ROBUST ATTRACTOR SETS WITHOUT DOMINATED SPLITTING ON MANIFOLDS WITH BOUNDARY

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Abstract. In this paper we prove that there exists a positive integer k with the following property: Every compact 3-manifold with boundary carries a C^∞ vector field exhibiting a C^k-robust attractor set without dominated splitting.

1. Introduction

Let M be a compact 3-manifold with boundary ∂M. Denote by $\mathscr{X}^k(M, \partial M)$, $k \geq 1$ (or $k = \infty$), the space of C^k vector fields in M tangent to ∂M (if nonempty) endowed with the standard C^k topology. We fix $X \in \mathscr{X}^1(M, \partial M)$ and denote by X_t, $t \in \mathbb{R}$, the flow generated by X in M. A compact invariant set Λ of X is *isolated* if there is an open set $U \supset \Lambda$, called an *isolating block*, such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U).$$

An *attracting set* is an isolated set with a positively invariant isolating block U, i.e., $X_t(U) \subset U$ for all $t > 0$. Given $p \in M$ we define its *omega-limit* set,

$$\omega(p) = \{q \in M : q = \lim_{n \to \infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty\}.$$

An compact invariant Λ of X is *transitive* if $\Lambda = \omega(p)$ for some $p \in \Lambda$. An *attractor* is a transitive attracting set (further definitions of attractors can be found in [7]). An invariant set Λ of X is *non-trivial* if it is not a single orbit.

If $k \in \mathbb{N}^+$ and $X \in \mathcal{X}^k(M, \partial M)$ we say that an isolated set Λ of X is a C^k-*robust transitive set* if it exhibits an isolating block U such that the continuation

$$\Lambda_Y = \bigcap_{t \in \mathbb{R}} Y_t(U)$$

of Λ for Y C^k-close to X is a non-trivial transitive set of Y. A C^k-*robust attractor* is a C^k-robust transitive set which is simultaneously an attracting set. A singularity σ of X is called *Lorenz-like* if it has three real eigenvalues $\lambda_{ss}, \lambda_s, \lambda_u$ satisfying $\lambda_{ss} < \lambda_s < 0 < -\lambda_s < \lambda_u$ up to some order.

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Let Λ be a compact invariant set of X. A continuous invariant splitting $T_\Lambda M = E_\Lambda \oplus E_\Lambda$ over Λ is dominated if there are positive constants C, λ such that

$$\frac{||DX_t(x)/_{E_x}||}{||m(DX_t(x)/_{F_x})||} \leq C\lambda^t \text{ for all } t > 0 \text{ and for all } x \in \Lambda.$$

The motivation of this paper is the work [11] dealing with C^1 robust transitive sets for vector fields on compact boundaryless 3-manifolds. Indeed, it was proved that these sets are robust attractors, exhibit a dominated splitting and that their singularities are Lorenz-like up to flow reversing. This result certainly suggests the same for vector fields on compact 3-manifolds with boundary, but it seems that it is not so. In fact the recent work [3] proved that there is $r \in \mathbb{N}^+$ with the following property: Every compact 3-manifold with boundary carries a C^∞ vector field X exhibiting a C^r-robust transitive set with a singularity that is not Lorenz-like for X or $-X$. Nevertheless the C^r-robust transitive sets obtained there are not attractors, so, it is still possible that the singularities of a robust attractor for vector fields on compact 3-manifolds with boundary be Lorenz-like ones. Moreover also it is still possible that robust attractor for vector fields on compact 3-manifolds with boundary exhibit dominated splitting. The result below given a negative answer for this last question.

Theorem 1.1. *There is a positive integer k such that every compact 3-manifold with boundary carries a C^∞ vector field exhibiting a C^k-robust attractor without dominates splitting.*

The integer k above and r in [3] may be different (this will be clear in the context).

2. Proof

Let consider a vector field $X \in \mathscr{X}^\infty(M, \partial M)$ satisfying the following properties:

(a) X has three hyperbolic singularities σ_0, σ_1 and σ_2 such that $\sigma^i \in \partial M$, for $i = 0, 1, 2$.

(b) If the singularities σ^i have real eigenvalues λ^i_{ss}, λ^i_s and λ^i_u with $\lambda^i_{ss} < \lambda^i_s < 0 < \lambda^i_u$, $\alpha_i = -\frac{\lambda^i_s}{\lambda^i_u}$ and $\beta_i = -\frac{\lambda^i_{ss}}{\lambda^i_u}$ for $i = 0, 1, 2$, then

  (b-1) $\beta_0\alpha_i < 1$ and $\alpha_0 + \beta_0\beta_i > 1$, for $i = 1, 2$.

