A DICHOTOMY FOR HIGHER-DIMENSIONAL FLOWS

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Abstract. We analyze the dichotomy between the *sectional-Axiom A flows* (c.f. [18]) and the flows with points accumulated by periodic orbits of different indices. Indeed, we prove such a dichotomy for C^1 generic flows whose singularities accumulated by periodic orbits have codimension one. This improves [20].

1. Introduction

In his celebrated proof of the stability conjecture [15], Mañé asked if the *star property*, i.e., the property of being far away from systems with non-hyperbolic periodic orbits, is sufficient to guarantee that a system be Axiom A. Actually, the answer is positive for diffeomorphisms [10] but not for flows by the geometric Lorenz attractor [1], [8], [9]. On the other hand, if singularities are not allowed then the star property for flows implies Axiom A, as showed in [5]. Previously, Mañé connects the star property with the nowadays called *Newhouse phenomenon* at least for surfaces. In fact, he proved that a C^1-generic surface diffeomorphism either is Axiom A or displays infinitely many sinks or sources [16].

In the extension of the aforementioned Mañé's work on surfaces, [20] obtained the following results about C^1-generic flows for closed 3-manifolds: Any C^1-generic star flow is singular-Axiom A and, consequently, any C^1-generic flow is singular-Axiom A or displays infinitely many sinks or sources. The notion of *singular-Axiom A* was introduced in [21] inspired on the dynamical properties of both Axiom A flows and the geometric Lorenz attractor.

It is then natural to investigate such generic phenomena in higher dimensions. At a first sight, the natural questions are: Is a C^1-generic star flow in a closed n-manifold singular-Axiom A? Does a C^1-generic vector field in a closed n-manifold is singular-Axiom A or has infinitely many sinks or sources? Unfortunately, we only know that the second question has negative answer for $n \geq 5$ as counterexamples can be obtained by suspending the diffeomorphisms in Theorem C of [2] (but the answer may be positive for $n = 4$).

A new light comes from the *sectional-Axiom A flows* introduced in [18]. Indeed, the first author replaced the term singular-Axiom A by sectional-Axiom A above in order to formulated the following conjecture on closed n-manifolds, $n \geq 3$ (compare with one in p. 947 of [7]):

Conjecture 1. *C^1-generic star flows are sectional-Axiom A.*

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Analogously we can ask if a C^1-generic vector field in a closed n-manifold is sectional-Axiom A or display infinitely many sinks or sources. But in this case the answer is negative not only for $n = 5$, by the suspension of [2] as above, but also for $n = 4$ by [24] and the suspension of certain diffeomorphisms [17] (which are singular-Axiom A but not sectional-Axiom A). Nevertheless, in all these counterexamples there are *points accumulated by hyperbolic periodic orbits of different Morse indices.* Since it is possible to observe that phenomenon in a number of well-known examples of non-hyperbolic systems and since, in dimension three, that phenomenon implies existence of infinitely many sinks or sources, it is possible to propose the following dichotomy (which, in virtue of Proposition 14, follows from Conjecture 1):

Conjecture 2. *C^1-generic vector fields X satisfy (only) one of the following properties:*

1. *X has a point accumulated by hyperbolic periodic orbits of different Morse indices;*
2. *X is sectional-Axiom A.*

In this paper we prove Conjecture 2 but in a case very close to the three-dimensional one, namely, when the *singularities accumulated by periodic orbits have codimension one* (i.e. Morse index 1 or $n - 1$). Observe that our result implies the dichotomy in [20] since the assumption about the singularities is automatic for $n = 3$. It also implies Conjecture 2 in large classes of vector fields as, for instance, those whose singularities (if any) have codimension one.

