would imply that $f/(\tilde{V}^+, V^+]$ is orientation-reversing with $f(\tilde{V}^+) = V^+$. So, $f/(\tilde{V}^+, V^+]$ has a fixed point which represents an invariant leaf of Π_W. Consequently $(\tilde{V}^+, V^+]$ intersects a periodic orbit. Since $(\tilde{V}^+, V^+] \subset V$ we arrive to a contradiction since V does not intersect such orbits. This contradiction proves that (6.7) cannot occur and then the result follows. □

Next we extend the previous theorem to the singular curves.

Theorem 6.7. *For every singular cross section $\mathscr{S}$ close to $M(X)$ and $S \in \mathscr{S}$ there is an adapted band around (and close to) l_S.*

Proof. We can assume that l_S is accumulated by periodic orbits for, otherwise, we are done by Theorem 6.6 applied to $L_0 = l_S$.

Fix a closed band Q around l_S such that Q^+ and Q^- are equidistant to l_S. Since l_S is accumulated by periodic orbits we can select a leaf $L' \subset Q^o$ containing a periodic orbit. Clearly $L' \neq l_S$ and then either $L' < l_S$ or $L' > l_S$ in the leaf space. We shall assume $L' < l_S$ for the treatement in the remaining case is similar.

Notice that $L' \subset Dom(\Pi_Q)$ and there is a finite set of leaves $L'_0, \cdots, L'_k \subset Dom(\Pi_Q)$ with $L'_0 = L'_k = L'$ such that $\Pi^i_Q(L') = L'_i$ for all $i = 0, \cdots, k$.

We have the following two possibilities,

- $L'_i < l_S$ for all i;
- there is L'_j such that $L'_j > l_S$.

In the latter case we take L'_{j_0} realizing the minimal distance to l_S of those leaves $L'_j > l_S$; and L'_{i_0} realizing the minimal distance to l_S of those leaves $L'_i < L'$. Hence the band $V = [L'_{i_0}, L'_{j_0}]$ satisfies the conclusion of the theorem.

Now we assume that $L'_i < l_S$ for all i. Take L^* as the leaf in $\{L'_0, \cdots, L'_k\}$ realizing the minimal distance to l_S. Such a minimality certainly implies

$$L^* \cap \Pi^{-1}_{[L^*, Q^+]}((L^*, Q^+)) = \emptyset.$$

Now we proceed as in the proof of Theorem 6.6. If either $Q^+ \cap Dom(\Pi_{[L^*, Q^+]}) = \emptyset$ or else both $Q^+ \subset Dom(\Pi_{[L^*, Q^+]})$ and $\Pi_{[L^*, Q^+]}(Q^+) \subset L^* \cup Q^+$ hold, then we take $V = [L^*, Q^+]$ and we are done. Hence we can assume

$$Q^+ \subset Dom(\Pi_{[L^*, Q^+]}) \quad \text{and} \quad \Pi_{[L^*, Q^+]}(Q^+) \subset (L^*, Q^+).$$

From this we can select a maximal leaf $L^{**} \subset (L^*, Q^+)$ such that

$$(L^{**}, Q^+] \subset Dom(\Pi_{[L^*,Q^+]}) \quad \text{and} \quad \Pi_{[L^*,Q^+]}/(L^{**}, Q^+] \quad \text{is } C^1.$$

The equidistance hypothesis on Q implies $l_S < L^{**}$. Again if $L^{**} \cap Dom(\Pi_{[L^*,Q^+]}) = \emptyset$ or else $L^{**} \subset Dom(\Pi_{[L^*,Q^+]}$ and $\Pi_{[L^*,Q^+]}(L^{**}) \subset L^*$ we are done by taking $V = [L^*, L^{**}]$. Since $\Pi_{[L^*,Q^+]}(L^{**}) \not\subset (L^*, Q^+)$ whenever $L^{**} \subset Dom(\Pi_{[L^*,Q^+]})$ by the maximality of L^{**} we are left to consider the case

$$L^{**} \subset Dom(\Pi_{[L^*,Q^+]}) \quad \text{and} \quad \Pi_{[L^*,Q^+]}(L^{**}) \subset Q^+.$$

This together with $\Pi_{[L^*,Q^+]}(Q^+) \subset (L^*, Q^+)$ implies that $\Pi_{[L^*,Q^+]}/(L^{**}, Q^+]$ (or its induced one-dimensional map) is *orientation reversing*. Hence such a map has a fixed leaf $L_{fix} \subset (L^{**}, Q^+]$ (i.e. $\Pi_{[L^*,Q^+]}(L_{fix}) \subset L_{fix}$). Then we choose $V = [L^*, L_{fix}]$ and we are done. This proves the result. □

6.2 Dynamical properties

In this section we apply the results of the previous section to analyze the dynamics of sectional-Anosov flows X on compact 3-manifolds M.

6.2.1 Characterizing omega-limit sets

This subsection is motivated by two interesting properties related to the ordinary differential equation in Figure 6.3. The first one is that the omega-limit set $\omega(q)$ of the point q in the figure is a hyperbolic singularity of saddle-type. The second one is that there is a *closed* subset Σ (the vertical segment in the figure) such that q has Property $(P)_\Sigma$.

It is natural to ask how these properties are related *among those points q having saddle type hyperbolic omega-limit set*. For example if $n \geq 2$ and $\omega(q)$ is a *closed orbit* (i.e. a singularity or a periodic orbit), then q satisfies $(P)_\Sigma$ for some *closed* subset Σ. The question is then whether the satisfaction of $(P)_\Sigma$ for some Σ closed implies that $\omega(q)$ is a closed orbit. Indeed, this is true for $n = 2$ (e.g. [87] p. 145-146) but false for $n \geq 4$ by the following counterexample:

Example 6.8. *Let D^2 and S^1 be the two-dimensional closed unit disk and the unit circle respectively. Consider the vector field X^0 in the solid torus $ST = D^2 \times S^1$ obtained from the suspension of the Smale Horseshoe in D^2 (see [87]). As is well known there is $x_0 \in ST$ whose omega-limit set H respect to X^0 is a saddle type hyperbolic set but not a closed orbit. Now define the vector field X in $ST \times [-1, 1]$ by $X(x, y) = (X^0(x), 2y)$, $\forall (x, y) \in ST \times [-1, 1]$. Fix $q = (x_0, 0)$. Then, $\omega(q) = H \times 0$ hence $\omega(q)$ is not a closed orbit but a saddle type hyperbolic set. However, q satisfies*

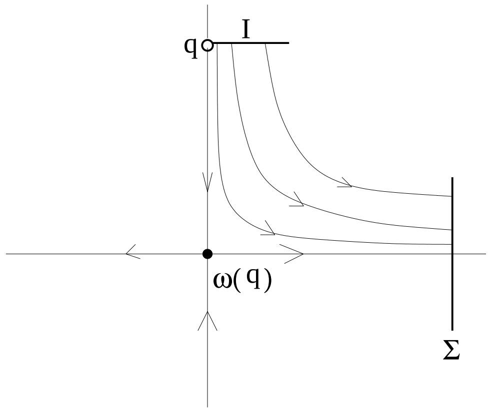


Fig. 6.3

$(P)_\Sigma$ for some closed subset Σ (e.g. take $\Sigma = ST \times 1$ with $I = \{(x_0, y) \in ST \times [-1, 1] : 0 < y \le \frac{1}{2}\}$). Analogous counterexample can be constructed for $n \ge 4$.

Here we give positive answer for the question above when $n = 3$. Actually we do it *among those points q having sectional-hyperbolic omega-limit set*. More precisely, we prove the following result whose original proof can be found in [20].

Theorem B. *Let X be a C^1 vector field in a compact 3-manifold M. If $q \in M$ has sectional-hyperbolic omega-limit set $\omega(q)$, then the following properties are equivalent:*

1. *$\omega(q)$ is a closed orbit.*
2. *q satisfies $(P)_\Sigma$ for some closed subset Σ.*

Proof. Let X a C^1 vector field in a compact 3-manifold M and $q \in M$. Suppose that $\omega(q)$ is a sectional-hyperbolic set. We shall prove that $\omega(q)$ is a closed orbit if q satisfies $(\mathrm{P})_\Sigma$ for some closed subset Σ.

To start with we fix a neighborhood U of $\omega(q)$ where the sectional-hyperbolic splitting $T_{\omega(q)}M = E^s_{\omega(q)} \oplus E^c_{\omega(q)}$ of $\omega(q)$ extends to a continuous splitting $T_U M = \hat{E}^s_U \oplus \hat{E}^c_U$. Let $W^{ss} = \{W^{ss}(x) : x \in U\}$ be the corresponding strong stable foliation. As U is a neighborhood of $\omega(q)$ we can assume that $q \in U$.

Let I be the interval in the definition of $(\mathrm{P})_\Sigma$. *We can assume that I is both tangent to $\hat{E}^c_U$ transverse to X*. Indeed, observe that there is $\varepsilon > 0$ small such that the local strong stable manifold $W^{ss}_\varepsilon(q)$ satisfies

$$I \cap \left(\bigcup_{-1 \le t \le 1} X_t(W^{ss}_\varepsilon(q)) \right) = \emptyset.$$

(Otherwise it would exist $x \in I$ such that $O^+(x) \cap \Sigma = \emptyset$ as $O^+(x)$ is asymptotic to $O^+(q)$.) Then, we can use W^{ss} to project I onto an open interval $\hat{I}$, with q as a boundary point, such that $\hat{I}$ is tangent to $\hat{E}^c_U$ and transverse to X. As $Cl(O^+(q))$ and Σ are disjoint we can enlarge Σ a bit using W^{ss} to obtain a closed subset $\hat{\Sigma}$ with $Cl(O^+(q)) \cap \hat{\Sigma} = \emptyset$ such that $O^+(x) \cap \hat{\Sigma} \neq \emptyset$ for all $x \in \hat{I}$. Then, we can replace I by $\hat{I}$ and Σ by $\hat{\Sigma}$ if necessary in order to assume that I is tangent to $\hat{E}^c_U$ and transverse to X.

We have that $\omega(q)$ has a singular partition with arbitrarily small diameter $\mathscr{R} = \{S_1, \cdots, S_k\}$ by Theorem 2.14. We have $\mathscr{R} \subset U$ (since $\mathscr{R}$ has small diameter) so the projection $\mathscr{F}^s(\cdot, S_i)$ of $\mathscr{F}^{ss}$ into S_i is well defined for every $i = 1, \cdots, k$.

As $Cl(O^+(q))$ and Σ are disjoint there is a compact neighborhood $W \subset U$ of $\omega(q)$ such that

$$W \cap \Sigma = \emptyset.$$

Furthermore we can assume that

$$O^+(q) \subset W.$$

Because the diameter of the partition is small we can further assume that

$$\mathscr{R} \subset Int(W).$$

Now assume that $\omega(q)$ is not a singularity. As I is tangent to $\hat{E}^c_U$ and transverse to X we obtain $S, \hat{q}_i, \hat{J}_i$ and δ from Theorem 2.15. Let $x \in S$ be a limit point of $\hat{q}_i$.

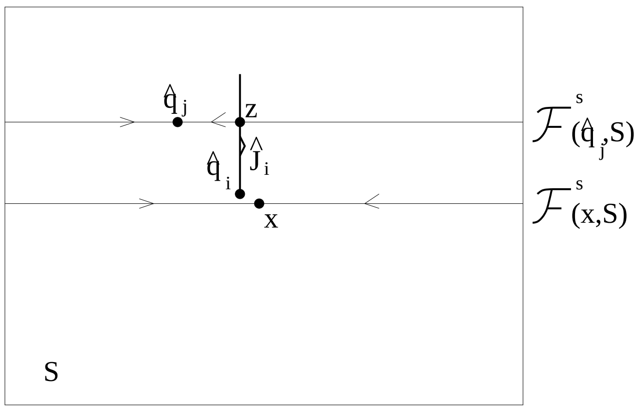


Fig. 6.4

If $\hat{q}_i \notin \mathscr{F}^s(x,S)$ for infinitely many i's we have a situation which is similar to that in Figure 2 of [110] p. 371: The splitting $T_{\omega(q)}M = E^s_{\omega(q)} \oplus E^c_{\omega(q)}$ is dominated and $\hat{J}_i$ is both tangent to $\hat{E}^c_U$ and transverse to X for all i. Therefore, the angle between the arcs $\hat{J}_i$ and the leaves $\{\mathscr{F}^s(y,S),: y \in S\}$ is bounded away from 0. As Length$(\hat{J}_i)$ is also bounded away from 0 and $\hat{q}_i \to x$ we eventually obtain an intersection point

$$z \in \hat{J}_i \cap \mathscr{F}^s(\hat{q}_j, S)$$

between $\hat{J}_i$ and $\mathscr{F}^s(\hat{q}_j,S)$ for some $i, j \in \mathbb{N}$ (see Figure 6.4).

As $z \in \hat{J}_i$ we have that z is in the positive orbit of I so

$$O^+(z) \cap \Sigma \neq \emptyset.$$

But $z \in \mathscr{F}^s(\hat{q}_j,S)$ as well so $O^+(z)$ is asymptotic to $O^+(q)$ hence $O^+(z)$ cannot escape from W because $O^+(q) \subset W$. As $W \cap \Sigma = \emptyset$ we conclude that

$$O^+(z) \cap \Sigma = \emptyset$$

yielding a contradiction.

Therefore we can assume that $\hat{q}_i \in \mathscr{F}^s(x,S)$ for all i large. In this situation we can apply Lemma 5.6 in [110] p. 369 to obtain that $\omega(q)$ is a periodic orbit. The result follows. □

6.2.2 Existence of periodic orbits

It follows from the Anosov closing lemma that every Anosov flow flow on a closed manifold has a periodic orbit. The result below extends it to the sectional-Anosov flows on compact 3-manifolds.

Theorem C. *Every sectional-Anosov flow on a compact* 3*-manifold has a periodic orbit.*

Proof. Let X be a sectional-Anosov flow on a compact 3-manifold M. By Proposition 4.20 we can assume that X has a Lorenz-like singularity. Hence the set $LSing(X) = \{\sigma \in Sing(X) : \sigma \text{ is Lorenz-like}\}$ is not empty. Clearly every $\sigma \in LSing(X)$ is hyperbolic and so it is equipped with the stable and unstable manifolds $W^s(\sigma)$, $W^u(\sigma)$ tangent at σ to the eigenspace associated to the set of eigenvalues $\{\lambda_2, \lambda_3\}$ and $\{\lambda_1\}$. In particular, $dim(W^s(\sigma)) = 2$ and $dim(W^u(\sigma)) = 1$. A further invariant manifold, the strong stable manifold $W^{ss}(\sigma)$, is both well defined and tangent at σ to the eigenspace associated the $\{\lambda_2\}$. Consequently $dim(W^{ss}(\sigma)) = 1$.

Take a linearizing coordinate system (x_1, x_2, x_3) in a neighborhood of σ. Note that $W^{ss}(\sigma)$ separates $W^s(\sigma)$ in two connected components, namely, the top and the bottom ones. In the top component we consider a rectangle S^t_σ of X_t together with a curve l^t_σ. Similarly we consider a rectangle S^b_σ and a curve l^b_σ in the bottom component. We take these rectangles so that the curve l^*_σ are contained in $W^s(\sigma)\setminus W^{ss}(\sigma)$ for $*=t,b$. Moreover, both rectangles are foliated rectangles of X. The positive flow lines starting at $S^t_\sigma \cup S^b_\sigma \setminus (l^t_\sigma \cup l^b_\sigma)$ exit a small neighborhood of σ.

On the other hand, the positive orbits starting at $l^t_\sigma \cup l^b_\sigma$ goes directly to σ. We note that the boundary of S^*_σ is formed by four curves, two of them transverse to l^*_σ and two of them parallel to l^*_σ. The union of the curves in the boundary of S^*_σ which are parallel (resp. transverse) to l^*_σ is $\partial^v S^*_\sigma$ (resp. $\partial^h S^*_\sigma$).

By Lemma 2.8 we can choose S^t_σ, S^b_σ satisfying $M(X)\cap \partial^h S^*_\sigma=\emptyset$ for $*=t,b$. Then, the collection

$$\mathscr{S}=\bigcup_{\sigma\in LSing(X)}\{S^t_\sigma,S^b_\sigma\}$$

is a singular cross-section with singular curve

$$l_{\mathscr{S}}=\bigcup_{\sigma\in LSing(X)}(l^t_\sigma\cup l^b_\sigma).$$

Clearly $\mathscr{S}$ can be constructed close to $M(X)$.

Choose δ as in Theorem 6.3 with $\lambda>4$ for such a cross-section. For each $\sigma\in LSing(X)$ we select two closed bands V^t_σ around l^t_σ and V^b_σ around l^b_σ with $V^*_\sigma\subset B_\delta(l^*_\sigma)$ for $*=t,b$. It follows that the band

$$\mathscr{V}=\bigcup_{\sigma\in LSing(X)}\{V^t_\sigma,V^b_\sigma\}$$

satisfies $\mathscr{V}\subset B_\delta(l_{\mathscr{S}})$. It follows from Theorem 6.3 that $\Pi_{\mathscr{V}}$ is a λ-hyperbolic triangular map with associated foliation $\mathscr{F}^s$ satisfying **(H1)-(H2)**.