(c) The unstable manifold $W^u(\sigma_0)$, stable manifold $W^s(\sigma_i)$ and strong stable manifold $W^{ss}(\sigma_i)$ satisfy $W^u(\sigma_0) \cap (W^s(\sigma_i) \backslash W^{ss}(\sigma_i)) \neq \emptyset$, for $i = 1, 2$.

(d) There are two positive real numbers $\bar{a}, \tilde{a}$ such that X is C^2-linear in the cubes

$$\begin{aligned} Q_0 &= \{(x, y, x) : |x| \leq 1, |y| \leq 1, 0 \leq z \leq 1\}; \\ Q_1 &= \{(\bar{x}, \bar{y}, \bar{z}) : |\bar{x}| \leq \bar{a}, |\bar{y}| \leq \bar{a}, 0 \leq \bar{z} \leq 1\} \text{ and} \\ Q_2 &= \{(\tilde{x}, \tilde{y}, \tilde{z}) : |\tilde{x}| \leq \tilde{a}, |\tilde{y}| \leq \tilde{a}, 0 \leq \tilde{z} \leq 1\} \end{aligned}$$

containing to σ_i, for $i = 0, 1, 2$, respectively. Moreover, the trajectories of the unstable manifold of σ_i, for $i = 1, 2$, intersect the top rectangle $\Sigma_0 = \{(x, y, x) : |x| \leq 1, |y| \leq 1, z = 1\}$ of the cube Q_0.

(e) The corresponding eigenspace associate to λ^i_{ss}, $E^{ss}_{\sigma_i}$, and $T_{\sigma_i}\partial M$, for $i = 0, 1, 2$, are transversals.

(f) X has a trapping region.

(g) There exists a vertical invariant contracting stable C^1 foliation in Σ_0 for the first return Poincaré map.

(h) The dynamic in the space of the leaf is expansive.

Note that rectangle Σ_0 is divided by the stable manifold of σ_0 in two subrectangles Σ^+_0 and Σ^-_0. We let us consider $\Sigma^*_0 = \Sigma^+_0 \cup \Sigma^-_0$, $\Sigma^+_1 = \{(x, y, z) : |x| \leq 1, y = 1, 0 \leq z \leq 1\}$, $\Sigma^-_1 = \{(x, y, z) : |x| \leq 1, y = -1, 0 \leq z \leq 1\}$, $\Sigma^+_2 = \{(\bar{x}, \bar{y}, \bar{z}) : |\bar{x}| \leq \bar{a}, y = -\bar{a}, 0 \leq \bar{z} \leq 1\}$, $\Sigma^-_2 = \{(\tilde{x}, \tilde{y}, \tilde{z}) : |\tilde{x}| \leq \tilde{a}, y = \tilde{a}, 0 \leq \tilde{z} \leq 1\}$, $\Sigma^+_3 = \{(\bar{x}, \bar{y}, \bar{z}) : |\bar{x}| \leq \bar{a}, |\bar{y}| \leq \bar{a}, \bar{z} = 1\}$ and $\Sigma^-_3 = \{(\tilde{x}, \tilde{y}, \tilde{z}) : |\tilde{x}| \leq \tilde{a}, |\tilde{y}| \leq \tilde{a}, \tilde{z} = 1\}$.

Figure 1 show the principal features of the vector field X.