2. Statement of the Main Theorem

In what follows M is a compact connected boundaryless Riemannian manifold of dimension $n \geq 3$ (or a *closed n-manifold* for short). If X is a C^1 vector field in M we will denote by X_t the flow generated by X in M. A subset $\Lambda \subset M$ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in I\!R$. By a *closed orbit* we mean a periodic orbit or a singularity. We define the *omega-limit set* of $p \in M$ by

$$\omega(p) = \left\{ x \in M : x = \lim_{n\to\infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty \right\}$$

and call Λ *transitive* if $\Lambda = \omega(p)$ for some $p \in \Lambda$. Clearly every transitive set is compact invariant. As customary we call Λ *nontrivial* it it does not reduce to a single orbit. We also say that Λ is *robustly transitive* if there is a neighborhood U of it such that $\bigcap_{t\in\mathbb{R}} Y_t(U)$ is a nontrivial transitive set of Y, for every vector field Y that is C^1 close to X.

Denote by $\|\cdot\|$ and $m(\cdot)$ the norm and the minimal norm induced by the Riemannian metric and by $Det(\cdot)$ the jacobian operation. A compact invariant set Λ is *hyperbolic* if there are a continuous invariant tangent bundle decomposition

$$T_\Lambda M = \hat{E}^s_\Lambda \oplus E^X_\Lambda \oplus \hat{E}^u_\Lambda$$

and positive constants K, λ such that E^X_Λ is the subbundle generated by X,

$$\|DX_t(x)/\hat{E}^s_x\| \leq Ke^{-\lambda t} \quad \text{and} \quad m(DX_t(x)/\hat{E}^u_x) \geq K^{-1}e^{\lambda t},$$

for all $x \in \Lambda$ and $t \geq 0$. Sometimes we write $\hat{E}^{s,X}_x$, $\hat{E}^{u,X}_x$ to indicate dependence on X.

A closed orbit O is hyperbolic if it does as a compact invariant set. In such a case we define its *Morse index* $I(O) = dim(\hat{E}^s_O)$, where $dim(\cdot)$ stands for the dimension operation. If O reduces to a singularity σ, then we write $I(\sigma)$ instead of $I(\{\sigma\})$ and say that σ has *codimension one* if $I(\sigma) = 1$ or $I(\sigma) = n - 1$. It is customary to call hyperbolic closed orbit of maximal (resp. minimal) Morse index *sink* (resp. *source*).

On the other hand, an invariant splitting $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ over Λ is *dominated* (we also say that E_Λ *dominates* F_Λ) if there are positive constants K, λ such that

$$\frac{\|DX_t(x)/E_x\|}{m(DX_t(x)/F_x)} \leq Ke^{-\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0.$$

In this work we agree to call a compact invariant set Λ *partially hyperbolic* if there is a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ with *contracting* dominating subbundle E^s_Λ, namely,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0.$$

We stress however that this is not a standard usage (specially due to the lack of symmetry in this definition). Anyway, in such a case, we say that Λ has *contracting dimension* d if $dim(E^s_x) = d$ for all $x \in \Lambda$. Moreover, we say that the central subbundle E^c_Λ is *volume expanding* or *sectionally expanding* depending on whether

$$|Det(DX_t(x))| \geq K^{-1}e^{\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0$$

or

$$dim(E^c_x) \geq 2 \quad \text{and} \quad |Det(DX_t(x)/L_x)| \geq K^{-1}e^{\lambda t}, \qquad \forall x \in \Lambda \text{ and } t \geq 0$$

and all two-dimensional subspace L_x of E^c_x. Notice that sectionally expansiveness implies volume expansiveness but not conversely.

We call a partially hyperbolic set whose singularities (if any) are hyperbolic *singular-hyperbolic* (resp. *sectional-hyperbolic*) if its central subbundle is volume (resp. sectionally) expanding ([1]).