Now suppose that X has no periodic orbits. If there is $x\in\mathscr{V}\setminus l_{\mathscr{S}}$ whose positive orbit does not intersect $\mathscr{V}$, then $\omega(x)$ has no singularities and then it is hyperbolic by the hyperbolic lemma. By the shadowing lemma for flows [62] applied to the positive orbit of x we would have that X has a periodic orbit, a contradiction. It follows that $Dom(\Pi_{\mathscr{V}}=\mathscr{V}\setminus l_{\mathscr{S}}$. Hence $\Pi_{\mathscr{V}}$ has large domain and then it has a periodic point by Theorem 1.45. Hence X would have a periodic orbit which is a contradiction. This contradiction proves the result. □

Let us present some corollaries of Theorem C. Recall that by the stable manifold theory [68], if O is a hyperbolic closed orbit of X with splitting $T_OM=E^s_O\oplus E^X_O\oplus E^u_O$, then its unstable manifold

$$W^u(O)=\{q\in M:\mathrm{dist}(X_t(q),\mathscr{O})\to 0, t\to-\infty\},$$

is indeed an immersed submanifold tangent at $\mathscr{O}$ to the subbundle $E^X_O\oplus E^u_O$. Denote by $Cl(B)$ the closure of $B\subset M$. The following corollary improves [100].

Corollary 6.9. *The maximal invariant set of a transitive sectional-Anosov flow with singularities on a compact* 3*-manifold is an expanding attractor.*

Proof. All sectional-Anosov flows on compact 3-manifolds are codimension one by Proposition 4.21 Then, the result follows from Corollary 4.23 since all such flows have a periodic orbit by Theorem C. □

Recall that $X \in \mathscr{X}^1(M)$ is *transitive* if $M(X) = \omega(x)$ for some $x \in M(X)$. This definition generalizes the classical definition of transitive vector fields on closed manifolds.

Corollary 6.10. *If X is a transitive sectional-Anosov flow on a compact 3-manifold M, then $M(X) = Cl(W^u(O))$ for some periodic orbit O.*

Proof. By Theorem C there is a periodic orbit O. As before we have that O is saddle-type and so $dim(W^u(O)) = 2$. In addition, $W^u(O) \subset M(X)$ since $M(X)$ is attracting. To prove that $W^u(O)$ is dense in $M(X)$ we proceed as in the proof of Theorem 4.1 p. 365 in [110] using $dim(W^u(O)) = 2$, the dense orbit and the contracting direction of X. This completes the proof. □

We say that $X \in \mathscr{X}^1(M)$ is C^1*robust transitive* if every C^1 vector field which is C^1 close to X is transitive with non-trivial maximal invariant set.

Corollary 6.11. *All C^1 robust transitive flows on compact 3-manifolds have a hyperbolic periodic orbit.*

Proof. This is a direct consequence of the Theorem C since every flow as in the statement is sectional-Anosov. □

Corollary 6.12. *A sectional-Anosov flow X on a compact 3-manifold M is transitive if and only if there is $x \in M$ such that $\omega(x) = M(X)$.*

Proof. The direct implication is obvious so we only need to prove the converse one. Hence suppose that there is some $x \in M$ such that $\omega(x) = M(X)$. Again by Theorem C we can choose a periodic orbit O. Note that $W^u(O) \subset M(X)$. Hence the positive orbit of x passes close to $W^u(O)$ and so we can assume that x itself is close to $W^u(O)$. Then, the strong stable leaf $W^{ss}(x)$ intersects $W^u(O)$ at some point x^*. Then, $x^* \in M(X)$ and $\omega(x^*) = \omega(x) = M(X)$ and so X is transitive. □

6.2.3 Sectional-Anosov closing and connecting lemmas

To state our next result we use the following definition. Given $X \in \mathscr{X}^1(M)$ and $p, q \in M$ we write $p \prec q$ if for all $\varepsilon > 0$ there is $t > 0$ such that $X_t(B_\varepsilon(p)) \cap B_\varepsilon(q) \neq \emptyset$, where $B_\varepsilon(x)$ denotes the open ε-ball around x.

A well known property of Anosov flows X on closed manifolds M is that for all $p, q \in M$ with $p \prec q$ there is $x \in M$ such that $\alpha(p) = \alpha(x)$ and $\omega(x) = \omega(q)$. It is natural to ask if the sectional-Anosov flows satisfy this conclusion as well. Nevertheless, consider the transitive sectional-Anosov flow in the solid bitorus depicted in Figure 6.5 (this is a variation of an example in Chapter 2) which by Theorem C has

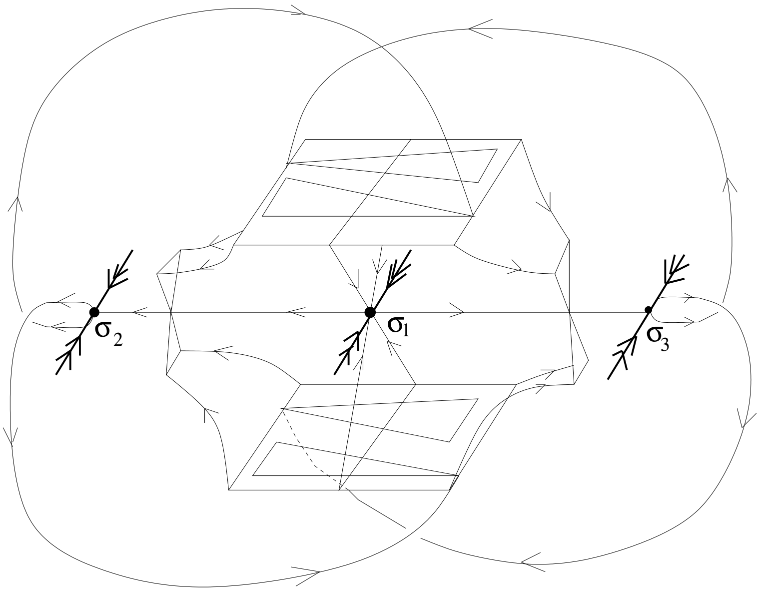


Fig. 6.5

a periodic poit p. Then, p and $q = \sigma_2$ (or σ_3) in the figure satisfies the hypotheses but not the conclusion of this lemma.

However, it is possible to observe in this example that both $\alpha(p)$ has no singularities and there is x such that $\alpha(p) = \alpha(x)$ and $\omega(x) = \{\sigma_1\}$.

We generalize this observation to all sectional-Anosov flows on compact 3-manifolds. See [19] for the original proof.

Theorem D (Sectional-Anosov connecting lemma)**.** *Let X a sectional-Anosov flow on a compact 3-manifold M and $p, q \in M$ be such that $\alpha(p)$ has no singularities. If $p \prec q$, then there is $x \in M$ such that $\alpha(p) = \alpha(x)$ and $\omega(x)$ is either $\omega(q)$ or a singularity of X.*

Proof. The proof will be done through three steps.

The first one consists of assuming that p is a periodic point and $q = \sigma$ is a singularity. Denote by $O = O(p)$ the periodic orbit containing p. Let U and V be fixed (but arbitrary) neighborhoods of p and σ respectively. Since $p \prec \sigma$ there are $t > 0$ and $z \in X_t(U) \cap V$. Choosing U close to p we ensure that the strong stable manifold through $X_{-t}(z)$ intersects $W^u(O)$ at some point z'. On the other hand, $t > 0$ can be made large since $\sigma \notin O$. Then, as z and z' belongs to the same strong stable manifold, we have that $X_t(z') \in V$. But $z' \in W^u(O)$ which is invariant, hence $z' \in W^u(O) \cap V$.

Thus, $W^u(O) \cap V \neq \emptyset$ and so $\sigma \in Cl(W^u(O))$ since V is arbitrary. We conclude that σ is Lorenz-like by Theorem 4.19.

By Corollary 2.7 we have $M(X) \cap W^{ss}(\sigma) = \{\sigma\}$. From this we can select as in the proof of Theorem C a singular cross-section S associated and close to σ (hence disjoint from O) such that $M(x) \cap \partial^h S = \emptyset$ and $W^u(O) \cap Int(S)$ has an accumulation point in the singular leaf l_S. Applying Theorem 6.7 to this section we can select two adapted bands $V_0, V_1 \subset S$ around l_S with $V_0 \subset V_1^o$.

On the other hand, $W^u(O)$ accumulates on l_S which is contained in V_0^o. Then, we can select a point $a^* \in W^u(O) \cap Int(V_0)$. Taking the backward orbit of a^* we obtain a *fundamental domain* (e.g. [87]) say $D^u = [a,b]$ of $W^{uu}(p)$ such that $D^u \cap S = \emptyset$. Moreover, b is in the positive orbit of a and a is in the negative orbit of a^*.

Notice that the intersection of the negative orbit of a^* with V_1^o yields a finite sequence $\{a_0^*, a_1^*, \cdots, a_k^*\}$ with $a_k^* = a^*$ in a way that $a_{i+1}^* = \Pi_{V_1^o}(a_i^*)$ for all $i = 0, \cdots k-1$. We can assume that $a^* = a_k^*$ is the unique intersection point between such a negative orbit and V_0^o for otherwise we replace it by the first point with this property.

Denote by $\Pi : Dom(\Pi) \subset D^u \to V_1^o$ the composition

$$\Pi = \Pi_{V_1^o}^k \circ \Pi_{D^u, V_1^o}.$$

Since V_1 adapted we have from Lemma 6.3 that $Dom(\Pi_{V_1^o})$ is open in V_1^o and $\Pi_{V_1^o}$ is C^1. By the same reason we have from Lemma 6.5 that $Dom(\Pi_{D^u, V_1^0})$ is open in D^u and Π_{D^u, V_1^o} is continuous. Therefore $Dom(\Pi)$ is open in D^u and Π is continuous. On the other hand, it follows from the construction that $a, b \in Dom(\Pi)$ and $\Pi(a) = \Pi(b) = a^*$. Hence

$$\Pi(a) = \Pi(b) \in Int(V_0).$$

Now define q^* as the supremum of the following set,

$$\{s \in [a,b] : [a,s] \subset Dom(\Pi), \Pi([a,s]) \subset Int(V_0), \Pi/[a,s] \text{ is continuous}\}.$$

Since Π is continuous with open domain and $\Pi(a) \in Int(V_0)$ we have that q^* is well defined and $a < q^*$.

If $q^* \notin Dom(\Pi_{D^u, V_1^o})$ then q^* would satisfy Property $(P)_{V_0}$ with $I = (a, q^*)$. Since $\omega(q^*)$ sectional-hyperbolic we would have from Theorem B that $\omega(q^*)$ is a periodic orbit or a singularity. But $\omega(q^*)$ cannot be a periodic orbit by the argument in the last part of the proof of Lemma 5.4 in [110]. Then, $q^* \in W^s(\sigma^*)$ for some singularity σ^*. As $q^* \in D^u \subset W^u(O)$ we have that $x = q^*$ works. In this case we are done. Hence we can assume that $q^* \in Dom(\Pi_{D^u, V_1^o})$.

Denote $q_0^* = \Pi_{D^u, V_1^o}$. If $q_0^* \notin Dom(\Pi_{V_1^o})$ then q_0^* would satisfy $(P)_{V_0}$ with $I = \Pi_{D^u, V_1^o}(a, q^*)$ and then $\omega(q_0^*)$ would be a periodic orbit or a singularity by Theorem B once more. Again the periodic orbit case cannot happen so $\omega(q_0^*)$ would be a singularity and we are done as before. Hence we can assume that $q_0^* \in Dom(\Pi_{V_1^o})$ thus $q_1^* = \Pi_{V_1^o}(q_0^*)$ is well defined. Repeating this argument with the resulting sequence $q_0^*, \cdots, q_k^*$ we can assume $q^* \in Dom(\Pi)$ and so $[a, q^*] \subset Dom(\Pi)$.

Since Π is continuous we have $\Pi([a,q^*]) \subset V_0$ and also that $\Pi([a,q^*])$ is a connected arc transverse to the stable subbundle joining $\Pi(a)$ to $\Pi(q^*) \in \partial^v V_0$.

By replacing a by b in the above argument we obtain another point $q^{**} < b$ such that $\Pi([q^{**},b])$ is a connected arc transverse to the stable subbundle joining $\Pi(b)$ to $\Pi(q^{**}) \in \partial^v V_0$.

But $\Pi(a) = \Pi(b)$ so $l = \Pi([q^{**},b] \cup [a,q^*])$ is a connected arc transverse to the stable subbundle which joints two points in $\partial^v V_0$. It follows from such a transversality that these two points belong to *different* connected components of $\partial^v V_0$. From this and the connecteness of l we obtain an intersection point $x \in l \cap l_S$. As $l_S \subset W^s(\sigma)$ and $l \subset W^u(p)$ we have $x \in W^s(\sigma) \cap W^u(p)$. It follows that $\alpha(p) = \alpha(x)$ and $\omega(x) = \sigma$ proving the result when p is periodic and $q = \sigma$ is a singularity.

The second step consists of proving the result when $q = \sigma$ is a singularity. We have that $\alpha(p)$ is a hyperbolic set since it is non-singular [114]. Fix $y \in \alpha(p)$ and a real number sequence $t_n \to \infty$ such that $X_{-t_n}(p) \to y$.

As the negative orbit of p remains close $\alpha(p)$ which is hyperbolic, we can apply graph transformed techniques ([68], [69]) to find $\varepsilon > 0$ and a sequence of open intervals $I_n = (X_{t_n}(p) - \varepsilon, X_{t_n}(p) + \varepsilon) \subset W^{uu}(X_{-t_n}(p))$ converging to the open interval $I = (y - \varepsilon, y + \varepsilon) \subset W^{uu}(y)$.

Applying the shadowing lemma for flows to the negative orbit of p we can construct a sequence of periodic points $p_n \to y$ whose strong unstable manifolds $W^{uu}(p_n)$ has uniformly large size and approaches to I as $n \to \infty$. Then, both $W^{uu}(p_n)$ and I_n approach to the common interval I. This allows us to fix positive integers n_0, n_1 with the following property (to be applied twice below):

(Q) The stable manifold of every point close to $X_{-t_{n_0}}(p)$ intersects $W^{uu}(p_{n_1})$ and, conversely, the stable manifold of every point close to p_{n_1} intersects I_{n_0} (see Figure 6.6).

As $p \prec \sigma$ we have $X_{t_{n_0}}(p) \prec \sigma$ so there are sequences $z_m \to X_{t_{n_0}}(p)$ and $t_m > 0$ such that $X_{t_m}(z_m) \to \sigma$. Then, **(Q)** implies that there is a corresponding sequence $z'_m \in W^{uu}(p_{n_1})$ in the stable manifold of z_m. Then, $X_{t_m}(z'_m) \to \sigma$ and so $p_{n_1} \prec \sigma$). Since p_{n_1} is periodic it follows from the first step that there is x^* such that $\alpha(p_{n_1}) = \alpha(x^*)$ and $\omega(x^*)$ is a singularity σ^* (see Figure 6.6). By taking the backward orbit of x^* if necessary we can assume x^* to be close to p_{n_1}. Then, **(Q)** once more implies that the stable manifold of x^* intersects I_{n_0} at some point x (see Figure 6.6). Hence $\alpha(x) = \alpha(p)$ (for $I_{n_0} \subset W^{uu}(p)$) and $\omega(x) = \omega(x^*) = \sigma^*$ is a singularity. This proves the result when $q = \sigma$ is a singularity.

Now we prove the sectional-Anosov connecting lemma in the general case. If q belongs to the positive orbit of p, then $x = p$ satisfies the conclusion of the lemma, so, we can assume that q does not belong to the positive orbit of p.

If either $\omega(p)$ or $\omega(q)$ contains a singularity σ, then $p \prec \sigma$ so the second step applies. If $\alpha(q)$ contains a singularity σ, then the continuity of X_t and the fact that q is not in the positive orbit of p imply $p \prec \sigma$ hence step two applies once more.

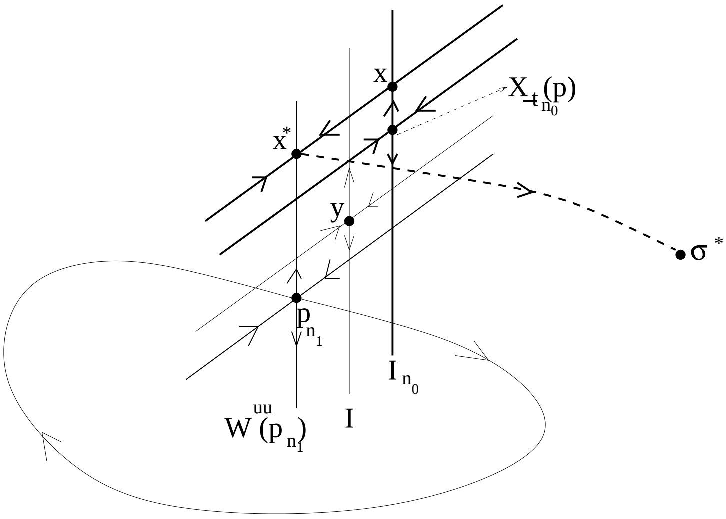


Fig. 6.6

We are left to consider the case when the union $\omega(p) \cup \alpha(q) \cup \omega(q)$ is non-singular. In this case we shall prove that there is x such that $\alpha(x) = \alpha(p)$ and $\omega(x) = \omega(q)$.