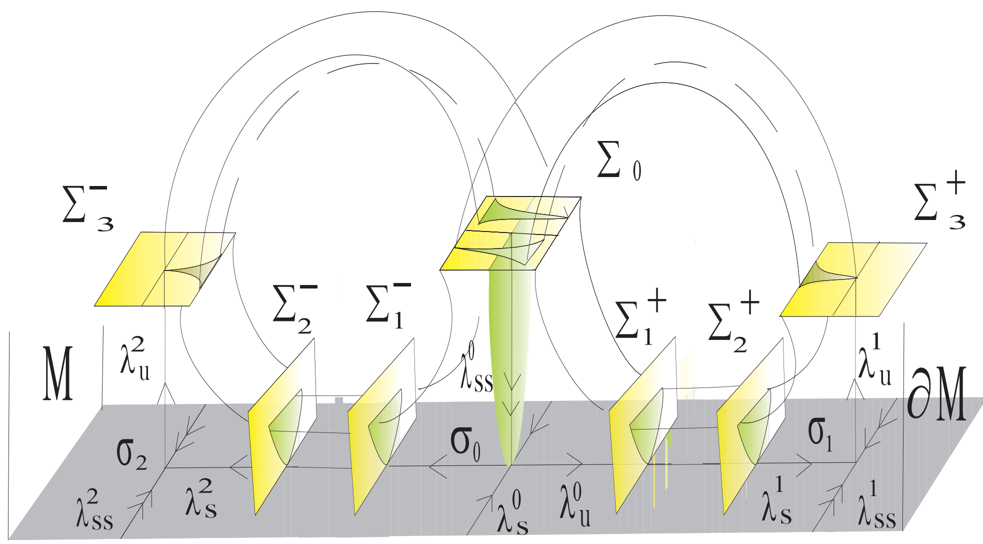

Figure 1

As X is C^2-linear in the cube Q_0, X is the three model linear differential equations:

$$\begin{cases} \dot{x} &= \lambda^0_s x \\ \dot{y} &= \lambda^0_u y \\ \dot{z} &= \lambda^0_{ss} z. \end{cases}$$

Which with initial conditions $(x_0, y_0, 1)$ in Σ^*_0, the solution is given by

$$\begin{cases} x(t) &= e^{\lambda^0_s} x_0 \\ y(t) &= e^{\lambda^0_u} y_0 \\ z(t) &= e^{\lambda^0_{ss}}. \end{cases}$$

The trajectories of points $(x_0, y_0, 1) \in \Sigma_0^*$ meets the plane Σ_1^+ when

$$\left\{ \begin{array}{lcl} x & = & x_0 y_0^{\alpha_0} \\ y & = & 1 \\ z & = & y_0^{\beta_0}. \end{array} \right. \tag{2.1}$$

In similar form to (2.1) we obtain for points $(x_0, y_0, 1) \in \Sigma_0^*$ meets the plane Σ_1^-.

As X is C^2-linear in the cube Q_1, X is the three model linear differential equations:

$$\left\{ \begin{array}{lcl} \dot{\bar{x}} & = & \lambda_{ss}^1 \bar{x} \\ \dot{\bar{y}} & = & \lambda_s^1 \bar{y} \\ \dot{\bar{z}} & = & \lambda_u^1 \bar{z}. \end{array} \right.$$

Which with initial conditions $(\bar{x}_0, -\bar{a}, \bar{z}_0)$ in Σ_1^+, the solution is given by

$$\left\{ \begin{array}{lcl} \bar{x}(t) & = & e^{\lambda_{ss}^1} \bar{x}_0 \\ \bar{y}(t) & = & e^{\lambda_s^1} (-\bar{a}) \\ \bar{z}(t) & = & e^{\lambda_u^2}. \end{array} \right.$$

The trajectories of points $(\bar{x}_0, -\bar{a}, \bar{z}_0) \in \Sigma_1^+$ meets the plane Σ_2^+ when

$$\left\{ \begin{array}{lcl} \bar{x} & = & \bar{x}_0 \bar{z}_0^{\beta_1} \\ \bar{y} & = & -\bar{a} \bar{z}_0^{\alpha_1} \\ \bar{z} & = & 1. \end{array} \right. \tag{2.2}$$

Finally, as X is C^2-linear in the cube Q_2, X is the three model linear differential equations:

$$\left\{ \begin{array}{lcl} \dot{\tilde{x}} & = & \lambda_{ss}^2 \tilde{x} \\ \dot{\tilde{y}} & = & \lambda_s^2 \tilde{y} \\ \dot{\tilde{z}} & = & \lambda_u^2 \tilde{z}. \end{array} \right.$$

Which with initial conditions $(\tilde{x}_0, \tilde{a}, \tilde{z}_0)$ in Σ_1^-, the solution is given by

$$\left\{ \begin{array}{lcl} \tilde{x}(t) & = & e^{\lambda_{ss}^2} \tilde{x}_0 \\ \tilde{y}(t) & = & e^{\lambda_s^2} (\tilde{a}) \\ \tilde{z}(t) & = & e^{\lambda_u^2}. \end{array} \right.$$