Now we recall the concept of sectional-Axiom A flow [18]. Call a point $p \in M$ *nonwandering* if for every neighborhood U of p and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. We denote by $\Omega(X)$ the set of nonwandering points of X (which is clearly a compact invariant set). We say that X is an *Axiom A flow* if $\Omega(X)$ is both hyperbolic and the closure of the closed orbits. The so-called *Spectral Decomposition Theorem* [12] asserts that the nonwandering set of an Axiom A flow X splits into finitely many disjoint transitive sets *with dense closed orbits* (i.e. with a dense subset of closed orbits) which are hyperbolic for X. This motivates [21] to define *singular-Axiom A flow* as a vector field X whose nonwandering set splits into finitely many disjoint transitive sets with dense closed orbits which are either hyperbolic or singular-hyperbolic for X or singular-hyperbpolic for $-X$. Analogously [18] stated the following definition:

Definition 3. A C^1 vector field X in M is called *sectional-Axiom A flow* if there is a finite disjoint decomposition $\Omega(X) = \Omega_1 \cup \cdots \cup \Omega_k$ formed by transitive sets with dense periodic orbits $\Omega_1, \cdots, \Omega_k$ such that, for all $1 \leq i \leq k$, Ω_i is either a hyperbolic set for X or a sectional-hyperbolic set for X or a sectional-hyperbolic set for $-X$.

Let $\mathcal{X}^1$ denote the space of C^1 vector fields X in M. Notice that it is a Baire space if equipped with the standard C^1 topology. The expression C^1*-generic vector field* will mean a vector field in a certain residual subset of $\mathcal{X}^1$. We say that a point

[1]Some authors prefer to keep the same name (i.e. singular-hyperbolicity) for both concepts.

is *accumulated by periodic orbits*, if it lies in the closure of the union of the periodic orbits, and *accumulated by hyperbolic periodic orbits of different Morse index* if it lies simultaneously in the closure of the hyperbolic periodic orbits of Morse index i and j with $i \neq j$. With these definitions we can state our main result settling a special case of Conjecture 2.

Main Theorem. *A C^1-generic vector field $X \in \mathcal{X}^1$ for which the singularities accumulated by periodic orbits have codimension one satisfies (only) one of the following properties:*

1. *X has a point accumulated by hyperbolic periodic orbits of different Morse indices;*
2. *X is sectional-Axiom A.*

Standard C^1-generic results [3] imply that the sectional-Axiom A flows in the second alternative above also satisfy the no-cycle condition.

The proof of our result follows that of Theorem A in [20]. However, we need a more direct approach bypassing Conjecture 1. Indeed, we shall use some methods in [20] together with a combination of results [6], [7], [18] for nontrivial transitive sets (which were originally proved for robustly transitive sets).

3. Proof

Hereafter we fix a closed n-manifold M, $n \geq 3$, $X \in \mathcal{X}^1$ and a compact invariant set Λ of X. Denote by $Sing(X, \Lambda)$ the set of singularities of X in Λ. We shall use the following concept from [6].

Definition 4. We say that Λ has a definite index $0 \leq Ind(\Lambda) \leq n-1$ if there are a neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1$ and a neighborhood U of Λ in M such that $I(O) = Ind(\Lambda)$ for every hyperbolic periodic orbit $O \subset U$ of every vector field $Y \in \mathcal{U}$. In such a case we say that Λ is *strongly homogeneous (of index $Ind(\Lambda)$)*.

It turns out that the strongly homogeneous property imposes certain constraints on the Morse indices of the singularities [7]. To explain this we use the concept of *saddle value* of a hyperbolic singularity σ of X defined by

$$\Delta(\sigma) = Re(\lambda) + Re(\gamma)$$

where λ (resp. γ) is the stable (resp. unstable) eigenvalue with maximal (resp. minimal) real part (c.f. [25] p. 725). Indeed, based on the Hayashi's connecting lemma [11] and well-known results about unfolding of homoclinic loops [25], Lemma 4.3 in [7] proves that, if Λ is a robustly transitive set which is strongly homogeneous with hyperbolic singularities, then $\Delta(\sigma) \neq 0$ and, furthermore, $I(\sigma) = Ind(\Lambda)$ or $Ind(\Lambda) + 1$ depending on whether $\Delta(\sigma) < 0$ or $\Delta(\sigma) > 0$, $\forall \sigma \in Sing(X, \Lambda)$. However, we can observe that the same is true for nontrivial transitive sets (instead of robustly transitive sets) for the proof in [7] uses the connecting lemma only once. In this way we obtain the following lemma.