For all $\delta > 0$ we denote by $B_\delta(A)$ the open δ-ball centered at A. Recall that $Sing(X)$ denotes the set of singularities of X.

As the union $\alpha(p) \cup \omega(p) \cup \alpha(q) \cup \omega(q)$ is non-singular we have that there is $\delta_1 > 0$ such that p and q belongs to H_1 defined by

$$H_1 = \bigcap_{t \in \mathbb{R}} X_t(M \setminus B_{\delta_1}(Sing(X)).$$

Clearly H_1 is a compact invariant set. Moreover, it is a hyperbolic set since it is non-singular [114]. So, the strong unstable manifold $W^{uu}(p)$ is a well defined one-dimensional manifold containing p.

Since $p \prec q$ there are sequences $z_n \to p$ and $t_n > 0$ such that $X_{t_n}(z_n) \to q$. We can take $t_n \to \infty$ since q is not in the positive orbit of p. But the size of the strong stable manifold $W^{ss}(z_n)$ is bounded away from zero and $W^{uu}(p)$ is a one-dimensional manifold containing p. Then, we can assume that there is a sequence $z'_n \in W^{ss}(z_n) \cap W^{uu}(p)$ so that $z'_n \to p$. As $t_n \to \infty$ we obtain

$$X_{t_n}(z'_n) \to q. \tag{6.8}$$

Suppose for a while that for every $k \in \mathbb{N}$ there is $\sigma_k \in Sing(X)$ such that

$$\sigma_k \in Cl\left(\bigcup_{n=k}^{\infty} O^+(z'_n)\right).$$

As the number of equilibria in Λ is finite we can assume that $\sigma = \sigma_k$ does not depend on k. As $z'_n \to p$ we conclude that $p \prec \sigma$ so step two applies.

Then, we can assume that there are $k_0 \in \mathbb{N}$ and $0 < \delta_2 < \delta_1$ such that

$$\left(\bigcup_{n=k}^{\infty} O^+(z'_n)\right) \cap B_{\delta_2}(Sing(X)) = \emptyset.$$

Observe that $O^+(z'_n) \subset U$ since U is positively invariant. This together with the above equality imply

$$O^+(z'_n) \subset U \setminus B_{\delta_2}(Sing(X)), \quad \forall n \geq k_0.$$

On the other hand, $z'_n \in W^{uu}(p)$ so $\alpha(z'_n) = \alpha(p)$ is non-singular. Then, as $z'_n \to p$, we conclude that there is $0 < \delta_3 < \delta_2$ such that

$$O^-(z'_n) \subset U \setminus B_{\delta_3}(Sing(X)), \quad \forall n \geq k_0.$$

(Recall that $O^-(z)$ is the negative orbit of z.) Consequently each z'_n (with $n \geq k_0$) as well as both p and q belong to

$$H = \bigcap_{t \in \mathbb{R}} X_t(U \setminus B_{\delta_3}(Sing(X))$$

which is non-singular and so hyperbolic by the hyperbolic lemma (c.f. Chapter 1). Then, (6.8) and well known properties of hyperbolic sets yield x such that $\alpha(x) = \alpha(p)$ and $\omega(x) = \omega(q)$. The result is proved. □

The sectional-Anosov connecting lemma is false without the hypothesis that $\alpha(p)$ be non-singular. Indeed, consider a transitive sectional-Anosov flow X with a unique singularity σ on a compact 3-manifold M whose unstable branches are both dense in $M(X)$ (e.g. a generic flow in the solid bitorus as in Theorem 3.22, see for instance [36]). Take $p = \sigma$ and q a point in a periodic orbit of X. Since X is transitive we have p and q satisfies the hypotheses of the theorem except that $\alpha(p)$ is singular. If $x \in M(X)$ satisfies $\alpha(p) = \alpha(x)$, then $x \in W^u(\sigma)$. If $x \neq \sigma$ hence x belongs to one of the unstable branches of σ. From this we get that $\omega(x)$ (which is $M(X)$) is neither $\omega(q)$ (which a periodic orbit) nor a singularity([1]).

A direct corollary of the sectional-Anosov connecting lemma is the following. Given an invariant set H we define its *stable* and *unstable* manifolds by

$$W^s(H) = \{x \in M : \omega(x) \subset H\} \quad \text{and} \quad W^u(H) = \{x \in M : \alpha(x) \subset H\}$$

respectively. Denote by $Cl(\cdot)$ the closure operation.

---

[1] We thank Professor A. Arbieto for providing us this counterexample.

Corollary 6.13. *If H is a compact invariant set without singularities of a sectional-Anosov flow on a compact 3-manifold, then $Cl(W^u(H)) \cap Sing(X) \neq \emptyset$ if and only if $W^u(H) \cap W^s(\sigma) \neq \emptyset$ for some $\sigma \in Sing(X)$.*

Proof. We only have to prove the direct implication since the converse one is trivial. It follows from the hypothesis that there are a singularity q and a sequence $x_n \to q$ such that $\alpha(x_n) \subset H$ for all n. Take a closure point p of $\cup_n \alpha(x_n)$. Then, p and q satisfy the hypothesis of the sectional-Anosov connecting and q is a singularity. So, there is $x \in M$ such that $\alpha(x) = \alpha(p)$ and $\omega(x)$ is a singularity σ. In particular, $x \in W^s(\sigma)$ and, since H is compact, we have $p \in H$ therefore $\alpha(x) = \alpha(p) \subset H$ so $x \in W^u(H)$. Hence $x \in W^u(H) \cap W^s(\sigma)$ and the result follows. □

Applying this corollary to a periodic orbit O we obtain the following statement:

Corollary 6.14. *If O is a periodic orbit of a sectional-Anosov flow X on a compact manifold satisfying $Cl(W^u(O)) \cap Sing(X) \neq \emptyset$, then there is $\sigma \in Sing(X)$ such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$.*

Such a corollary in turns is closely related to Theorem 4.1 in [110] which claims that for every sectional-Anosov flow on a compact 3-manifold *M with a dense orbit in $M(X)$* one has $W^u(O) \cap W^s(\sigma) \neq \emptyset$ for every periodic orbit O and every $\sigma \in Sing(X)$. Indeed, such a theorem is false for a counterexample can be obtained by taking the periodic orbit O and the equilibrium $\sigma = \sigma_2$ in Figure 6.5. Despite of this, the main results in [110] are correct since they are based on Theorem C in [110] whose proof uses the conclusion of Corollary 6.14 only (see p. 364 of [110]). Further applications of the above statement can be found in [35], [99], [108] or [107].

We finish this chapter with the following corollary. Recall Definition 4.25 of Property (P).

Corollary 6.15. *Every sectional-Anosov flow with singularities and dense periodic orbits on a compact 3-manifold has the Property (P).*

Proof. Let X be a sectional-Anosov flow on a compact 3-manifold M as in the statement. Take a periodic orbit O of X. If $Cl(W^u(O)) \cap Sing(X) = \emptyset$ then $Cl(W^u(O))$ is a hyperbolic set by the hyperbolic lemma. In such a case $Cl(W^u(O))$ is an attracting set and clearly it is closed in $M(X)$. On the other hand, if $M(X) \setminus Cl(W^u(O))$ were closed then $Cl(W^u(O))$ would be a connected component of $M(X)$. Since $M(X)$ is connected we would have $Cl(W^u(O)) = M(X)$ and so X has no singularities which contradicts the hypothesis. Then, $M(X) \setminus Cl(W^u(O))$ cannot be closed so there is a sequence $x_n \in M(X) \setminus Cl(W^u(O))$ converging to some $x \in Cl(W^u(O))$. But X has dense periodic orbits so each x_n is accumulated by periodic points which necessarily belong to $Cl(W^u(O))$ for this last set is attracting. This yields a contradiction unless $Cl(W^u(O)) \cap Sing(X) \neq \emptyset$ and then Corollary 6.14 applies. □

The second result of this section is motivated by the so-called *Anosov closing lemma* which states that every recurrent point of an Anosov flow on a closed manifold is approximated by periodic points [62]. It is natural to ask if this is true for sectional-Anosov flows instead of Anosov flows, but this is false by the example in Theorem A. There is however a version the Anosov closing lemma for sectional-Anosov flows which is a reformulation of the main result in [95].

Theorem E (Sectional-Anosov closing lemma)**.** *A recurrent point of a sectional-Anosov flow on a compact* 3*-manifold can be approximated either by periodic points or by* points for which the omega-limit set is a singularity.

Proof. First we show that if $q \in \Omega(X)$ and $\omega(q)$ has a singularity, then

$$W^{ss}(q) \cap Cl(Per(X) \cup W^s(Sing(X))) \neq \emptyset. \tag{6.9}$$

We can assume that q is not contained in the stable manifolds of the singularities for, otherwise, $q \in W^s(Sing(X))$ and then we are done. Choose $\sigma \in \omega(q)$. Hence q is regular and $q \notin W^s(\sigma)$. As $\sigma \in \omega(q)$ we conclude that σ is Lorenz-like. From this we can assume that q belongs to a singular cross-section as in the proof of Theorem C.

Now, to prove $W^{ss}(q) \cap Cl(Per(X) \cup W^s(Sing(X))) \neq \emptyset$, we shall assume by contradiction that

$$W^{ss}(q) \cap Cl(Per(X) \cup W^s(Sing(X))) = \emptyset.$$

Therefore

$$\mathscr{F}^s_q \cap Cl(Per(X) \cup W^s(Sing(X))) = \emptyset. \tag{6.10}$$

In particular, $\mathscr{F}^s_q$ is not accumulated by periodic orbits. Then, by Theorem 6.6 applied to $L_0 = \mathscr{F}^s_q$, we can choose an adapted band V around (and arbitrarily close to) $\mathscr{F}^s_q$. *Consequently we can assume that* V *is not accumulated by periodic orbits*.

On the other hand, since V is close to $\mathscr{F}^s_q$, we can use (6.10) to assume that V does not intersect the stable manifold of the singularities. Since V contains q which is non-wandering we get from Theorem 6.5 that V *intersects a periodic orbit*. This is a contradiction which proves $\mathscr{F}^s_q \cap Cl(Per(X) \cup W^s(Sing(X))) \neq \emptyset$ yielding (6.9).

Now assume that q be a recurrent point. We must prove

$$q \in Cl(Per(X) \cup W^s(Sing(X))) \tag{6.11}$$

For this we consider two cases, namely, $\omega(q)$ contains a singularity or not. If $\omega(q)$ has no singularities, then $\omega(q)$ is hyperbolic by the hyperbolic lemma and then (6.11) holds by the Shadowing Lemma. Now assume that $\omega(q)$ contains a singularity. Note that $q \in \Omega(X)$ since q is recurrent. Hence there is $z \in W^{ss}(q) \cap Cl(Per(X) \cup W^s(Sing(X)))$ by (6.9). As $z \in W^{ss}(q)$ we have $\omega(z) = \omega(q)$. But

$$\omega(z) \subset Cl(Per(X) \cup W^s(Sing(X)))$$

since $z \in Cl(Per(X) \cup W^s(Sing(X)))$ which is compact invariant. So,

$$\omega(q) \subset Cl(Per(X) \cup W^s(Sing(X)))$$

and then (6.11) holds because $q \in \omega(q)$. This proves the result. □

Observe that the sectional-Anosov connecting lemma doesn't imply the sectional-Anosov closing lemma due to the second alternative in the former lemma. Let us explain why the conclusion of the sectional-Anosov closing lemma is sharp. First of all we observe that every sectional-Anosov flow on a compact 3-manifold has a periodic orbit by Theorem C, so, all such flows have recurrent points which can be approximated by periodic points. Secondly, there is a sectional-Anosov flow on a certain compact 3-manifold all of whose recurrent points are approximated by both periodic points and by points for which omega-limit set is a singularity (e.g. the one in Theorem 3.22).

The sectional-Anosov closing lemma is related to the aforementioned work [10] whose main result would imply the three-dimensional case of the Homoclinic class conjecture. See also the early version [11] of [10] or [97]. It would follow from such a result that the periodic orbits are dense in the maximal invariant set of any *transitive* sectional-Anosov flow on a compact 3-manifold. The sectional-Anosov closing lemma doesn't assume transitivity, but yields instead denseness of periodic points or points for which the omega-limit set is a singularity. Anyway the conclusion of the sectional-Anosov closing lemma is sharp in the general case.

6.2.4 Dynamics of venice masks

It is easy to see that the examples of venice mask in Subsection 3.4.3 satisfy that the maximal invariant set is a non-disjoint union of two different homoclinic classes (represented by H_1 and H_2 in the example of Figure 3.24). The goal of this section is to prove that property holds for all venice masks with only one singularity on compact 3-manifolds. In other words, we shall prove that the maximal invariant set of these flows is a non-disjoint union of two different homoclinic classes. The result of this section were originally proved in [107].

To start with we present a classical criterium for the transitivity of compact invariant sets due to Birkhoff.

Lemma 6.6. *Let X a vector field and T be a compact invariant set of X. If for all open sets U,V intersecting T there is $t > 0$ such that $X_t(U \cap T) \cap V \neq \emptyset$, then T is transitive.*

This criterium will be used together with the following lemma. Recall the notation $W^{s,+}(\sigma)$ and $W^{s,-}(\sigma)$ introduced in Theorem 4.27.

Lemma 6.7. *Let X sectional-Anosov flow on a compact 3-manifold M, σ a Lorenz-like singularity and $p,r \in Per(X)$ be such that $W^u(p) \cap W^{s,+}(\sigma) \neq \emptyset$ and $W^u(r) \cap W^{s,+}(\sigma) \cap Cl(Per(X)) \neq \emptyset$. Then, for all neighborhoods U and V of p and r respectively there is $z \in W^u(p) \cap U$ and $t > 0$ such that $X_t(z) \in V$. A similar result holds replacing $+$ by $-$.*

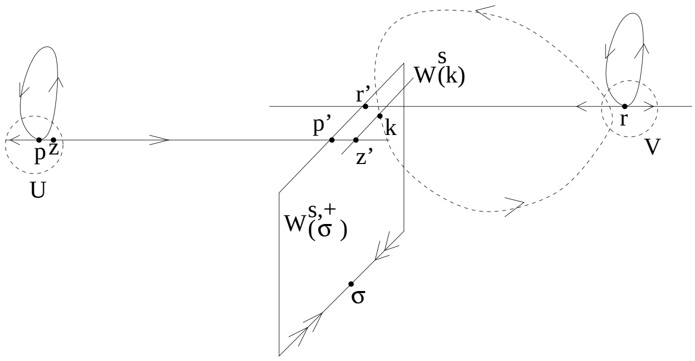


Fig. 6.7 Proof of Lemma 6.7.

Proof. The proof is described in Figure 6.7. Pick $p' \in W^u(p) \cap W^{s,+}(\sigma)$ and $r' \in W^u(r) \cap W^{s,+}(\sigma) \cap Per(X)$. We can choose both p' and r' close to σ (and hence close one to another) for they are in $W^s(\sigma)$. Since $r' \in Per(X)$ we can choose a periodic point k nearby r'. It follows from the uniform size of the stable manifolds that there is $z' \in W^s(k) \cap W^u(p)$. Therefore the positive orbit of z' converges to the periodic orbit through k. On the other hand, k is close to r' which belongs to $W^u(r)$ so the orbit through k enters into V. Hence there is a point $z \in U$ in the negative orbit of z' whose positive orbit passes through V. □

Next we recall the sufficient condition (4.4) for existence of singular partitions in Theorem 4.27.

Proposition 6.16. *For every venice mask X with a unique singularity σ on a compact 3-manifold there are periodic points p and r satisfying (4.4).*

Proof. Suppose by contradiction that no such periodic points exist. Take two open sets U, V intersecting $M(X)$. Since X has dense periodic orbits we can select $p \in Per(X) \cap U$ and $r \in Per(X) \cap V$. We have three cases to consider, namely, either $W^u(p) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$ or $W^u(p) \cap W^s(\sigma) \subset W^{s,-}(\sigma)$ or $W^u(p) \cap W^{s,+}(\sigma) \neq \emptyset$ and $W^u(p) \cap W^{s,-}(\sigma) \neq \emptyset$.

If $W^u(p) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$, then $W^u(r) \cap W^{s,+}(\sigma) \neq \emptyset$ since p and r do not satisfy (4.4). Then, by Lemma 6.7, there are $z \in W^u(p) \cap U$ and $t > 0$ such that $X_t(z) \in V$. Since $W^u(p) \subset M(X)$ we conclude that there is $t > 0$ such that $X_t(U \cap M(X)) \cap V \neq \emptyset$.

If $W^u(p) \cap W^s(\sigma) \subset W^{s,-}(\sigma)$, then again we have $W^u(r) \cap W^{s,-}(\sigma) \neq \emptyset$ because r and p do not satisfy (4.4). Then, by Lemma 6.7, there are $z \in W^u(p) \cap U$ and $t > 0$ such that $X_t(z) \in V$, so, there is $t > 0$ such that $X_t(U \cap M(X)) \cap V \neq \emptyset$.