The trajectories of points $(\tilde{x}_0, \tilde{a}, \tilde{z}_0) \in \Sigma_1^-$ meets the plane Σ_2^- when

$$\left\{ \begin{array}{lcl} \tilde{x} & = & \tilde{x}_0 \tilde{z}_0^{\beta_2} \\ \tilde{y} & = & \tilde{a} \tilde{z}_0^{\alpha_2} \\ \tilde{z} & = & 1. \end{array} \right. \tag{2.3}$$

There exists fourth non-linear return maps: $\Pi_{out}^{1,+} : \Sigma_1^+ \to \Sigma_2^+$, $\Pi_{out}^{3,+} : \Sigma_3^+ \to \Sigma_0$, $\Pi_{out}^{1,-} : \Sigma_1^- \to \Sigma_2^-$ and $\Pi_{out}^{3,-} : \Sigma_3^- \to \Sigma_0$. We let consider

$$\Pi_{loc}^* = \Pi_{loc}^{2,*} \circ \Pi_{out}^{1,*} \circ \Pi_{loc}^{0,*}, \text{for } * = +, -.$$

Therefore, if we suppose that $\Pi^{1,*}_{out} = Id$, we Id is the is the identity map, for $* = +, -$ then from (2.1), (2.2) and (2.3) we obtain

$$\Pi^{+}_{loc}(x, y) = (axy^{\alpha_0+\beta_0\beta_1}, by^{\beta_0\alpha_1}) \text{ if } y > 0;$$

for some real constants a and b (depending only of X) and

$$\Pi^{-}_{loc}(x, y) = (cxy^{\alpha_0+\beta_0\beta_2}, dy^{\beta_0\alpha_2}) \text{ if } y < 0;$$

for some real constants c and d (depending only of X). So two Poincaré maps are defined: $R^* = \Pi^{3,*}_{out} \circ \Pi^{*}_{loc} : \Sigma^{*}_{0} \to \Sigma_0$, for $* = +, -$. As by hyphoteses there exists a invariant contracting C^1 foliation in Σ_0 which is invariant by return map R were ($R(x, y) = R^+(x, y)$ if $y > 0$ and $R(x, y) = R^-(x, y)$ if $y < 0$), then using this foliation, R can by defined by

$$R(x, y) = (F(x, y), f(y))$$

where

$$\begin{cases} (a)F(x, y) > \frac{1}{4} & \text{for } y > 0 \\ (b)F(x, y) < \frac{1}{4} & \text{for } y < 0. \end{cases}$$

and where f satisfies the properties:

$$\begin{cases} (a)f(0^+) = -\frac{1}{2} & f(0^+) = -\frac{1}{2} \\ (b)f'(x) > \sqrt{2} & \text{for } x \in [-1, 1]\backslash\{0\} \\ (c) -\frac{1}{2} < f(x) < \frac{1}{2} & \text{for } x \in [-1, 1]\backslash\{0\}. \end{cases} \tag{2.4}$$

(b) and (c) holding throughout the range $-\frac{1}{2} \le x \le \frac{1}{2}$.

Moreover there exists a trapping region (isolated block) U of the cube above. Define

$$\Lambda = \bigcap_{t \ge 0} X_t(U).$$

This finish the construction of X and Λ. Now we will prove that Λ is C^k-robust attractor set.

The assumption of C^2 linearing coordinates nearby $\sigma_i(Y)$, imply $Y \in \mathscr{X}^{k}_{i}(M, \partial M)$ where

$$k_i > 2 + \frac{4 \cdot min\{\lambda^i_{ss}, -\lambda^i_u\} - Log(56)}{max\{\lambda^i_s, -\lambda^i_u\}}.$$

We choose $k = \min\{k_i\}$, $i = 0, 1, 2$. Now we fix such k.

Take any neighborhood $\mathscr{U}$ of the vector field X in the C^k topology. Now fix such k and $Y \in \mathscr{U}$, and let us consider the continuations $\sigma_i(Y)$, $i = 0, 1, 2$, of singularities σ_i as well defined. The vector field Y is C^2-linearizable nearby $\sigma_i(Y)$, $i = 0, 1, 2$.

We can assume that the cross-sections Σ_0, Σ^*_1, Σ^*_2, Σ^*_3, for $* = +, -$, remain transverse to any Y C^k-close to X. Moreover we can assume that any Y C^k-close to X is C^2-linear in the cubes Q_i, for $i = 0, 1, 2$. In the same way $\lambda^i_{ss}(Y)$, $\lambda^i_s(Y)$ and $\lambda^i_u(Y)$, for $i = 0, 1, 2$, the respective continuations of the eigenvalues λ^i_{ss}, λ^i_s and λ^i_u. Denote $\alpha^i_Y = -\frac{\lambda^i_s(Y)}{\lambda^i_u(Y)}$ and $\beta^i_Y = -\frac{\lambda^i_{ss}(Y)}{\lambda^i_u(Y)}$, for $i = 0, 1, 2$.