Lemma 5. *Let Λ be a nontrivial transitive set which is strongly homogeneous with singularities (all hyperbolic) of X. Then, every $\sigma \in Sing(X, \Lambda)$ satisfies $\Delta(\sigma) \neq 0$ and one of the properties below:*

- *If $\Delta(\sigma) < 0$, then $I(\sigma) = Ind(\Lambda)$.*
- *If $\Delta(\sigma) > 0$, then $I(\sigma) = Ind(\Lambda) + 1$.*

On the other hand, the following inequalities for strongly homogeneous sets Λ where introduced in [6]:

$$I(\sigma) > Ind(\Lambda), \qquad \forall \sigma \in Sing(X, \Lambda). \tag{1}$$

$$I(\sigma) \leq Ind(\Lambda), \qquad \forall \sigma \in Sing(X, \Lambda). \tag{2}$$

We shall use the above lemma to present a special case where one of these inequalities can be proved.

Proposition 6. *Let Λ be a nontrivial transitive set which is strongly homogeneous with singularities (all hyperbolic of codimension one) of X. If $n \geq 4$ and $1 \leq Ind(\Lambda) \leq n-2$, then Λ satisfies either (1) or (2).*

Proof. Otherwise there are $\sigma_0, \sigma_1 \in Sing(X, \Lambda)$ satisfying $I(\sigma_0) \leq Ind(\Lambda) < I(\sigma_1)$. Since both σ_0 and σ_1 have codimension one and $1 \leq Ind(\Lambda) \leq n-2$ we obtain $I(\sigma_0) = 1$ and $I(\sigma_1) = n-1$. If $\Delta(\sigma_0) \geq 0$ then $I(\sigma_0) = Ind(\Lambda) + 1$ by Lemma 5 so $Ind(\Lambda) = 0$ which contradicts $1 \leq Ind(\Lambda)$. Then $\Delta(\sigma_0) < 0$ and so $Ind(\Lambda) = I(\sigma_0) = 1$ by Lemma 5. On the other hand, if $\Delta(\sigma_1) < 0$ then $Ind(\Lambda) = I(\sigma_1) = n-1$ by Lemma 5. As $Ind(\Lambda) = 1$ we get $n = 2$ contradicting $n \geq 4$. Then $\Delta(\sigma_1) \geq 0$ so $I(\sigma_1) = Ind(\Lambda) + 1$ by Lemma 5 thus $n = 3$ contradicting $n \geq 4$. The proof follows. □

The importance of (1) and (2) relies on the the following result proved in [6], [7], [18]: A C^1 robustly transitive set Λ with singularities (all hyperbolic) which is strongly homogeneous satisfying (1) (resp. (2)) is sectional hyperbolic for X (resp. $-X$). However, we can observe that the same is true for nontrivial transitive sets (instead of robustly transitive sets) as soon as $1 \leq Ind(\Lambda) \leq n-2$. The proof is similar to that in [6],[7], [18] but using the so-called *preperiodic set* [26] instead of the natural continuation of a robustly transitive sets. Combining this with Proposition 7 we obtain the following corollary in which the expression *up to flow-reversing* means either for X or $-X$.

Corollary 7. *Let Λ be a nontrivial transitive set which is strongly homogeneous with singularities (all hyperbolic of codimension one) of X. If $n \geq 4$ and $1 \leq Ind(\Lambda) \leq n-2$, then Λ is sectional-hyperbolic up to flow-reversing.*

A direct application of this corollary is as follows. We say that Λ is *Lyapunov stable* for X if for every neighborhood U of it there is a neighborhood $W \subset U$ of it such that $X_t(p) \in U$ for every $t \geq 0$ and $p \in W$.