Finally assume that $W^u(p) \cap W^{s,+}(\sigma) \neq \emptyset$ and $W^u(p) \cap W^{s,-}(\sigma) \neq \emptyset$. Since X has Property (P) by Corollary 6.15 we have that $W^u(r) \cap W^s(\sigma) \neq \emptyset$. Since $M(X) \cap W^{ss}(\sigma) = \{\sigma\}$ by Corollary 2.7 applied to $\Lambda = M(X)$ we have either $W^u(r) \cap W^{s,+}(\sigma) \neq \emptyset$ or $W^u(r) \cap W^{s,-}(\sigma) \neq \emptyset$. It follows that either $W^u(p) \cap W^{s,+}(\sigma) \neq \emptyset$

and $W^u(r)\cap W^{s,+}(\sigma)\neq\emptyset$ or $W^u(p)\cap W^{s,-}(\sigma)\neq\emptyset$ and $W^u(r)\cap W^{s,-}(\sigma)\neq\emptyset$. In each case we conclude as before that there is $t>0$ such that $X_t(U\cap M(X))\cap V\neq\emptyset$.

Since U and V are arbitrary we conclude from Birkhoff's that $M(X)$ is a transitive set. Therefore X is transitive which is a contradiction since X is a venice mask. This proves the result. □

Proposition 6.17. *If X is a venice mask with a unique singularity σ on a compact 3-manifold, then there is no sequence $p_n\in Per(X)$ converging to some point in $W^{s,+}(\sigma)$ such that $W^u(p_n)\cap W^{s,-}(\sigma)\neq\emptyset$ for all n. Similarly interchanging the roles of $+$ and $-$.*

Proof. By Proposition 6.16 we can fix $p,r\in Per(X)$ satisfying (4.4). In particular, $W^u(p)\cap W^s(\sigma)\subset W^{s,+}(\sigma)$. Since X satisfies Property (P) we can fix $a\in W^u(p)\cap W^{s,+}(\sigma)$. Suppose by contradiction that there is sequence $p_n\in Per(X)$ which both converges to some point $z\in W^{s,+}(\sigma)$ and satisfies $W^u(p_n)\cap W^{s,-}(\sigma)\neq\emptyset$ for all n. By taking forward orbits if necessary we can assume that both z and a are close to σ. Since $p_n\to z$ we have from the uniformly large size of the stable manifolds that there is n large such that $W^s(p_n)\cap W^u(p)\neq\emptyset$ (indeed, such an intersection contains a point close to a). Applying the Inclination-lemma [87] and the fact that $W^u(p_n)\cap W^{s,-}(\sigma)\neq\emptyset$ we have that $W^u(p)\cap W^{s,-}(\sigma)\neq\emptyset$ which contradicts (4.4). This contradiction proves the assertion for $+$. The assertion for $-$ follows analogously by considering the inclusion $W^u(r)\cap W^s(\sigma)\subset W^{s,-}(\sigma)$ in (4.4). □

Proposition 6.18. *If X is a venice mask with a unique singularity σ on a compact manifold, then $\sigma\notin\omega(q)$ for all $q\in W^u(\sigma)\setminus\{\sigma\}$.*

Proof. Suppose by contradiction that $\sigma\in\omega(q)$. Without loss of generality we can assume that there is $z\in\omega(q)\cap W^{s,+}(\sigma)$. Then we can choose a sequence q_n in the positive orbit of q such that $q_n\to z$.

By Proposition 6.16 we can take $r\in Per(X)$ as in (4.4). Choose $r'\in W^u(r)\cap W^{s,-}(\sigma)$ and a sequence $r_n\in W^u(r)$ converging to r in a way that the positive orbit of r_n accumulates on q as indicated in Figure 6.8. Since X has dense periodic orbits and $W^u(r)\subset M(X)$ we can select a sequence $p'_n\in Per(X)$ close to r_n. Therefore, $p'_n\to r$. Since $r\in W^{s,-}(\sigma)$, $p'_n\in Per(X)$ and $p'_n\to r$ we get from Proposition 6.17 that $W^u(p'_n)\cap W^{s,+}(\sigma)=\emptyset$ and so

$$W^u(p'_n)\cap W^s(\sigma)\subset W^{s,-}(\sigma),\quad \forall n \text{ large.} \tag{6.12}$$

On the other hand, since p'_n close to r_n and the orbit of r_n passes close to q we can arreange p_n in the positive orbit of p'_n such that p_n is close to q_n. Therefore $p_n\to z$. Since $z\in W^{s,+}(\sigma)$, $p_n\in Per(X)$ and $p_n\to z$ we get from Proposition 6.17 that $W^u(p_n)\cap W^{s,-}(\sigma)=\emptyset$ and so $W^u(p_n)\cap W^s(\sigma)\subset W^{s,+}(\sigma)$ for all n large. But p_n belongs to the orbit of p'_n so $W^u(p'_n)=W^u(p_n)$. We conclude that

$$W^u(p'_n)\cap W^s(\sigma)\subset W^{s,+}(\sigma),\quad \forall n \text{ large}$$

which clearly contradicts (6.12). This ends the proof. □

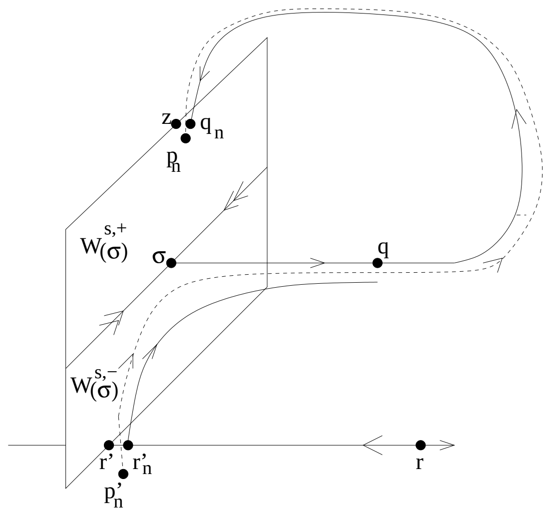


Fig. 6.8 Proof of Proposition 6.18.

Theorem 6.19. *If X is a venice mask with a unique singularity σ on a compact manifold, then $\omega(q)$ is a periodic orbit* with positive expanding eigenvalue *for all $q \in W^u(\sigma) \setminus \{\sigma\}$.*

Proof. Fix $q \in W^u(\sigma) \setminus \{\sigma\}$ and assume by contradiction that $\omega(q)$ is *not* a periodic orbit. We have from Proposition 6.18 that $\sigma \notin \omega(q)$ so q cannot be a recurrent point. Moreover, by Proposition 6.16 there are $p, r \in Per(X)$ satisfying (4.4). On the other hand we have that X satisfies Property (P) by Corollary 6.15 so $W^u(z) \cap W^s(\sigma)$ is dense in $W^u(z)$ for every $z \in Per(X)$. In particular, $W^u(p) \cap W^{s,+}(\sigma)$ is dense in $W^u(p)$ and $W^u(p) \cap W^{s,-}(\sigma)$ is dense in $W^u(r)$. It then follows from Theorem 4.27 that $\omega(q)$ has singular partitions $\mathscr{R}$ close to it.

Using that $W^u(p) \cap W^{s,+}(\sigma) \neq \emptyset$ and $W^u(r) \cap W^{s,-}(\sigma) \neq \emptyset$ and a linear coordinate around σ we can construct an open interval $I = I_q$, contained in a suitable cross-section throught q such that $I \setminus \{q\}$ is formed by two intervals $I^+ \subset W^u(p)$ and $I^- \subset W^u(r)$ in a way that I is tangent to the central subbundle E^c of X. Since $\omega(q) \cap Sing(X) = \emptyset$ (for $Sing(X) = \{\sigma\}$ and $\sigma \notin \omega(q)$) we can apply Theorem 2.15 to such an interval I in order to obtain $S \in \mathscr{R}$, a sequence $q_1, q_2, \cdots \in S$ of points in the positive orbit of q and a sequence of intervals $J_1, J_2, \cdots \subset S$ in the positive orbit of I with $q_n \in J_n$ such that $\{\text{Length}(J_n) : n = 1, 2, 3, \cdots\}$ is bounded away from 0. For all n we let J_n^+ and J_n^- denote the two connected components of $J_n \setminus \{q_n\}$ in a way that J_n^+ is in the positive orbit of I^+ and J_n^- is in the positive orbit of I^-.

Since $W^u(p) \cap W^s(\sigma)$ is dense in $W^u(p)$ and $W^u(r) \cap W^s(\sigma)$ is dense in $W^u(r)$ we have that $W^{s,+}(\sigma) \cap I^+$ is dense in I^+ and $W^{s,-}(\sigma) \cap I^-$ is dense in I^-. So,

$W^{s,+}(\sigma) \cap J_n^+$ is dense in J_n^+ and $W^{s,-}(\sigma) \cap J_n^-$ is dense in J_n^-. From this and the uniform size of the stable manifolds we have the following property for all curve $c \subset S$:

$$c \pitchfork \mathscr{F}^s(q_n, S) \neq \emptyset \quad \Longrightarrow \quad c \cap W^{s,+}(\sigma) \neq \emptyset \quad \text{and} \quad c \cap W^{s,-}(\sigma) \neq \emptyset. \tag{6.13}$$

Now take a limit point $x \in S$ of q_n. Then $x \in \omega(q) \cap Int(S)$. Because I is tangent to E^c the interval sequence J_n converges to an interval $J \subset W^u(x)$ in the C^1 topology ($W^u(x)$ exists since $\omega(q)$ is hyperbolic).

We have that J is not trivial since the length of J_n is bounded away from 0. If $q_n \in \mathscr{F}^s(x, S)$ for n large we would obtain that x is a periodic point by [110, Lemma 5.6], a contradiction since $\omega(a)$ is not a periodic orbit. Therefore $q_n \notin \mathscr{F}^s(x, S)$, $\forall n$.

Consequently, as $J_n \to J$ and $q_n \to x$, we eventually have either $J_{n+1}^+ \cap \mathscr{F}^s(q_n, S) \neq \emptyset$ or $J_{n+1}^- \cap \mathscr{F}^s(q_n, S) \neq \emptyset$. Since both intersections are transversal we conclude by taking $c = J_{n+1}^+$ or $c = J_{n+1}^-$ respectively in (6.13) that either I^+ or I^- intersects both $W^{s,+}(\sigma)$ and $W^{s,-}(\sigma)$. Since $I^+ \subset W^u(p)$ and $I^- \subset W^u(r)$ we get a contradiction by (4.4). This contradictions shows that $\omega(q)$ is a periodic orbit O.

Now we prove that the expanding eigenvalue of O is positive. Suppose by contradiction that it is not so. From the Property (P) we have that $W^u(O)$ intersects $W^s(\sigma)$ and so either $W^u(O) \pitchfork W^{s,+}(\sigma) \neq \emptyset$ or $W^u(O) \pitchfork W^{s,-}(\sigma) \neq \emptyset$. Suppose that $W^u(O) \pitchfork W^{s,+}(\sigma) \neq \emptyset$ (the other case can be handled similarly). As $W^u(O) \pitchfork W^{s,+}(\sigma) \neq \emptyset$, $\omega(q) = O$ and O has negative eigenvalues we have from the Inclination-lemma that $I^- \cap W^{s,+}(\sigma) \neq \emptyset$. But $I^- \subset W^u(r)$ so $W^u(r) \cap W^{s,+}(\sigma) \neq \emptyset$ and then $W^u(r) \cap W^s(\sigma) \not\subset W^{s,-}(\sigma)$ which contradicts the second inclusion in (4.4). This contradiction proves the result. □

In the sequel we shall assume that X is a venice mask with a unique singularity σ *on a compact* 3*-manifold M.*

Let $q \in W^s(\sigma) \setminus \{\sigma\}$ be fixed. By Theorem 6.19 we have that $q \in W^s(O)$ for some periodic orbit with positive expanding eigenvalue O. Since the expanding eigenvalue of O is positive we have that $W^u(O)$ is a cylinder with generating curve O and so O separates $W^u(O)$ in two connected components. It follows from the proof of Theorem 6.19 that none of these components can intersect $W^{s,+}(\sigma)$ and $W^{s,-}(\sigma)$ simultaneously (otherwise we would contradict (4.4) using the curve I and the Inclination-Lemma). Then, we can denote them by $W^{u,+}$ and $W^{u,-}$ in a way that $W^{u,+} \cap W^{s,-}(\sigma) = \emptyset$ and $W^{u,-} \cap W^{s,+}(\sigma) = \emptyset$ or, equivalently,

$$W^{u,+} \cap W^s(\sigma) \subset W^{s,+}(\sigma) \quad \text{and} \quad W^{u,-} \cap W^s(\sigma) \subset W^{s,-}(\sigma). \tag{6.14}$$

A basic property of these components is given below.

Proposition 6.20. *If* $z \in Per(X)$ *and* $W^s(z) \cap W^{u,+} \neq \emptyset$*, then* $W^s(z) \cap W^{u,+}$ *is dense in* $W^{u,+}$*. Similarly replacing* $+$ *by* $-$*.*

Proof. First observe that $W_X^{s,+}(\sigma) \cap W^{u,+}$ is dense in $W^{u,+}$. Choose $x \in W^{u,+}$. As $W^{s,+}(\sigma) \cap W^{u,+}$ is dense in $W^{u,+}$ we can choose an interval $I_x \subset W^{u,+}$ arbitrarily close to x such that $I_x \cap W^{s,+}(\sigma) \neq \emptyset$.

The positive orbit of I_x first passes through q and, afterward, it accumulates on $W^{u,+}$ by the Inclination-lemma (it cannot accumulates on $W^{u,-}$ for otherwise we would have $W^{u,+} \cap W^{s,-}(\sigma) \neq \emptyset$ which contradicts (6.14)).

But $W^s(z) \cap W^{u,+} \neq \emptyset$ by assumption, and such an intersection is transversal, so the Inclination Lemma implies that the positive orbit of I_x intersects $W^s(z)$. By taking the backward flow of the last intersection we arrive to $W^s(z) \cap I_x \neq \emptyset$. Since I_x is close to x and $I_x \subset W^{u,+}$ we get that $W^s(z) \cap W^{u,+}$ is dense in $W^{u,+}$. This proves the result. □

Now we define

$$H^+ = Cl(\{p \in Per(X) : W^u(p) \cap W^{s,+}(\sigma) \neq \emptyset\}) \tag{6.15}$$

and

$$H^- = Cl(\{r \in Per(X) : W^u(r) \cap W^{s,-}(\sigma) \neq \emptyset\}). \tag{6.16}$$

Since $W^{u,+} \cup W^{u,-} \subset W^u(O)$ we have from (6.14) and the Property (P) of X respectively that

$$O \subset H^+ \cap H^- \quad \text{and} \quad M(X) = H^+ \cup H^-. \tag{6.17}$$

More properties of these sets are given below.

Proposition 6.21. $H^+ = Cl(W^{u,+})$ *and* $H^- = Cl(W^{u,-})$.

Proof. Fix $p \in Per(X)$ such that $W^u(p) \cap W^{s,+}(\sigma) \neq \emptyset$. Choose $z \in W^u(p) \cap W^{s,+}(\sigma)$ close to σ. Since X has dense periodic orbits we can choose $r \in Per(X)$ close to z. Note that $W^{u,+} \cap W^{s,+}(\sigma) \neq \emptyset$ so we can choose $t \in W^{u,+} \cap W^{s,+}(\sigma)$ close to σ and then r and t are close one to another. Then, since the stable manifold has uniform size we obtain that $W^s(r) \cap W^{u,+} \neq \emptyset$ and so $W^{u,+}$ accumulates on r by the Inclination-lemma. From this we get that z (and so p) belong to $Cl(W^{u,+})$. Therefore

$$H^+ \subset Cl(W^{u,+}).$$

Conversely fix $x \in W^{u,+}$. Since X has dense periodic orbits and $W^{u,+} \subset M(X)$ there is $p \in Per(X)$ close to x. In particular, $W^s(p) \cap W^{u,+} \neq \emptyset$ because stable manifolds have uniformly size. If $W^u(p) \cap W^{s,-}(\sigma) \neq \emptyset$ then the Inclination Lemma and $W^s(p) \cap W^{u,+} \neq \emptyset$ would imply $W^{u,+} \cap W^{s,-}(\sigma) \neq \emptyset$ which contradicts (6.14). So $W^u(p) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$ and then $x \in H^+$ which proves

$$Cl(W^{u,+}) \subset H^+$$

therefore $H^+ = Cl(W^{u,+})$. Analogously we prove $H^- = Cl(W^{u,-})$. □

Recall that if $z \in Per(X)$ then $H(z)$ denotes the homoclinic class of z.

Proposition 6.22. *If* $z \in Per(X)$ *is close to some point in* $W^{u,+}$*, then* $H(z) = Cl(W^{u,+})$*. Similarly replacing* $+$ *by* $-$.