Also we denote by $\Pi^{0,*}_{loc,Y}$, $\Pi^{2,*}_{loc,Y}$, $\Pi^{1,*}_{out,Y}$, $\Pi^{3,*}_{out}$, R^*_Y and R_Y, for $* = +,-$, the continuation of $\Pi^{0,*}_{loc}$, $\Pi^{2,*}_{loc}$, $\Pi^{1,*}_{out}$, $\Pi^{3,*}_{out}$, R^* and R, respectively.

Now by choice of k for any vector filed Y C^k-close to X there are C^2 linearizing coordinates at the singularity $\sigma(Y)_i$, for $i = 0,1,2$, so the Poincaré map P_Y is a C^2 map. Additionally, we suppose that $\Pi^{1,*}_{out}$ preserves the "horizontal "lines and $\Pi^{3,*}_{out}$, for $* = +,-$ put Σ^*_3 into Σ_0 expanding in "vertical"direction. So, again the techniques in [2] give us a R_Y-invariant contracting C^1 foliation $\mathscr{F}$ (see also [12]). This construction can be made in a such way that the set $\{y = 0\}$, $\{y = -\frac{1}{2}\}$ and $\{y = \frac{1}{2}\}$ are leaves of this foliation.

We can use this foliation to put new coordinates (x, y) on Σ_0, still linearizing, such that for all $(x, y) \in \Sigma^*_0$

$$R_Y(x, y) = (H_Y(x, y), f_Y(y)), \tag{2.5}$$

for some C^1 maps $f_Y(\cdot\,,\cdot)$ and $g_Y(\cdot)$. Moreover, $g_Y(a)$ and $g_Y(b)$ are greater than 1.

It follows from the above that

$$\Lambda_Y = \bigcap_{t\in\mathbb{R}} Y_t(U) = Cl\left(\bigcup_{t\in\mathbb{R}} Y_t\left(\bigcap_{n\in\mathbb{Z}} R^n_Y(\Sigma_0)\right)\right).$$

So, in order to prove that Λ_Y is a transitive set, we only need to prove that the maximal invariant set

$$\bigcap_{n\in\mathbb{Z}} R^n_Y(\Sigma_0) \tag{2.6}$$

is a transitive set for R_Y and for this purpose essentially we follow the arguments given in [3]. Other argument can be find in [4].

Definition 2.1. *Define $\mathscr{A}_L$ as the set of C^1-maps $f : [-1/2, 1/2]\backslash\{0\} \to [-1/2, 1/2]$ satisfying the following properties:*

(a) $f'(y) > \sqrt{2}$ *on* $[-1/2, 1/2]\backslash\{0\}$;
(b) f *is strictly increasing on* $[-1/2, 1/2]\backslash\{0\}$;
(c) $f(0^-) = 1/2$ *and* $f(0^+) = -1/2$;
(d) $f'(y) \to \infty$ *as* $y \to 0$ *(from right and left).*
(e) *There are* $\alpha^1_f < 1$, $\alpha^2_f < 1$ *and two* C^1 *function* $H^1_f : [-1/2, 0) \to \mathbb{R}$, $H^2_f : (0, 1/2] \to \mathbb{R}$ *with* $\lim_{y\to 0} \frac{DH^1_f(y)}{y} = 0$ *and* $\lim_{y\to 0} \frac{DH^2_f(y)}{y} = 0$ *such that* $f(x) = a_1 + |y|^{\alpha^1_f} H^1_f(y)$, *for all* $y \in [-1/2, 0)$ *and* $f(y) = a_2 + |y|^{\alpha^2_f} H^2_f(x)$, *for all* $y \in (0, 1/2]$, *for some real numbers* a_1 *and* a_2.