It was proved in Theorem C of [20] that, for C^1 generic three-dimensional star flows, every nontrivial Lyapunov stable set with singularities is singular-hyperbolic. We will need a similar result for higher dimensional flows, but with the term singular-hyperbolic replaced by sectional-hyperbolic. The following will supply such a result.

Corollary 8. *Let Λ be a nontrivial transitive set which is strongly homogeneous with singularities (all hyperbolic of codimension one) of X. If $n \geq 4$, $1 \leq Ind(\Lambda) \leq n-2$ and Λ is Lyapunov stable, then Λ is sectional-hyperbolic for X.*

Proof. By Corollary 7 it suffices to prove that Λ cannot be sectional-hyperbolic for $-X$. Assume by contradiction that it does. Then, by integrating the corresponding contracting subbundle, we obtain a strong stable manifold $W^{ss}_{-X}(x)$, $\forall x \in \Lambda$. But

Λ is Lyapunov stabile for X so $W^{ss}_{-X}(x) \subset \Lambda$, $\forall x \in \Lambda$, contradicting p. 556 in [19]. Then, Λ cannot be sectional-hyperbolic for $-X$ and we are done. □

The following lemmas will be used only to prove Proposition 11. Denote by $CO(X, \Lambda)$ the union of the closed orbits of X in Λ. In the case when $CO(X, \Lambda)$ consists of hyperbolic closed orbits one has a natural splitting

$$T_x M = \hat{E}^s_x \oplus E^X_x \oplus \hat{E}^u_x, \qquad \forall x \in CO(X, \Lambda),$$

where E^X_x is the flow direction at x (thus $E^X_x = 0$ if x is a singularity). This applies for instance when Λ is sectional-hyperbolic. In such a case we have the following lemma.

Lemma 9. *Let Λ be a sectional-hyperbolic set of a vector field X in a closed n-manifold, $n \geq 3$. Then,*

$$\hat{E}^s_x = E^s_x \oplus (\hat{E}^s_x \cap E^c_x), \qquad \forall x \in CO(X, \Lambda).$$

Proof. Since E^s_x is contracting one has $E^s_x \subseteq \hat{E}^s_x$ and $E^s_x \cap \hat{E}^u_x = \{0\}$. On the other hand, every $\hat{v}^s \in \hat{E}^s_x$ can written as $\hat{v}^s = v^s + v^c \in E^s_x \oplus E^c_x$. It follows that $v^c = \hat{v}^s - v^s \in \hat{E}^s_x \cap E^c_x$ yielding $\hat{E}^s_x = E^s_x + (\hat{E}^s_x \cap E^c_x)$. As $E^s_x \cap \hat{E}^s_x \cap E^c_x \subset E^s_x \cap E^c_x = \{0\}$ we obtain the result. □

Lemma 10. *If Λ is a nontrivial transitive sectional-hyperbolic set of a vector field X in a closed n-manifold, $n \geq 3$, then $dim(\hat{E}^s_\sigma \cap E^c_\sigma) = 1$, $\forall \sigma \in Sing(X, \Lambda)$.*

Proof. Clearly $E^s_\sigma \cap \hat{E}^u_\sigma = \{0\}$ so $\hat{E}^u_\sigma \subset E^c_\sigma$ since $\hat{E}^u_\sigma$ is invariant and E^s_Λ dominates E^c_Λ. As $T_\sigma M = \hat{E}^s_\sigma \oplus \hat{E}^u_\sigma$ we conclude that $T_\sigma M = \hat{E}^s_\sigma + E^c_\sigma$.