Proof. Take z close to some point in $W^{u,+}$ hence $W^s(z) \cap W^{u,+} \neq \emptyset$. On the one hand we have that $H(z) \subset Cl(W^{u,+})$ by the Inclination-lemma. On the other hand, $W^{u,+} \cap W_X^{s,-}(\sigma) = \emptyset$ by (6.14) we have $W_X^u(z) \cap W_X^s(\sigma) \subset W_X^{s,+}(\sigma)$ by the Inclination-lemma. In addition $W^u(z) \cap W^s(\sigma) \neq \emptyset$ by Property (P), so, $W^u(z) \cap W^{s,+}(\sigma) \neq \emptyset$. Then, $W^u(z)$ accumulates on $W^{u,+}$ and so $H(z)$ contains $W^s(z) \cap W^{u,+}$ by the Inclination-lemma once more. But, $W^s(z) \cap W^{u,+}$ is dense in $W^{u,+}$ by Proposition 6.20 since $W^s(z) \cap W^{u,+} \neq \emptyset$. So, $Cl(W^{u,+}) \subset H(z)$ since homoclinic classes are closed set. This proves the result. □

Corollary 6.23. *H^+ and H^- are homoclinic classes.*

Proof. We only explain the proof for H^+ since that for H^- is similar. By Proposition 6.21 it suffices to prove that $Cl(W^{u,+})$ is a homoclinic class. Since X has dense periodic orbit and clearly $W^{u,+} \subset M(X)$ we can choose a periodic point z close to some point in $W^{u,+}$. Then $Cl(W^{u,+}) = H(z)$ by Proposition 6.22 so $Cl(W^{u,+})$ is a homoclinic class. □

Now we state the main result of this section.

Theorem F. *The maximal invariant set of a venice mask with a unique singularity on a compact 3-manifold is a non disjoint union of two different homoclinic classes.*

Proof. Let X be a venice mask with a unique singularity σ on a compact 3-manifold M. By (6.17) we can write $M(X) = H^+ \cup H^-$ with $H^+ \cap H^-$ containing a periodic orbit O. We have that H^+ and H^- are homoclinic classes by Corollary 6.23. Since $M(X) = H^+ \cup H^-$, X is not transitive and both H^+, H^- are homoclinic classes (hence transitive) we see that $H^+ \neq H^-$. Finally $H^+ \cap H^- \neq \emptyset$ since this intersection contains O. □

6.3 Perturbing sectional-Anosov flows

In this section we present some result about the perturbation theory of sectional-Anosov flows on compact 3-manifolds.

6.3.1 A bound for the number of attractors

Recall that an *attractor* of a vector field X is a transitive set equals to $\bigcap_{t>0} X_t(V)$ for some compact neighborhood V. In this chapter r will denote either ∞ or a positive integer.

Theorem G. *Every attractor of every vector field C^r close to a C^r transitive sectional-Anosov flow with singularities on a compact 3-manifold has a singularity.*

Proof. Suppose by contradiction that there is a C^r transitive sectional-Anosov flow X on a compact 3-manifold M which is the C^r limit of a sequence of vector fields X^n each one exhibiting an attractor without singularities A^n. It follows from the hyperbolic lemma that all such attractors are hyperbolic for the corresponding flow.

We claim that

$$Sing(X) \cap Cl\left(\bigcup_{n\in\mathbb{N}} A^n\right) \neq \emptyset.$$

Otherwise there is $\delta > 0$ such that

$$B_\delta(Sing(X)) \cap \left(\bigcup_{n\in\mathbb{N}} A^n\right) = \emptyset. \tag{6.18}$$

Define

$$H = \bigcap_{t\in\mathbb{R}} X_t\left(M \setminus B_{\delta/2}(Sing(X))\right).$$

Obviously $Sing(X) \cap H = \emptyset$ and then H is a hyperbolic set. Denote by $E^s \oplus E^X \oplus E^u$ the corresponding hyperbolic splitting.

It follows from the stability of hyperbolic sets that there are compact neighborhoods V, W

$$H \subset Int(V) \subset V \subset Int(W) \subset W$$

of H and $\varepsilon > 0$ such that if Y is a vector field that is C^1 close to X and H_Y is a compact invariant set of Y in W then:

(H1) H_Y is hyperbolic and its hyperbolic splitting $E^{s,Y} \oplus E^Y \oplus E^{u,Y}$ satisfies

$$dim(E^u) = dim(E^{u,Y}), \;\; dim(E^s) = dim(E^{s,Y}).$$

(H2) The local strong unstable manifolds $W^{uu}_Y(y,\varepsilon)$, $y \in H_Y$, are one-dimensional of uniform size ε.

We assert that $A^n \subset W$ for all n large. Indeed, suppose by contradiction that this is not true. Then, there are sequences $n_k \to \infty$ and $x^{n_k} \in A^{n_k}$ such that $x^{n_k} \notin W$ for all k. Since M is compact we can assume that $x^{n_k} \to x$ for some $x \in M$. Clearly $x \in M \setminus Int(W)$ and so $x \notin V$. Then, since $H = \bigcap_{t\in\mathbb{R}} X_t\left(M \setminus B_{\frac{\delta}{2}}(Sing(X))\right)$ and $H \subset V$, we can arrange $t \in \mathbb{R}$ such that $X_t(x) \in B_{\frac{\delta}{2}}(Sing(X))$. On the other hand, $X^n \to X$ and $x^{n_k} \to x$ so $X^{n_k}_t(x^{n_k}) \to X_t(x)$ hence $X^{n_k}_t(x^{n_k}) \in B_{\frac{\delta}{2}}(Sing(X))$ for k large. However, A^n is X^n-invariant so $X^{n_k}_t(x^{n_k}) \in A^{n_k}$ yielding $X^{n_k}_t(x^{n_k}) \in A^{n_k} \cap B_{\frac{\delta}{2}}(Sing(X))$ from which we get $A^{n_k} \cap B_{\frac{\delta}{2}}(Sing(X)) \neq \emptyset$ in contradiction with (6.18). This proves the assertion.

As $X^n \to X$ the assertion and (H2) with $Y = X^n$ and $H_Y = A^n$ imply that $W^{uu}_{X^n}(y,\varepsilon)$ has uniform size ε for all $y \in A^n$ and n large.

Take $x^n \in A^n$ converging to some $x \in M$. Clearly $x \in H$. Note that the tangent vectors of the curve $W^{uu}_{X^n}(x^n,\varepsilon)$ at every $c \in W^{uu}_{X^n}(x^n,\varepsilon)$ belongs to E^{u,X^n}_c.

As $X^n \to X$ we have that the angle between the directions E^{u,X^n} and E^u goes to zero as $n \to \infty$. Henceforth the manifolds $W^{uu}_X(x,\varepsilon)$ and $W^{uu}_{X^n}(x^n,\varepsilon)$ are almost parallel as $n \to \infty$. As $x^n \to x$ we conclude that

$$W^{uu}_{X^n}(x^n,\varepsilon) \to W^{uu}_X(x,\varepsilon)$$

in the sense of C^1 submanifolds [123].

Fix an open interval $I \subset W^{uu}_X(x,\varepsilon)$ containing x. Since $Sing(X) \neq \emptyset$ and X is transitive, we have that there are $q \in I$ and $T > 0$ such that

$$X_T(q) \in B_{\delta/5}(Sing(X)).$$

Then, by the Tubular Flow Box Theorem, there is an open set V_q containing q such that

$$X_T(V_q) \subset B_{\delta/5}(Sing(X)).$$

As $X^n \to X$ we have

$$X^n_T(V_q) \subset B_{\delta/4}(Sing(X)) \tag{6.19}$$

for all n large. But $W^{uu}_{X^n}(x^n,\varepsilon) \to W^{uu}_X(x,\varepsilon)$, $q \in I \subset W^{uu}_X(p,\varepsilon)$, $q \in V_q$ and V_q is open. So,

$$W^{uu}_{X^n}(x^n,\varepsilon) \cap V_q \neq \emptyset$$

for all n large. Applying (6.19) to X^n for n large we have

$$X^n_T(W^{uu}_{X^n}(x^n,\varepsilon)) \cap B_{\delta/4}(Sing(X)) \neq \emptyset.$$

As $W^{uu}_{X^n}(x^n,\varepsilon) \subset W^u_{X^n}(x^n)$ the invariance of $W^u_{X^n}(x^n)$ implies

$$W^u_{X^n}(x^n) \cap B_{\delta/2}(Sing(X)) \neq \emptyset.$$

Observe that $W^u_X(x^n) \subset A^n$ since $x^n \in A^n$ and A^n is an attractor. We conclude that

$$A^n \cap B_\delta(Sing(X)) \neq \emptyset$$

which contradicts (6.18). The claim follows.

Let us continue with the proof of the theorem. By the previous claim we can choose

$$\sigma \in Sing(X) \cap Cl\left(\bigcup_{n\in\mathbb{N}} A^n\right).$$

We have that σ is Lorenz-like and satisfies

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$

Let $S^t = S^t_\sigma$ and $S^b = S^b_\sigma$ be the singular cross-sections associated to σ as in the proof of Theorem C. In particular,

$$M(X) \cap \left(\partial^h S^t \cup \partial^h S^b\right) = \emptyset.$$

As $X^n \to X$ we have that S^t, S^b are singular-cross sections of X^n too. By implicit function reasons we can assume that $\sigma(X^n) = \sigma$, where $\sigma(X^n)$ is the continuation of $\sigma = \sigma(X)$ (c.f. [123]). Moreover,

$$l^t \cup l^b \subset W^s_{X^n}(\sigma), \ \ \forall n. \tag{6.20}$$

The one-dimensional subbundle E^s of X extends to a contracting invariant subbundle in M. Take a continuous (but not necessarily invariant) extension of E^c. We still denoted by $E^s \oplus E^c$ the above-mentioned extension.

By the Invariant Manifold Theory it follows that the splitting $E^s \oplus E^c$ persists by small perturbations of X. More precisely, for all n large the vector field X^n has an splitting $E^{s,n} \oplus E^{c,n}$ over U such that $E^{s,n}$ is invariant contracting, $E^{s,n} \to E^s$ and $E^{c,n} \to E^c$ as $n \to \infty$. In particular, $E^{s,n} \oplus E^{c,n}$ is defined in $S^t \cup S^b$ for all n large. In what follows we denote by E^Y the subbundle in TM generated by a vector field Y in M.

The dominance condition together with [42, Proposition 2.2] imply that for $* = t, b$ one has

$$T_x S^* \cap \left(E^s_x \oplus E^X_x\right) = T_x l^*,$$

for all $x \in l^*$.

Denote by $\angle(E,F)$ the angle between two linear subspaces. The last equality implies that there is $\rho > 0$ such that

$$\angle(T_x S^* \cap E^c_x, T_x l^*) > \rho,$$

for all $x \in l^*$ ($* = t, b$). But $E^{c,n} \to E^c$ as $n \to \infty$. So for all n large we have

$$\angle(T_x S^* \cap E^{c,n}_x, T_x l^*) > \frac{\rho}{2}, \tag{6.21}$$

for all $x \in l^*$ (again $* = t, b$).

Fix $* = t, b$ and a coordinate system $(x,y) = (x^*, y^*)$ in S^* such that

$$S^* = [-1,1] \times [-1,1], \quad l^* = \{0\} \times [-1,1]$$

with respect to (x,y).

Denote by $\Pi^* : S^* \to [-1,1]$ the projection

$$\Pi^*(x,y) = x$$

and for $\Delta > 0$ we define

$$S^{*,\Delta} = [-\Delta,\Delta] \times [-1,1].$$

Define the line field F^n in $S^{*,\Delta}$ by

$$F^n_x = T_x S^* \cap E^{c,n}_x, \;\; x \in S^{*,\Delta}.$$

The continuity of $E^{c,n}$ and (6.21) imply that $\exists \Delta_0 > 0$ such that $\forall n$ large the line F^n is *transverse to* Π^*. By this we mean that $F^n(z)$ is *not tangent to the curves* $(\Pi^*)^{-1}(c)$.

Now recall that A^n is a hyperbolic attractor of X^n for all n. It follows that the periodic orbits of X^n in A^n are dense in A^n ([123]). Then, as $\sigma \in Cl\,(\cup_{n\in\mathbb{N}} A^n)$, there is a periodic orbit sequence $O_n \in A^n$ accumulating on σ. It follows that there is $n_0 \in \mathbb{N}$ such that either

$$O_{n_0} \cap Int(S^{t,\Delta_0}) \neq \emptyset \;\text{ or }\; O_{n_0} \cap Int(S^{b,\Delta_0}) \neq \emptyset.$$

Because $O_{n_0} \subset A_{n_0}$ we conclude that either

$$A^{n_0} \cap Int(S^{t,\Delta_0}) \neq \emptyset \;\text{ or }\; A^{n_0} \cap Int(S^{b,\Delta_0}) \neq \emptyset.$$

We denote $Z = X^{n_0}, A = A^{n_0}, F = F^{n_0}$ for simplicity. Thus A is a hyperbolic attractor of Z and so it is not a singularity of Z.

We can assume that $A \cap Int(S^{t,\Delta_0}) \neq \emptyset$. Note that $\partial^h S^{t,\Delta_0} \subset \partial^h S^t$ by definition. Then,

$$A \cap \partial^h S^{t,\Delta_0} = \emptyset.$$

We denote $S = S^{t,\Delta_0}$, $(x,y) = (x^t, y^t)$ and $\Pi = \Pi^t$ for simplicity. Note that $A \cap S$ is a compact non-empty subset of S. Hence there is $p \in S \cap A$ such that

$$dist(\Pi(S^t \cap A), 0) = dist(\Pi(p), 0),$$

where $dist$ denotes the distance in $[-\Delta_0, \Delta_0]$.

Now, $p \in A$ and so $W^u_Z(p)$ is a well defined two-dimensional submanifold. The dominance condition of sectional-hyperbolicity implies that

$$T_z(W^u_Z(p)) = E^c_z, \quad \forall z \in W^u_Z(p).$$

Hence

$$T_z(W^u_Z(p)) \cap T_z S = E^c_z \cap T_z S = F_z$$

for every $z \in W^u_Z(p) \cap S$.

As $W^u_Z(p) \cap S$ is transversal, we have that $W^u_Z(p) \cap S$ contains a curve C whose interior contains p as in Figure 6.9. The last equality implies that C is tangent to F.

As F is transverse to Π we have that C is *transverse* to Π (i.e. C is transverse to the curves $\Pi^{-1}(c)$, for every $c \in [-1,1]$). We conclude that $\Pi(C)$ contains an open interval $I \subset [-\Delta_0, \Delta_0]$ with $\Pi(p) \in Int(I)$. So, there is $z_0 \in C$ such that

$$dist(\Pi(z_0), 0) < dist(\Pi(p), 0).$$

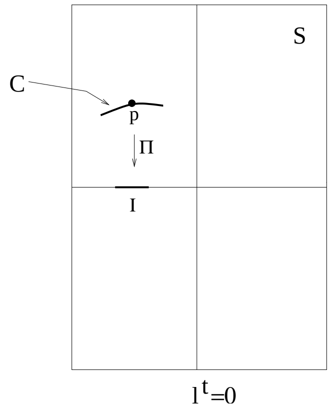


Fig. 6.9

Note that $C \subset S \cap A$ since A is an attractor of Z. Moreover, $p \in A$ and $C \subset W^u_Z(p)$. As $A \cap \partial^h S = \emptyset$ we conclude that

$$dist(\Pi(S \cap A), 0) = 0.$$

As A is closed, this last equality implies

$$A \cap l^t \neq \emptyset.$$

Since $l^t \subset W^s_Z(\sigma)$ and A is closed invariant for Z we conclude that $\sigma \in A$. However, this is impossible since A is a hyperbolic attractor. This contradiction implies the result. □

Let us present some corollaries of Theorem G.

Corollary 6.24. *Every C^r vector field that is C^r close to a C^r transitive sectional-Anosov flow with $k \geq 0$ singularities on a compact 3-manifold has at most $k+1$ attractors.*

Proof. Let X be a C^r transitive sectional-Anosov flow with $k \geq 0$ singularities on a compact 3-manifold M. If $k = 0$ then X is robustly transitive and then every vector field Y that is C^r close to X has $M(Y)$ as its unique attractor. Thus the result holds in this case. If $k \geq 1$, then Theorem G implies that every attractor of every vector field Y that is C^r close to X has a singularity. Taking Y nearby X we can assume that Y has also k singularities. As the family of attractors is pairwise disjoint, we conclude that Y has at most k attractors. This proves the result. □

We say that a singularity σ of a vector field X on M is *isolated from the nonwandering set* if $\Omega(X) \setminus \{\sigma\}$ is closed in M.

Corollary 6.25. *Every C^r vector field that is C^r close to a C^r transitive sectional-Anosov flow with singularities on a compact 3-manifold has a singularity non-isolated from the nonwandering set.*

Proof. Suppose by contradiction that there is a C^r transitive sectional-Anosov flow X on a compact 3-manifold M which is the limit of a sequence X^n all of whose singularities are isolated from the nonwandering set.

Define $C^n = \Omega(X^n) \setminus Sing(X^n)$ Then, C^n is a compact invariant set of X^n which is hyperbolic for it is non-singular. Now, there is a disjoint union

$$\Omega(X^n) = C^n \cup Sing(X^n)$$

and $Sing(X^n)$ is hyperbolic too. It follows that $\Omega(X^n)$ is a hyperbolic set of X^n. In particular, the positive limit set $L^+ = L^+(X^n)$ (which is the closure of the union of the ω-limit sets) is hyperbolic since it is contained in $\Omega(X^n)$. It follows from the Shadowing Lemma for flows that L^+ is the closure of its closed orbits. Hence there is a spectral decomposition $L^+ = L_1 \cup \cdots \cup L_k$, where each L_i is a hyperbolic basic set of X^n (see [123]). Note that

$$M = W^s_{X^n}(L^+) = \bigcup_{i=1}^{k} W^s_{X^n}(L_i)$$

so there is i_0 such that $W^s_{X^n}(L_{i_0})$ has non-empty interior. This implies that $A = L_{i_0}$ is a hyperbolic attractor of X^n. But there is not such attractors by Theorem G. This is a contradiction which proves the result. □

We say that a hyperbolic singularity σ of $X \in \mathscr{X}^r(M)$ is *C^r-stably non-isolated from the nonwandering set* if there is a neighborhood $\mathscr{U}$ of X in $\mathscr{X}^r(M)$ such that the continuation $\sigma(Y)$ of σ is non-isolated from the nonwandering set $\forall Y \in \mathscr{U}$. The following is a direct consequence of the above corollary.