If $\hat{E}^s_\sigma \cap E^c_\sigma = \{0\}$ we would have $T_\sigma M = \hat{E}^s_\sigma \oplus E^c_\sigma$ hence $dim(E^c_\sigma) = n - dim(\hat{E}^s_\sigma)$. But we also have $dim(E^c_\sigma) = n - dim(E^s_\sigma)$ so we would have $dim(E^s_\sigma) = dim(\hat{E}^s_\sigma)$.

Let $W^{ss}(\sigma)$ be the strong stable manifold obtained by integrating the strong stable subbundle E^s_Λ (c.f. [13]).

As $dim(W^{ss}(\sigma)) = dim(E^s_\sigma) = dim(\hat{E}^s_\sigma) = dim(W^s(\sigma))$ and $W^{ss}(\sigma) \subseteq W^s(\sigma)$ we get $W^{ss}(\sigma) = W^s(\sigma)$. But Λ is transitive and nontrivial so the dense orbit will accumulate at some point in $W^s(\sigma) \setminus \{\sigma\}$. As $W^{ss}(\sigma) = W^s(\sigma)$ such a point must belong to $(\Lambda \cap W^{ss}(\sigma)) \setminus \{\sigma\}$. On the other hand, it is well known that $\Lambda \cap W^{ss}(\sigma) = \{\sigma\}$ (c.f. [20]) so we obtain a contradiction. Therefore $\hat{E}^s_\sigma \cap E^c_\sigma \neq \{0\}$ and so $dim(\hat{E}^s_\sigma \cap E^c_\sigma) \geq 1$. As $dim(\hat{E}^s_\sigma \cap E^c_\sigma) \leq 1$ we are done. □

We use these lemmas to prove the following proposition.

Proposition 11. *Every nontrivial transitive sectional-hyperbolic set Λ of a vector field X in a closed n-manifold, $n \geq 3$, is strongly homogeneous and satisfies $I(\sigma) = Ind(\Lambda) + 1$, $\forall \sigma \in Sing(X, \Lambda)$.*

Proof. Since transitiveness implies connectedness we have that the strong stable subbundle E^s_Λ of Λ has constant dimension. From this and the persistence of the sectional-hyperbolic splitting we obtain that Λ is strongly homogeneous of index $Ind(\Lambda) = dim(E^s_x)$, for $x \in \Lambda$. Now fix a singularity σ. By Lemma 10 one has $dim(\hat{E}^s_\sigma \cap E^c_\sigma) = 1$ and so $dim(\hat{E}^s_\sigma) = dim(E^s_\sigma) + 1$ by Lemma 9. Therefore $Ind(\sigma) = dim(\hat{E}^s_\sigma) = dim(E^s_\sigma) + 1 = Ind(\Lambda) + 1$. □

We say that Λ is an *attracting set* if there is a neighborhood U of it such that

$$\Lambda = \bigcap_{t>0} X_t(U).$$

On the other hand, a *sectional-hyperbolic attractor* is a transitive attracting set which is also a sectional-hyperbolic set. An *unstable branch* of a hyperbolic singularity σ of a vector field is an orbit in $W^u(\sigma) \setminus \{\sigma\}$. We say that Λ has *dense singular unstable branches* if every unstable branch of every hyperbolic singularity on it is dense in Λ.

The following is a straightforward extension of Theorem D in [20] to higher dimensions (with similar proof).

Proposition 12. *Let Λ be a Lyapunov stable sectional-hyperbolic set of a vector field X in a closed n-manifold, $n \geq 3$. If Λ has both singularities, all of Morse index $n-1$, and dense singular unstable branches, then Λ is a sectional-hyperbolic attractor of X.*

We also use the star flow's terminology from [26].

Definition 13. A *star flow* is a C^1 vector field which cannot be C^1-approximated by ones exhibiting non-hyperbolic closed orbits.

Corollary 8 together with propositions 11 and 12 implies the key result below.

Proposition 14. *A C^1-generic vector field X on a closed n-manifold, $\forall n \geq 3$, without points accumulated by hyperbolic periodic orbits of different Morse indices is a star flow. If, in addition, $n \geq 4$, then the codimension one singularities of X accumulated by periodic orbits belong to a sectional-hyperbolic attractor up to flow-reversing.*

Proof. We will use the following notation. Given $Z \in \mathcal{X}^1$ and $0 \leq i \leq n-1$ we denote by $Per_i(Z)$ the union of the hyperbolic periodic orbits of Morse index i. The closure operation will be denoted by $Cl(\cdot)$.

Since X has no point accumulated by hyperbolic periodic orbits of different Morse indices one has

$$Cl(Per_i(X)) \cap Cl(Per_j(X)) = \emptyset, \qquad \forall i, j \in \{0, \cdots, n-1\}, \quad i \neq j. \tag{3}$$

Then, since X is C^1-generic, standard lower-semicontinuous arguments (c.f. [3]) imply that there are a neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1$ and a pairwise disjoint collection of neighborhoods $\{U_i : 0 \leq i \leq n-1\}$ such that $Cl(Per_i(Y)) \subset U_i$ for all $0 \leq i \leq n-1$ and $Y \in \mathcal{U}$.

Let us prove that X is a star flow. When necessary we use the notation $I_X(O)$ to indicate dependence on X. By contradiction assume that X is not a star flow. Then, there is a vector field $Y \in \mathcal{U}$ exhibiting a non-hyperbolic closed orbit O. Since X is generic we can assume by the Kupka-Smale Theorem [12] that O is a periodic orbit. Unfolding the eigenvalues of O is a suitable way we would obtain two vector fields $Z_1, Z_2 \in \mathcal{U}$ of which O is a hyperbolic periodic orbit with $I_{Z_1}(O) \neq I_{Z_2}(O)$, $1 \leq I_{Z_1}(O) \leq n-1$ and $1 \leq I_{Z_2}(O) \leq n-1$. Consequently, $O \subset U_i \cap U_j$ where $i = I_{Z_1}(O)$ and $j = I_{Z_2}(O)$ which contradicts that the collection $\{U_i : 0 \leq i \leq n-1\}$ is pairwise disjoint. Therefore, X is a star flow.

Next we prove that $Cl(Per_i(X))$ is a strongly homogeneous set of index i, $\forall 0 \leq i \leq n-1$. Take $Y \in \mathcal{U}$ and a hyperbolic periodic orbit $O \subset U_i$ of Morse index

$I_Y(O) = j$. Then, $O \subset Cl(Per_j(Y))$ and so $O \subset U_j$ from which we get $O \subset U_i \cap U_j$. As the collection $\{U_i : 0 \leq i \leq n-1\}$ is disjoint we conclude that $i = j$ and so every hyperbolic periodic orbit $O \subset U_i$ of every vector field $Y \in \mathcal{U}$ has Morse index $I_Y(O) = i$. Therefore, $Cl(Per_i(X))$ is a strongly homogeneous set of index i.

Now, we prove that every codimension one singularity σ accumulated by periodic orbits belongs to a sectional-hyperbolic attractor up to flow-reversing. More precisely, we prove that if $I(\sigma) = n-1$ (resp. $I(\sigma) = 1$), then σ belongs to a sectional-hyperbolic attractor of X (resp. of $-X$). We only consider the case $I(\sigma) = n-1$ for the case $I(\sigma) = 1$ can be handled analogously by just replacing X by $-X$.

Since $I(\sigma) = n-1$ one has $dim(W^u(\sigma)) = 1$ and, since X is generic, we can assume that both $Cl(W^u(\sigma))$ and $\omega(q)$ (for $q \in W^u(\sigma) \setminus \{\sigma\}$) are Lyapunov stable sets of X (c.f. [4]). As σ is accumulated by periodic orbits we obtain from Lemma 4.2 in [20] that $Cl(W^u(\sigma))$ is a transitive set.