Corollary 6.26. *If X is a C^r transitive sectional-Anosov flow with a unique singularity σ on a compact 3-manifold, then σ is C^r stably non isolated in the nonwandering set.*

See [107] for a sort of converse of this result when $r = 1$.

Now recall Definition 4.6 of Property (P). We prove the openess of this property among transitive sectional-Anosov flows.

Corollary 6.27. *Every vector field C^r close to a C^r transitive sectional-Anosov flow with with singularities on a compact 3-manifold has the Property (P).*

Proof. By Corollary 6.14 it suffices to show that $Cl(W^u_Y(O)) \cap Sin(Y) \neq \emptyset$ for every vector field Y close to X and every periodic orbit O of Y.

Suppose by contradiction that there is Y close to X with a periodic orbit O such that $Cl(W^u_Y(O)) \cap Sin(Y) = \emptyset$. It follows that $Cl(W^u_Y(O))$ is a hyperbolic set by the hyperbolic lemma. Since $W^u_Y(O)$ is a two-dimensional submanifold we can easily prove that $Cl(W^u_Y(O))$ is an attracting set of Y. This attracting set necessarily contains a hyperbolic attractor. However, no such attractors exist by Theorem G. □

6.3.2 Omega-limit sets for perturbed flow

The second main result of this section is the following.

Theorem H. *If X is a sectional-Anosov flow with the Property (P) on a compact 3-manifold M, then every point in M is approximated by points for which the omega-limit set with respect to X* has *a singularity.*

Proof. Suppose by contradiction there is $x \in M$ which is not approximated by points for which the omega-limit set *has* a singularity. Since X has the Property (P) we can apply Theorem 4.26 in order to find a singular partition $\mathscr{S} = \{S_1, \cdots, S_r\}$ of $\omega(x)$ close to it. Moreover, we can fix an open interval I around (and close to) x tangent to E^c and orthogonal to E^X such that *I does not intersect the stable manifold of any singularity*.

Clearly $\omega(x)$ is not a singularity, by the absurd supposition, so we can apply Theorem 2.15 to $q = x$ in order to obtain $S \in \mathscr{R}$, a sequence $x_n \in S$ of points in the positive orbit of x and a sequence of intervals $J_n \subset S$ in the positive orbit of I with $x_n \in J_n$ such that if J_n^+ and J_n^- are the connected components of $J_n \setminus \{x_n\}$ then both sequences $\{\text{Length}(J_n^+) : n = 1,2,3,\cdots\}$ and $\{\text{Length}(J_n^-) : n = 1,2,3,\cdots\}$ are bounded away from 0.

Take a limit point $w \in S$ of x_n. Then $w \in \omega(x) \cap Int(S)$ since $\mathscr{S}$ is a singular partition. Because I is tangent to E^c the interval sequence J_n converges to an interval $J \subset W^u(w)$ in the C^1 topology (notice that $W^u(w)$ exists because $w \in \omega(x)$ and $\omega(x)$ is hyperbolic by the hyperbolic lemma). J is not trivial since $\{\text{Length}(J_j^+) : j = 1,2,3,\cdots\}$ and $\{\text{Length}(J_j^-) : j = 1,2,3,\cdots\}$ are bounded away from 0. It follows from these lower bounds and $x_j \to w$ that J_n intersects $W^s(w)$ for some n large. Now, w is accumulated by periodic orbits which by Property (P) and Lemma 4.6 are accumulated by points for which the omega-limit set is a singularity. It then follows from the continuous dependence in compact parts of the stable manifolds that there is an intersection point between J_n and the stable manifold of a singularity (see Figure 6.10).

Since the stable manifolds of the singularities are flow-invariant we see that I itself intersects the stable manifold of a singularity which contradicts the choice of I. This contradiction proves the result. □

Let us present two corollaries of this theorem.

Corollary 6.28. *For every C^r vector field that is C^r close to a C^r transitive sectional-Anosov flow with singularities on a compact 3-manifold M every point in M is approximated by points for which the omega-limit set* has *a singularity.*

Proof. The result follows from Theorem H since every vector field as in the statement satisfies Property (P) by Corollary 6.27. □

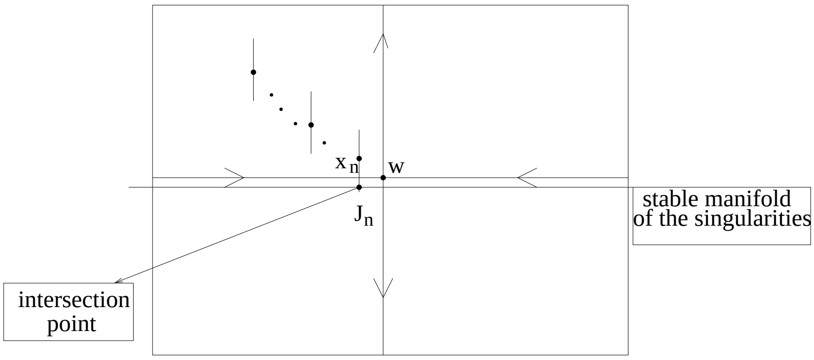


Fig. 6.10

The second corollary is the following one which is a reformulation of the main result in [35].

Corollary 6.29. *For every C^r vector field X that is C^r close to a C^r transitive sectional-Anosov flow with singularities on a compact 3-manifold M there is a residual subset R such that $\omega(x) \cap Sing(X) \neq \emptyset$ for all $x \in R$.*

Proof. First we recall a basic definition in topological dynamics [24, Chapter V]. Given a non-empty compact invariant set C of a vector field Z we define

$$A(Z,C) = \{z \in M : \omega_Z(z) \subset C\} \text{ and } A_w(Z,C) = \{z : \omega_Z(z) \cap C \neq \emptyset\}$$

It follows from the definition that

$$A(Z,C) \subset A_w(Z,C). \tag{6.22}$$

We claim that $A_w(Z,C)$ is dense in M if and only if $A_w(Z,C)$ is residual in M. Indeed, we only have to prove the direct implication, so, assume that $A_w(Z,C)$ is dense in M. Defining

$$W_n = \{x \in M : Z_t(x) \in B_{1/n}(C) \text{ for some } t > n\} \;\; \forall n \in \mathbb{N}$$

one has

$$A_w(Z,C) = \bigcap_n W_n.$$

In particular $A_w(Z,C) \subset W_n$, and so, since $A_w(Z,C)$ is dense, W_n is dense in M for all n. On the one hand, W_n is open by the Tubular Flow Box Theorem because $B_{1/n}(T)$ is. This proves that W_n is open and dense therefore the claim follows.

Now, let X be a C^r vector field that is C^r close to a C^r transitive sectional-Anosov flow with singularities on a compact 3-manifold M. If we define $R = A_w(X, Sing(X))$ it follows from (6.22) and Corollary 6.28 with $Z = X$ and $C = Sing(X)$ that R is dense in M, and so, it is residual in M by the claim. It follows from the definition that R satisfies the conclusion of the corollary. □

6.3.3 Small perturbations of venice masks

We start this subsection with the following result about small perturbations of venice masks.

Theorem 6.30. *Every C^r venice mask with a unique singularity on a compact 3-manifold M can be C^r approximated by C^r vector fields Y for which $M(Y) \not\subset \Omega(Y)$.*

Proof. Let X be a venice mask with a unique singularity σ on M. Fix $q \in W^u(\sigma)$ and observe that $\omega(q)$ is a periodic orbit O with positive expanding eigenvalues by Theorem 6.19. Let $W^{u,+}$ and $W^{u,-}$ be the two connected components of $W^u(O) \setminus O$.

We first claim that

$$Cl(W^{u,+}) \cap W^{s,-}(\sigma) = \emptyset. \tag{6.23}$$

Indeed, suppose by contradiction that $Cl(W^{u,+}) \cap W^{s,-}(\sigma) \neq \emptyset$. Choose $r \in Per(X)$ for which there is $r' \in W^u(r) \cap W^{s,-}(\sigma)$. Notice that such an intersection is transverse. Since $Cl(W^{u,+}) \cap W^{s,-}(\sigma)$ we can select an interval $I \subset W^{u,+}$ close to some point in $W^{s,-}(\sigma)$. Fix $i \in I$. It follows from the uniform size of the stable manifolds that the stable manifold through i intersects $W^u(r)$ at some point i' as described in Figure 6.11.

This point i' is in turns approximated by some $k \in Per(X)$ since X has dense periodic orbits and $i' \in W^u(r) \subset M(X)$. Since I is transverse to the stable foliations and k is close to i' we get $W^s(k) \cap I \neq \emptyset$ and so $W^s(k) \cap W^{u,+} \neq \emptyset$. Since k is periodic and close to $i' \in W^u(r)$ we have that there is a point k' in the orbit of k that is close to r. But $W^s(k) \cap W^{u,+} \neq \emptyset$ so $W^{u,+}$ accumulates on k' for it does in k. Since k' is close to r we conclude that $r \in Cl(W^{u,+})$. Since r is arbitrary we conclude from (6.16) that $H^- \subset Cl(W^{u,+})$. On the other hand, $H^+ = Cl(W^{u,+})$ by Proposition 6.21 hence $H^- \subset H^+$ and so $M(X) = H^+$ by the second equation in (6.17). But H^+ is a homoclinic class by Corollary 6.23 and homoclinic classes are always transitive sets due to the Birkhoff-Smale Theorem. So, $M(X)$ (and hence X) are transitive which is a contradiction since X is a venice mask. This contradiction ends the proof of the claim.

Now we fix cross-sections Σ_q and $\Sigma'_q = X_1(\Sigma_q)$ through q and $X_1(q)$ respectively which define the tubular neighborhood $\mathscr{O} = X_{[0,1]}(\Sigma_q)$ as indicated in Figure 6.12.

Since $\omega(q) = O$ we have that the positive orbit of q enters into $W^s(O)$ through some interval in $W^s(O)$ which is denoted by l''_s in the figure. We push backward this interval to obtain intervals $l_s \subset \Sigma_q$ and $l'_s \subset \Sigma'_q$ in a way that $l'_s = X_1(l_s)$. Inside

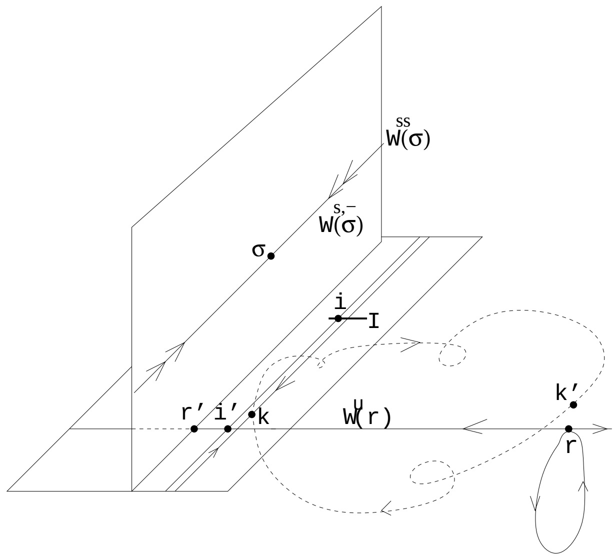


Fig. 6.11

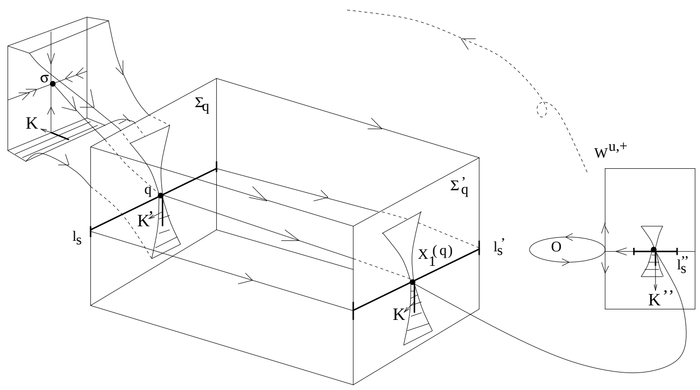


Fig. 6.12

Σ_q there is a cusped region through q corresponding to the positive orbits close to $W^{s,+}(\sigma)$ which enter inside Σ_q. Analogously for $-$.

By standard way [87] we can perform a C^r perturbation inside $\mathscr{O}$ as described in Figure 6.13 producing a vector field Y.

We claim that $M(Y) \not\subset \Omega(Y)$. Indeed, we first observe that since X is a venice mask there is $r \in Per(X)$ such that $W^u(r) \cap W^{s,-}(\sigma) \neq \emptyset$. Such a non-empty intersection gives rise an interval $K \subset W^u(r)$ intersecting $W^{s,-}(\sigma)$ transversally as described in Figure 6.12. The perturbation yielding Y produces the small interval $L_Y \subset K$ whose positive trajectory gives rise to the intervals $L'_Y, L''_Y, L'''_Y, L''''_Y$ in Fig-

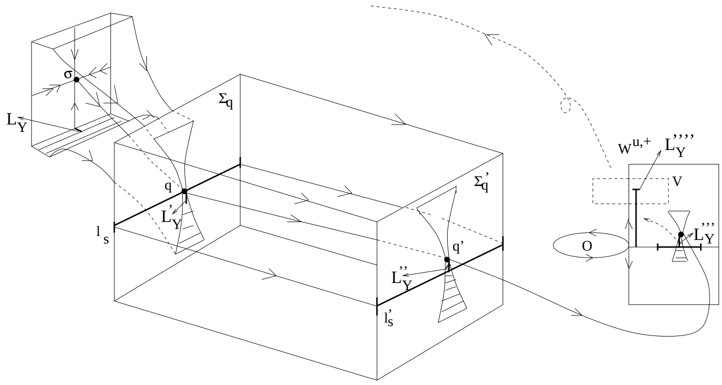


Fig. 6.13

ure 6.13. The basic property of these intervals is that every positive orbit through the final one L''''_Y passes close to a foliated rectangle V through $W^{u,-}$ which covers a fundamental domain. By (6.23) we have $Cl(W^{u,+}) \cap W^{s,-}(\sigma) = \emptyset$ so no positive orbit through L''''_Y passes close to $W^{s,-}(\sigma)$ for they pass through V first. From this we get $L_Y \cap \Omega(Y) = \emptyset$ and so $M(Y) \not\subset \Omega(Y)$ since $L_Y \subset M(Y)$. This proves the result. □

We finish this chapter with an application of Theorem 6.30. Recall that a C^r vector field X in M is C^r *robustly transitive* or C^r *robustly periodic* depending on whether every C^r vector field C^r close to it is transitive or has dense periodic orbits. It would follow from the main result in [10] that every C^r robustly transitive sectional-Anosov flow on a compact 3-manifold is C^r robustly periodic. The converse is true at least for flows with a unique singularity by the following corollary of Theorem 6.30.

Theorem I. *Every C^r robustly periodic sectional-Anosov flow with a unique singularity on a compact* 3*-manifold is C^r robustly transitive.*

Proof. It follows from the definitions of robustly periodic vector fields and venice masks that every C^r robustly periodic sectional-Anosov flow which is not C^r robustly transitive can be C^r approximated by C^r venice masks. In turns, by Theorem 6.30, every C^r venice mask with a unique singularity on a compact 3-manifold can be approximated by vector fields for which the maximal invariant set is not the nonwandering set. Then the result follows by contradiction since the closure of the periodic orbits is contained in the nonwandering set. □

6.4 Topological properties

In this section we shall be interested in the topological of codimension one sectional-Anosov flows on compact n-manifolds. The background in the nonsingular case is very extensive as, for instance, it is well known that all such flows satisfy the following properties:

1. Periodic orbits are not null homotopic;
2. The supporting manifold has infinite fundamental group, is covered by $\mathbb{R}^n$ and is irreducible (e.g. Proposition 5.3);
3. For $n = 3$ every transverse surface is an incompressible torus or Klein bottle (c.f. Theorem 5.9).

Another related result is Theorem 5.4. On the other hand, it is believed that for $n \geq 4$ the supporting manifolds of these flows fiber over the circle. This is part of the so-called *Verjovsky conjecture* which claims that all such flows are suspended (progress toward positive solution for this conjecture was done recently in [12]). The situation is completely different for closed 3-manifolds since there are Anosov flows like the geodesic flows on closed Riemannian surfaces, the Anomalous Anosov flow [46], the Bonatti-Langevin examples [30] with a transverse torus intersecting all orbits except one, the Handel-Thurston examples [61] or even the ones obtained by Dehn surgeries [55] whose ambient manifolds do not fiber over the circle (in fact all these manifolds are not homeomorphic one to another).