We claim that $Cl(W^u(\sigma))$ is strongly homogeneous. Indeed, since X is generic the General Density Theorem [23] implies $\Omega(X) = Cl(Per(X) \cup Sing(X))$. Denote by $Sing^*(X)$ is the set of singularities accumulated by periodic orbits. Then, there is a decomposition

$$\Omega(X) = \left(\bigcup_{0 \leq i \leq n-1} Cl(Per_i(X)) \right) \cup \left(\bigcup_{\sigma' \in Sing(X) \setminus Sing^*(X)} \{\sigma'\} \right)$$

which is disjoint by (3). In addition, $Cl(W^u(\sigma))$ is transitive and so it is connected and contained in $\Omega(X)$. As $\sigma \in Sing^*(X)$ by hypothesis we conclude that $Cl(W^u(\sigma)) \subset Cl(Per_{i_0}(X))$ for some $0 \leq i_0 \leq n-1$. But we have proved above that $Cl(Per_{i_0}(X))$ is a strongly homogeneous set of index i_0, so, $Cl(W^u(\sigma))$ is also a strongly homogeneous set of index i_0. The claim follows.

On the other hand, X is a star flow and so it has finitely many sinks and sources [14], [22]. From this we obtain $1 \leq i_0 \leq n-2$ and so $1 \leq Ind(Cl(W^u(\sigma))) \leq n-2$. Summarizing, we have proved that $Cl(W^u(\sigma))$ is a transitive set with singularities, all of them of codimension one, which is a Lyapunov stable strongly homogeneous set of index $1 \leq Ind(Cl(W^u(\sigma))) \leq n-2$. As certainly $Cl(W^u(\sigma))$ is nontrivial Corollary 8 applied to $\Lambda = Cl(W^u(\sigma))$ implies that $Cl(W^u(\sigma))$ is sectional-hyperbolic.

Once we have proved that $Cl(W^u(\sigma))$ is sectional-hyperbolic we apply Proposition 11 to $\Lambda = Cl(W^u(\sigma))$ yielding $I(\sigma') = i_0 + 1$, $\forall \sigma' \in Sing(X, Cl(W^u(\sigma)))$. But $\sigma \in Cl(W^u(\sigma))$ and $I(\sigma) = n-1$ so $i_0 = n-2$ by taking $\sigma' = \sigma$ above. Consequently, $I(\sigma') = n-1$ and so $dim(W^u(\sigma')) = 1$, $\forall \sigma' \in Cl(W^u(\sigma))$. This implies two things. Firstly that every singularity in $Cl(W^u(\sigma))$ has Morse index $n-1$ and, secondly, since X is generic, we can assume that $Cl(W^u(\sigma))$ has dense unstable branches (c.f. Lemma 4.1 in [20]). So, $Cl(W^u(\sigma))$ is a sectional-hyperbolic attractor by Proposition 12 applied to $\Lambda = Cl(W^u(\sigma))$. Since $\sigma \in Cl(W^u(\sigma))$ we obtain the result. □

The last ingredient is the proposition below whose proof follows from Theorem B of [5] as in the proof of Theorem B p. 1582 of [20].

Proposition 15. *If $n \geq 3$, every C^1-generic star flow whose singularities accumulated by periodic orbits belong to a sectional-hyperbolic attractor up to flow-reversing is sectional-Axiom A.*

Proof of the Main Theorem: Consider a C^1-generic vector field on a closed n-manifold, $n \geq 3$, such that every singularity accumulated by periodic orbits has codimension one. Suppose in addition that there is no point accumulated by hyperbolic periodic orbits of different Morse indices. Since X is C^1-generic we have by Proposition 14 that X is a star flow.

If $n = 3$ then, since X is generic, Theorem B in [20] implies that X is sectional-Axiom A.

If $n \geq 4$ then, by Proposition 14, since the singularities accumulated by periodic orbits have codimension one, we have that all such singularities belong to a sectional-hyperbolic attractor up to flow-reversing. Then, X is sectional-Axiom A by Proposition 15. □

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