In the following three subsections we investigate the relationship between the above results and the codimension one sectional-Anosov flows on compact manifolds. Note that none of the properties listed above is true for codimension one sectional-Anosov flows in general. Indeed there are sectional-Anosov flows not only in the 3-ball but also on reducible manifolds with the 2-sphere as a boundary component (e.g. Theorem 3.28).

6.4.1 Topology of the ambient manifold

An immediate question about codimension one sectional-Anosov flows is that if they exist on every compact manifold with boundary. As we already seem these flows exists not only in the 3-ball or the solid but also in every three-dimensional handlebody.

The situation may change if we demand additional properties to the flow. Indeed, the following result implies that there are no sectional-Anosov flows for which every singularity is Lorenz-like in the 3-ball or the solid torus.

Theorem 6.31. *Every sectional-Anosov flow on the* 3*-ball or the solid torus has a non Lorenz-like singularity.*

Proof. First we show that in each case the flow has a singularity. In the case of the 3-ball we appeal to the Poincaré-Hopf Index Theorem [40, 90]. This argument does not work for the solid torus since it has zero Euler number. However we can use the following alternative argument.

Hereafter we denote by $ST = S^1 \times D^2$ the solid torus. Suppose by contradiction that there is a nonsingular sectional-Anosov flow Z in ST. Since Z has no singularities we have that its stable foliation W^s is a non-singular codimension one foliation of ST transverse to $\partial(ST)$.

Next we apply the argument used in the proof of Theorem 5.9 based on the following definition: A *half-Reeb component* of W^s is a saturated subset, bounded by an annulus leaf A and an annulus $K \subset \partial(ST)$ with $\partial K = \partial A$, such that the double $2ST$ is a Reeb component [50] of the double foliation $2W^s$ (see Figure 6.14).

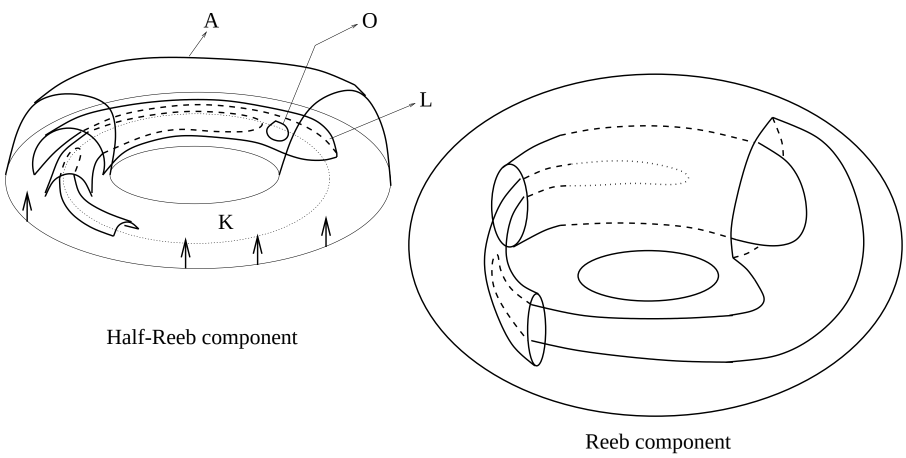


Fig. 6.14

We claim that W^s has neither Reeb components nor half-Reeb components. Indeed, since W^s is induced by stable manifolds we have that W^s has no compact leaves in $Int(ST)$. Consequently W^s has no Reeb components. Now suppose by contradiction that it has a half-Reeb component H. Let A, K be the boundary annuli of H with $K \subset \partial(ST)$. Pick $x \in Int(H)$. Note that the positive trajectory of x does not intersect A. As Z points inward to ST as indicated in Figure 6.14 we have that $\omega(x) \subset Int(H)$.

Now, $\omega(x)$ is contained in $ST(Z)$ which is hyperbolic. By using the orbit of x we can construct a periodic pseudo-orbit close to $\omega(x)$. By the Shadowing Lemma for flows [62] we have that such a pseudo-orbit is shadowed by a periodic orbit $O \subset Int(H)$. We have that O is contained in a leaf L of W^s and $L \neq A$. The last property implies that L is a half-plane, and so, it is simply connected as well. Consequently, O bounds a disk in L. Applying the Poincaré-Bendixon Theorem [87] to this disk we could find a singularity in $ST(Z)$ which is absurd. This contradiction proves the claim.

Now take the double foliation $2W^s$ defined on the double manifold $M = 2ST$. On the one hand, ST is a solid torus so M is diffeomorphic to $S^2 \times S^1$. Consequently, $\pi_2(M) \neq 0$. On the other hand, the claim says that W^s has neither Reeb nor half-Reeb components. Therefore, $2W^s$ has no Reeb components. Then, standard results in foliation theory (e.g. Theorem 1.10-(iii) p. 92 in [50]) imply that $2W^s$ is the product foliation $S^2 \times *$ of $M = S^2 \times S^1$. Then, W^s itself is the product foliation $D \times *$ by meridian disks on ST, and so, the leaves of W^s are invariant disks. But applying Poincare-Bendixon's to one of such disks as before we could find a singularity of Z in $Int(ST)$ which is absurd.

With this contradiction we prove that every sectional-Anosov flow on the 3-ball or the solid torus has a singularity. That one of these singularities is not Lorenz-like follows in both cases for, otherwise, the stable foliation of the flow would intersect the corresponding boundaries in a singular foliation with only saddle-type singularities, a fact which is againts the Poincaré-Hopf Theorem. □

Combining this theorem with Corollary 4.18 we obtain the following.

Corollary 6.32. *Neither the* 3*-ball nor the solid torus can support transitive sectional-Anosov flows or sectional-Anosov flows with dense periodic orbits.*

Notice that by Theorem 3.26 the above corollary is false for another handlebodies different from the 3-ball or the solid torus.

On the other hand Theorem 3.22 implies that there are transitive sectional-Anosov flows (and so with all its singularities of Lorenz-like type) in the solid bitorus. The fact that the the fundamental group of the solid bitorus has exponential growth (for it is the free product $\mathbb{Z} * \mathbb{Z}$) motivates the following conjecture (see also Problem 7.2).

Conjecture 6.1. The fundamental group of a compact manifold supporting codimension one sectional-Anosov flows for which every singularity (if any) is Lorenz-like has exponential growth.

This conjecture implies not only Theorem 6.31 but also the main result in [128]. Notice that if it were true, then every sectional-Anosov flow on a compact manifold for which the fundamental group has subexponential growth has a singularity.

6.4.2 Transverse surfaces

Some authors have studied transverse torus for codimension one Anosov flows on closed manifolds (e.g. Theorem 5.9 or [16]). But this case was benefited not only by the fact that the ambient manifold of the supporting manifolds are irreducible (Proposition 5.3) but also by the fact that the periodic orbits are not null homotopic. Since such properties are not true for sectional-Anosov flows in general we shall assume them in the next results.

Theorem J. *Every closed surface transverse to a sectional-Anosov flow* without null homotopic periodic orbits *on a compact irreducible* 3*-manifold has non-zero genus.*

Proof. Let X be a sectional-Anosov flow on a manifold M as in the statement. Consider a surface S in M transverse to X. We can assume that M is orientable for, otherwise, we pass to a double covering. It follows that S is orientable. Then, to prove that S non-zero genus, it suffices to prove that S is not a 2-sphere.

Suppose by contradiction that S is a 2-sphere. As M is irreducible we have that S bounds a 3-ball B in M. We have two possibilities for the vector field X at $S = \partial B$, namely, it points either inward or outward to B.

If X points inward to B at S have that X/B is a sectional-Anosov flow, so, X has a periodic orbit in B by Theorem C. But B is a ball so such a periodic orbit is null homotopic, contradiction.

Now suppose that X points outward to B at S and consider the foliation the singular stable manifold W^s of X. If the singularities of W^s do not intersect S, then W^s would induce a non-singular foliation in S since S is transverse to X. But a 2-sphere cannot support non-singular foliations, contradiction. So, there is a singular leaf $W^{ss}(\sigma)$ intersecting S. Obviously $\sigma \notin S$ because X is transverse to S. As X points outward to B in $S = \partial B$, we would obtain some point different from σ in $W^{ss}(\sigma)$ whose entire backward orbit does not exit M. It follows that such a point belongs to $M(X) \cap W^{ss}(\sigma)$ which contradicts Corollary 2.7. This contradiction proves the result. □

A counterexample for this theorem in the reducible case is the one in Theorem 3.28.

Recall that a surface S in a 3-manifold V is *incompressible* if it has two sides and the homomorphism $\pi_1(S) \to \pi_1(V)$ induced by the inclusion is injective. A 3-manifold is *∂-irreducible* if it has irreducible boundary [66], [72]. Recall that by a *regular point* of a vector field we mean a point where the vector field does not vanish.

Corollary 6.33. *For every Lorenz-like singularity σ of a sectional-Anosov flow without null homotopic periodic orbits on a compact irreducible ∂-irreducible* 3*-manifold M there is a regular point in $W^s(\sigma)$ whose backward orbit does not exit M.*

Proof. Let X be a sectional-Anosov flow on M as in the statement. We have to show that $M(X) \cap W^s(\sigma)$ contains regular points.

Suppose by contradiction that such a regular point does not exist. There is a fundamental domain in $W^s(\sigma)$ consisting of a circle C since $dim(W^s(\sigma)) = 2$ (see [132] for the corresponding definitions). We have that $C \cap M(X) = \emptyset$ since C consists of regular points. Hence the negative orbits through C exit M. By taking these negative orbits we obtain a Poincaré map $\Pi : C \to \partial M$ which is smooth. As C is a circle we conclude that $C' = \Pi(C)$ is a circle too contained in a single connected component $\overline{\partial}$ of ∂M. Note that C' is null homotopic in M for it can be homotoped to σ by taking the positive orbits through C'. As M is ∂-irreducible we conclude that C' is null homotopic in $\overline{\partial}$. So, it is the boundary of a disk D' in $\overline{\partial}$. As the positive orbits through $C' = \partial D'$ converges to σ we can push forward D' with the flow of X in order to obtain a transverse 2-sphere in M. But this contradicts Theorem J so the result follows. □

This corollary is false for non ∂-irreducible 3-manifolds. Indeed, the suspension of the Plykin attractor [124] in the three-punctured disk ([132]) is a hyperbolic attractor of saddle type having the manifold in Figure 6.15-(a) as basin of attraction. Such a manifold (throughout denote by N) is obtained by removing from the solid

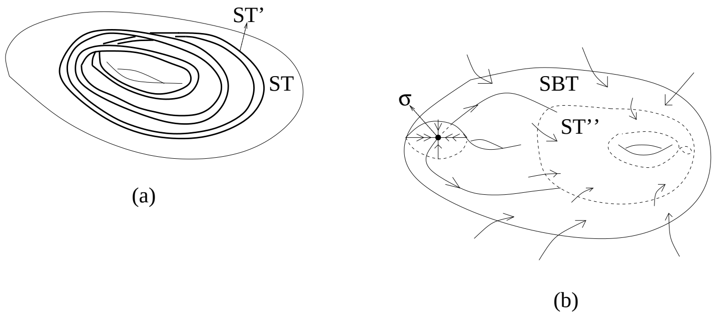


Fig. 6.15

torus ST in the figure the solid torus ST' which turns three times around the hole of ST. Now, consider the solid bitorus SBT described in Figure 6.15-(b) equipped with a vector field inwardly transverse to the boundary. Note that this vector field has a unique singularity σ of index 1 and an internal attracting solid torus ST'' turning once around the righ-hand handle of SBT. We also observe that the backward orbit of every regular point in $W^s(\sigma)$ exit SBT. Removing ST'' from SBT and gluing N to the resulting manifold along the torus boundary of ST in a suitable way we obtain the desired counterexample.

Our next result generalizes to the sectional-Anosov case a well known result about the incompressibility of torus transverse to Anosov flows on 3-manifolds [44].

Theorem K. *Let X be a sectional-Anosov flow on a compact irreducible 3-manifold all of whose singularities are Lorenz-like. If X has no null homotopic periodic orbits, then every torus transverse to X is incompressible.*

Proof. Suppose by contradiction that X exhibits a transverse torus which is not incompressible. As M is irreducible we have that T either bounds a solid torus ST or is contained in a 3-ball B in M (e.g. [63]). Let us analyse each case separately.

Suppose that T is the boundary of a solid torus ST and consider the foliation in T induced by the stable foliation W^s of X. Since every singularity is Lorenz-like (and so index 1) it follows from Poincaré-Hopf index that such an induced foliation is non-singular. From this we conclude that X has no singularities in ST. So, either X/ST or $-X/ST$ is a nonsingular sectional-Anosov flow in the solid torus contradicting Theorem 6.31.

Now consider the case when T is contained in a 3-ball B. Hence T separates M in two connected components one of which (V say) is contained in B (see for instance [100]). As before we can see that X has no singularities in V. So, applying the hyperbolic and shadowing lemmas to either X (if X points inward to V at T) or to $-X$ (otherwise) we can find a periodic orbit in V. However V is contained in B which is a ball hence such a periodic orbit would be null homotopic, contradiction. This contradiction proves the result. □

Remark 6.34. *This theorem is false for flows with non Lorenz-like singularities. In fact, by capping one of the holes of the basin of a geometric Lorenz attractor (which is a solid bi-torus) with a 3-ball, we obtain an example of a sectional-Anosov flow without null-homotopic periodic orbits (but with a non Lorenz-like singularity) on the solid torus.*

Recall that a 3-manifold is *atoroidal* if every incompressible torus on it is isotopic to a boundary component [66], [72].

Corollary 6.35. *A sectional-Anosov flow* with *singularities, all Lorenz-like, but without null homotopic periodic orbits in a compact atoroidal 3-manifold exhibits neither hyperbolic attractors nor hyperbolic repellers.*

Proof. Let X be a sectional-Anosov flow on a manifold M as in the statement. By hypothesis we have that X has singularities, so, $\partial M \neq \emptyset$.

Now, assume by contradiction that X has hyperbolic attractor A. We have that

$$A = \bigcap_{t \geq 0} X_t(V)$$

for some compact 3-manifold with boundary V. Applying X_t with sufficiently large t we see that V can be chosen close to A hence $V \subset Int(M)$, the interior of M. Since the ∂V's components support non-singular foliations we see that ∂V is formed by transverse tori T, all of which are incompressible by Theorem K. As M is atoroidal we see that each T is boundary parallel. So, for each T there is a torus boundary component T' of ∂M such that T and T' form the boundary of a submanifold W of M diffeomorphic to $T^2 \times I$. By attaching to V all these manifolds W along the corresponding tori T we obtain a compact 3-manifold U inside M such that $\partial U \subset \partial M$. As M is connected we conclude that $U = M$. As every boundary component of U is a torus we have the same for M. But X has singularities, all of index 1 for they are Lorenz-like. So, there is a torus supporting a one-dimensional foliation with singularities, all of saddle type. This contradicts the Poincaré-Hopf Index Theorem. Analogously we prove that X exhibits no hyperbolic repellers. This proves the result. □

Recall that if X is a vector field on a manifold M, then a *singularity non-isolated in the nonwandering set* of X is a singularity σ for which $\Omega(X) \setminus \{\sigma\}$ is not closed in M.

Corollary 6.36. *Every sectional-Anosov flow with singularities, all Lorenz-like, but without null homotopic periodic orbits on a compact atoroidal* 3*-manifold has a singularity non-isolated in the nonwandering set.*

Proof. Let X the flow and M be the manifold. By contradiction assume that X has no singularities non-isolated in the nonwandering set. Denoting by $Sing(X)$ the set of singularities of X we obtain that $\Omega(X) \setminus Sing(X)$ is closed. But $\Omega(X) \setminus Sing(X)$ is also invariant hence it is hyperbolic by the hyperbolic lemma. It then follows that $\Omega(X) = (\Omega(X) \setminus Sing(X)) \cup Sing(X)$ is disjoint union of hyperbolic set hence $\Omega(X)$ is a hyperbolic set. As M is three-dimensional we obtain that $\Omega(X)$ is also the closure of the closed orbits (this follows from the flow version of a theorem in [121], see also [35]). Then, we can apply the Smale's Spectral Decomposition Theorem to find a hyperbolic attractor. However no such attractors exist by Corollary 6.35. This contradiction proves the result. □

Remark 6.37. *This corollary is also false for non atoroidal manifolds.*

6.4.3 Existence of Lorenz-like singularities

As already seem in Theorem 6.31 every sectional-Anosov flow in the 3-ball has a non Lorenz-like singularity. In this section we present a sufficient condition for the existence of Lorenz-like singularities too. We believe however that this condition is unnecessary.

The following definition is motivated by both the definition of cube with knotted hole ([25] p. 218) and the definition of trivially embedded stable separatrices ([56]

p. 980). We denote by ∂A the boundary of A. A curve is called *simple* if it has no self-intersections.

Definition 6.38. *Let c be a simple* non-closed *compact curve in a 3-ball B satisfying $\partial B \cap c = \partial c$. We say that c is* unknotted in B *if there is a simple compact curve $\beta \in \partial B$ with $\partial \beta = \partial c$ such that the simple* closed *curve $\beta \cup c$ is unknotted in B (see Figure 6.16).*

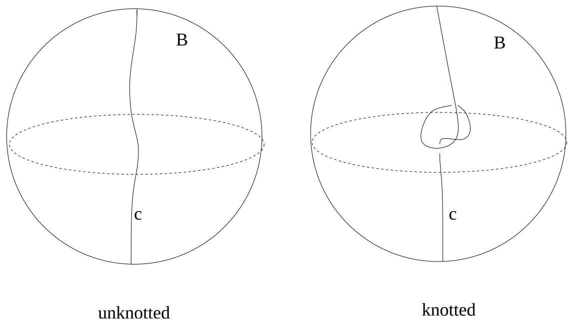


Fig. 6.16

We shall use this definition in the following context. If σ is a singularity of a sectional-Anosov flow X in B, then we have from Corollary 2.7 that $B(X) \cap W^{ss}_X(\sigma) = \{\sigma\}$, so, $W^{ss}(\sigma)$ is a simple curve satisfying $\partial B \cap W^{ss}(\sigma) = \partial W^{ss}(\sigma)$. With this in mind we can state the following definition.

Definition 6.39. *A sectional-Anosov flow X in B has* unknotted singular manifolds *if the curve $W^{ss}(\sigma)$ is unknotted in B for all $\sigma \in Sing(X)$.*

This definition is motivated by the following example.

Example 6.40. *There is $X \in \mathscr{X}^1(B)$ having a hyperbolic singularity $\sigma \in B$ with one-dimensional stable manifold $W^s(\sigma)$ such that $B(X) \cap W^s(\sigma) = \{\sigma\}$ but $W^s(\sigma)$ is* not *unknotted in B.*

It can be constructed in the following way: Take the vector field in Figure 6.17-(a) and the small tubular neighborhood described in Figure 6.17-(b). Remove this neighborhood from the ball and inserts the tubular flow depicted in Figure 1 p. 26 of [41] (or in Figure 6.17-(c)) instead. The resulting vector field in Figure 6.17-(d) is the desired one.

With the above definitions in mind we can state the following result (see [94]).

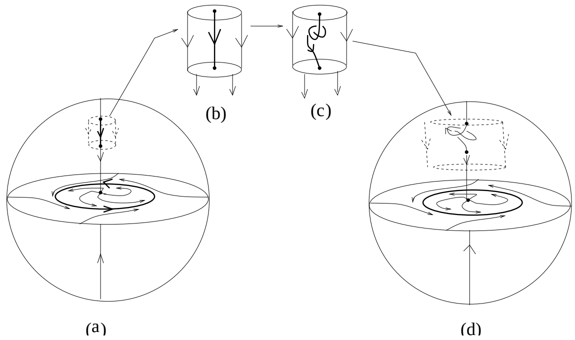


Fig. 6.17

Theorem L. *Every sectional-Anosov flow with unknotted singular manifolds in the* 3*-ball has both Lorenz-like and non Lorenz-like singularities.*

Proof. The existence of non Lorenz-like singularities follows at once from Theorem 6.31.

To prove that there are Lorenz-like singularities too we shall assume by contradiction that there is a sectional-Anosov flow without Lorenz-like singularities X in B. Since B has Euler number 1 and X points inward in ∂B we have from Poincaré-Hopf that X has only one singularity σ in B. It has index -1 because of Lemma 2.8. On the other hand, $B(X) \cap W^{ss}(\sigma) = \{\sigma\}$ by Corollary 2.7 so $B(X) \cap W^s(\sigma) = \{\sigma\}$. We also have that $W^s(\sigma) = W^{ss}(\sigma)$ is unknotted since X has unknotted singular manifolds by hypothesis.

We claim that there is a solid torus $ST \subset Int(B)$ such that X is inwardly transverse to $\partial(ST)$ and X has no singularities in ST. Indeed, since $(\bigcap_{t\geq 0} X_t(B)) \cap W^s(\sigma) = \{\sigma\}$ we have that the separatrices of $W^s(\sigma) \setminus \{\sigma\}$ exit B in the past as in Figure 6.18.

Then, by using the flow of X we can construct a torus T transverse to X in the interior of B by removing an small tubular neighborhood in B of the curve $c = W^s_X(\sigma) \cap B$. Note that T is the boundary of a compact manifold ST contained in the interior of B. Moreover, X points inward to ST in $T = \partial(ST)$. Since $W^s(\sigma)$ is unknotted we have that ST is a solid torus. The claim follows.

Since X is inwardly transverse to $\partial(ST)$ we have that X/ST is a nonsingular sectional-Anosov flow in the solid torus, a contradiction by Theorem 6.31. This contradiction proves the result. □

There are examples of a sectional-Anosov flow where the hypotheses of the above theorem are fulfilled.

A natural question is if the conclusion of Theorem L holds without the unknotted assumption. Note that Example 6.40 does not give negative answer for such

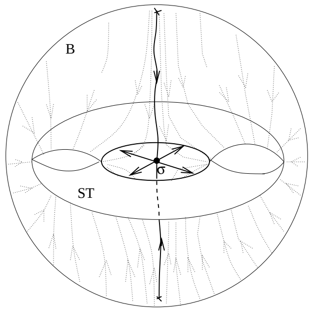


Fig. 6.18

a question because $B(X)$ in that example may not be partially hyperbolic. Indeed, $B(X)$ intersects the tubular neighborhood in Figure 6.17-(c) due to the Wazewski Principle (see p. 26 in [41]).

Further topological properties of sectional-Anosov flows but within the singular-hyperbolic attractor's terminology were obtained by E. Apaza in his thesis [4].

Chapter 7
Problems

In this chapter we shall present some problems for sectional-Anosov flows motivated by some of the results about Anosov flows mentioned in Part I.

Let M a compact manifold with possibly nonempty boundary ∂M and $X \in \mathscr{X}^1(M)$. A subset $\Lambda \subset M(X)$ is called *Lyapunov stable* if for every neighborhood U of Λ there is a neighborhood $W \subset U$ such that $X_t(p) \in U$ for every $t \geq 0$ and $p \in W$. It is clear that every attracting set is Lyapunov stable (but not conversely).

The first problem is motivated by known fact that every hyperbolic Lyapunov stable set is an attracting set too. It is then natural to ask if this is true as well replacing the term hyperbolic by sectional-hyperbolic. More precisely, we have the following question:

7.1. Is a sectional-hyperbolic Lyapunov stable set an attracting set?

The answer is unknown even for transitive sectional-hyperbolic Lyapunov stable sets. Results toward positive solution in dimension three have been reported by S. Hayashi [64]. This problem was formulated first in [35] but in the context of three-dimensional singular-hyperbolic flows. Under this dimensional restriction the problem was solved positively in [108] as soon as the unstable branches at the singularities are all dense in the set.

Our second problem is motivated by Theorem 5.4. Indeed, in light of this result we can ask if compact manifolds supporting sectional-Anosov flows of codimension one also have fundamental group of exponential growth (by *codimension one sectional-Anosov flow* we mean a sectional-Anosov flow for which the central subbundle is two-dimensional). However, this is false since the the unit ball B^n supports such flows for all $n \geq 3$. Nevertheless, since there are no *transitive* codimension one sectional-Anosov flows in these balls we still can ask the following,

7.2. Is the fundamental group of a compact manifold carrying *transitive* sectional-Anosov flows of exponential growth?

The next problem is motivated by Proposition 5.3 which says that every closed manifold carrying codimension one Anosov flows is irreducible. It is then natural to believe that a similar result holds for sectional-Anosov flows, but this is false by some examples reported in [97]. However, the verification of the following problem is still possible.

7.3. Is a compact manifold carrying *transitive* sectional-Anosov flows of codimension one irreducible?

We know by Theorem 3.26 that there are sectional-Anosov flows on every three-dimensional orientable handlebody. This result was the motivation for the following problem posed in [138]:

7.4. Are there sectional-Anosov flows on *every* compact manifold of dimension greater than 2?

To support positive answer [138] proved that there are sectional-Anosov flows in *punctured* handlebodies too. On the other hand, we have seen in Subsection 3.2.5 that Dehn surgery can be used to construct Anosov flows from older ones. As already said such a method was used successfully in [55] to construct the first examples of Anosov flows on closed hyperbolic three-manifolds (see also [14] or [45]). What we expect is that the same method can be applied to sectional-Anosov flows on compact three-manifolds as well. More precisely, we pose the following problem:

7.5. Can we use Dehn surgery to obtain new examples of (transitive) sectional-Anosov flows on compact 3-manifolds?

In particular, we would like to know what kind of sectional-Anosov flows can arise from Dehn surgery on geometric Lorenz attractors.

For the next question we recall that a vector field X is *sensitive to initial conditions* if there is $\delta > 0$ such that for every $x \in M$ and every neighborhood U of x there are $y \in U$ and $t \geq 0$ satisfying $d(X_t(x), X_t(y)) \geq \delta$ (here d stands for the metric induced by the Riemannian structure of M).

7.6. Is every sectional-Anosov flow on a compact manifold sensitive to initial conditions?

A positive solution in dimension three was obtained in [7] but for vector fields close to nonwandering sectional-Anosov flows.

A concept related to sensitiveness to initial conditions is a kind of expansiveness introduced by Komuro in [76]. More precisely, we say that X is *K-expansive* if for every $\varepsilon > 0$ there is $\delta > 0$ such that, for any surjective increasing continuous functions $h : I\!R \to I\!R$, if $x, y \in M(X)$ satisfy $d(X_t(x), X_{h(t)}(y)) \leq \delta$ for all $t \in I\!R$, then $X_{h(t_0)}(y) \in X_{[t_0-\varepsilon, t_0+\varepsilon]}(x)$, for some $t_0 \in I\!R$. (Notice however that K-expansiveness does not imply sensitiveness to initial conditions and viceversa.)

7.7. Is every sectional-Anosov flow on a compact manifold K-expansive?

A positive solution in the three-dimensional transitive case have been claimed in [8].

The motivation for the next problem is Theorem C about existence of periodic points for sectional-Anosov flows on compact three-manifolds. Of course this implies immediately the existence of infinitely many periodic *points* but not the existence of infinitely many periodic *orbits*. Therefore, we can suggest the following problem.

7.8. Are there infinitely many periodic *orbits* for sectional-Anosov flows on compact manifolds?

Results toward positive solution in codimension one have been reported in [131].

For the next problem we recall that a *homoclinic point of* X is a homoclinic point $q \neq p$ associated to a hyperbolic periodic point p of X.

7.9. Are there homoclinic points for *every* sectional-Anosov flow on a compact manifold?

(Notice that positive solution for this problem implies immediately positive solution for Problem 7.8.)

Recall that a *homoclinic class* of X is the closure of the homoclinic points associated to a given periodic point. Homoclinic classes play fundamental role in the hyperbolic theory of dynamical systems as, for instance, by the *Spectral Theorem* (Theorem 2.5), the nonwandering set of every Axiom A vector field on a compact manifold is a finite disjoint union of homoclinic classes. The following problem is directly related to this fact.

7.10. Is the nonwandering set of every sectional-Anosov flow on a compact manifold a finite union of transitive sets?

This problem was solved in dimension three but for venice masks with only one singularity [106]. Notice that the union in this problem may be non-disjoint [22].

The resent results [135], which can be seen as a weak spectral theorem, implies the existence of finitely many ergodic components for every C^2 sectional-Anosov flow of codimension one on compact manifolds. This result implies that Problem 7.8 has positive solution for C^2 sectional-Anosov flows of codimension one (notice however that Problem 7.9 is still open even for such flows). For a similar result but for transitive sectional-Anosov flows on compact three-manifolds see [8].

A question dealing with homoclinic classes is related to a result by Newhouse asserting that the sole area-preserving diffeomorphisms on closed surface exhibiting a hyperbolic nontrivial homoclinic class are the transitive Anosov ones (c.f. Proposition 2.3 p. 135 in [118]). C. Carballo expect a similar result for volume preserving flows on closed three-manifolds too. In the contrary direction we expect negative answer for the following problem:

7.11. Are there volume-preserving vector fields on closed 3-manifolds exhibiting sectional-hyperbolic homoclinic classes *with singularities*?

On the other hand, a direct consequence of Theorem 2.5 is that every Axiom A flow (including Anosov) on compact manifolds have finitely many homoclinic classes. Then, we have the following natural question

7.12. Are there finitely many homoclinic classes for sectional-Anosov flows on a compact manifold?

For the next problem we recall the well-known result that the maximal invariant set of a transitive Anosov flow on a compact manifold consists of a single homoclinic class. It is then natural to believe that this is true also for sectional-Anosov flows:

7.13. Is the maximal invariant set of a *transitive* sectional-Anosov flow on a compact manifold a homoclinic class?

This problem was referred to as the *Homoclinic Class Conjecture* in Subsection 3.4.6. It was solved positively for the geometric Lorenz attractor in [18]. The versions of this problem obtained by replacing the term transitive by nonwandering (or even periodic) are all false [22].

The next problem is motivated by Theorem I.

7.14. Is a robustly periodic sectional-Anosov flow on a compact manifold robustly transitive?

This together with Theorem 3.23 suggests the following problem.

7.15. Find sufficent conditions for a sectional-Anosov flow on a compact manifold to be transitive or robustly transitive.

For the next problem we introduce an additional terminology:

We say that X satisfies the *singular general density theorem* if the set of points which either are periodic or whose omega-limit set is a singularity is dense in the nonwandering set. We take this name from the well known Pugh's general density theorem [130] asserting that the set of points which are either periodic or singular is dense in the nonwandering set of a generic C^1 vector field (notice that the singular general density theorem reduces to the denseness of periodic points in the nonwandering set in abscence of singularities).

On the other hand, we also say that X satisfies the *singular connecting lemma* if the following is true: Let p,q be a pair of points with the alpha-limit set of p being non-singular. If for all $\varepsilon > 0$ there is a trajectory from a point ε-close to p to a point ε-close to q, then there is a point whose alpha-limit set is that of p and whose omega-limit set is either that of q or a singularity. Notice that, in abscence of singularities, the singular connecting lemma reduces to the property that if two points p,q satisfies that for every $\varepsilon > 0$ there is a trajectory from a point ε-close to p to a point ε-close to q, then there is a point whose omega (resp. alpha) limit set is that of p (resp. that of q).

7.16. Does every sectional-Anosov flow on a compact manifold satisfy the singular general density theorem and the singular connecting lemma?

Theorems D and E give positive answer in dimension three.

The following problems are motivated by results in [127]. Given $p \in M$ of X we denote by $W^{uu}(p)$, the strong unstable manifold throught p, i.e., the set of points whose negative orbit is asymptotic to that of p. More precisely,

$$W^{uu}(p) = \left\{ q \in M : \lim_{t \to -\infty} d(X_t(q), X_t(p)) = 0 \right\}.$$

Evidently $W^{uu}(p) \subset M(X)$ whenever $p \in M(X)$ and by Theorem 4.11 every transitive Anosov flows on compact manifolds satisfies either $W^{uu}(p)$ is dense in $M(X)$ for all $p \in M(X)$ or the flow is a constant-time suspension. As suspensions cannot occur in the presence of singularities we can suggest the following problem:

7.17. Is $W^{uu}(p)$ dense in $M(X)$ for every $p \in Per(X)$ and every transitive sectional-Anosov flow *with singularities* X of a compact manifold M?

(The corresponding problem for periodic sectional-Anosov flows instead of transitive ones has negative answer [22].)

It is also well known that if X is an Anosov flow on a closed manifold M, then $\Omega(X)$ has nonempty interior if and only if $\Omega(X)$ ([127]). We call a vector field X *nonwandering* if $M(X) = \Omega(X)$. It is then natural to consider the following:

7.18. Is the interior of the nonwandering set empty for every sectional-Anosov flow *with singularities* of a compact manifold?

Our next problem is concerned with the result in [124] that the sole Anosov flows on compact manifolds for which the (weak) stable foliation is transversely affine are the suspended ones. *Transversely affine* means roughly that the manifold can be covered by distinguished open sets for which the submersions defining the foliation have affine transition functions.

Since the sectional-Anosov flows also display (weak) stable foliations we can ask whether this foliation is transversely affine or not for the geometric or multidimensional Lorenz attractors [3], [29], [57], [60]. In fact, we believe negative answer not only for this but also for the next question:

7.19. Can the stable foliation of a sectional-Anosov flow *with singularities* of a compact manifold be transversely affine?

One more question is motivated by Corollary 6.28. This result was improved very recently in [7] by proving that every vector field close to a nonwandering sectional-Anosov flow with singularities on a compact three-manifold satisfies that the union of the stable manifolds of the singularities is dense. These results motivate the following question:

7.20. Is the union of the stable manifolds of the singularities dense for every vector field close to a nonwandering sectional-Anosov flow with singularities of a compact manifold?

Another class of problems can be also derived Corollary 4.23 and from [79] where it was proved that in certain geometric Lorenz attractors (more precisely, in the homoclinic case) the Hausdorff dimension is greater than 2. Noticing that in all these cases the corresponding central subbundle is two-dimensional we introduce the following question:

7.21. Is the topological (resp. Hausdorff) dimension of the maximal invariant set of a transitive sectional-Anosov flow with singularities on compact manifolds equal to (resp. *greater than*) the dimension of the central subbundle?

We still ask if it is possible to estimate the Hausdorff dimension as in Corollary 1 p. 700 of [79].

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