To every $f \in \mathscr{A}_L$ we called the *Lorenz-like map*. In $\mathscr{A}_L$ we define a C^1 topology induced by the following metric:

$$\begin{aligned} d_{\mathscr{A}_L}(f,g) \;=\; & \max\Big\{ \sup_y |f(y)-g(y)|, \sup_y |Df(y)-Dg(y)|, \sup_y |H^1_f(y)-H^1_g(y)|, \\ & \sup_y |H^2_f(y)-H^2_g(y)|, \sup_y \frac{1}{|y|}|DH^1_f(y)-DH^1_g(y)|, \\ & \sup_y \frac{1}{|y|}|DH^2_f(y)-DH^2_g(y)|, |\alpha^1_f-\alpha^1_g|, \\ & |\alpha^2_f-\alpha^2_g| : y \in [-1/2,1/2]\backslash\{0\}\Big\}, \end{aligned}$$

for all $f, g \in \mathscr{A}$.

Proposition 2.2. *(Eventually onto,* [4]*). Let $f : [-1/2,1/2]\backslash\{0\} \to [-1/2,1/2]$ be a Lorenz-map. If $J \subset [-1/2,1/2]$ is a subinterval, then, for all g C^1-close to f there exists a integer $\hat{n} = \hat{n}(g,J)$ and a subinterval $\hat{J}$ such that $g^{\hat{n}}(\hat{J}) = [-1/2,1/2]$.*

Proof. Fix the Lorenz-like map $f : [-1/2,1/2]\backslash\{0\} \to [-1/2,1/2]$ and $J \subset I$ and g C^1-close to f.

Let $I_0 = J$ if $0 \notin J$. If $0 \in J$, let I_0 be the bigger of the two intervals 0 splits J into. Define $I_1 = g(I_0)$ if $0 \notin g(I_0)$. If $0 \in g(I_0)$, let I_1 be the bigger of the two intervals 0 splits $g(I_0)$ into.

Suppose that for each i such that I_i is well defined. Let

$$I_{i+1} \begin{cases} f(I_i), & \text{if } 0 \notin f(I_i) \\ \text{bigger of two parts } 0 \text{ splits } f(I_i) \text{ into}, & \text{if } 0 \in f(I_i). \end{cases}$$

Let $\lambda = \min_{x \in I} f'(x) > \sqrt{2}$. By Mean Value Theorem,

$$length(g(I_{i+1})) = g'(\xi_0) length(I_{i+1}), \text{for some } \xi_0 \in I_{i+1}.$$

Therefore for all $g \in \mathscr{V}_\eta$ (η-neighborhood), we get $g'(\xi_0) > f'(\xi_0) - \eta$

$$length(g(I_{i+1})) \geq (\lambda - \eta) length(I_{i+1}).$$

Therefore if $0 \notin g(I_i)$ and $0 \in g(I_{i+1})$ we get

$$length(I_{i+2}) > \frac{(\lambda-\eta)^2}{2} length(I_i).$$

Define $\hat{\lambda}$ such that
$lim_{\eta \to 0^+}(\lambda - \eta) > \hat{\lambda} > \sqrt{2}$. Thus

$$length(I_{i+2}) \geq \frac{\hat{\lambda}^2}{2} length(I_i).$$

But as $\frac{\hat{\lambda}^2}{2} > 1$, this last inequality cannot always hold.

Then there exists an integer $n = n(J, g)$ such that $f(I_{n-1})$ contain 0 and an end point of I. So, either $g(I_{n-1}) = [-1/2, 0]$ or $g(I_{n-1}) = [0, 1/2]$. We suppose that $g(I_{n-1}) = [0, 1/2]$ then $g(I_n)$ contains to $[-1/2, 0]$. Therefore, $g^3(I_n) = I$. Thus, there exists an integer $\hat{n}$ and a subinterval $\hat{J} \subset J$ such that $g^{\hat{n}}(\hat{J}) = [-1/2, 1/2]$. The proof follows. □

Proposition 2.3. *Let $f : [-1/2, 1/2]\backslash\{0\} \to [-1/2, 1/2]$ be a Lorenz-map. Then all g C^1-close to f is hyperbolic, i.e., there exists $\lambda > 1$ such that for all $n \in \mathbb{N}$, for all y with $g^j(y) \in [-1/2, 1/2]$. $0 \le j \le n-1$, $|Dg^n(y)| \ge \lambda^n$.*

Proof. This is a consequence of the hyperbolicity of f and by the fact that its is a open property. □

Proposition 2.4. *Let $X \in \mathscr{X}^\infty(M, \partial M)$ and k as above. Then for all Y C^k-close to X, the map $g_Y(\cdot)$ satisfies the following properties:*

(a) *$g_X(\cdot) \in \mathscr{A}_L$;*
(b) *$g_Y(\cdot)$ is $d_{\mathscr{A}_L}$ close to $g_X(\cdot)$.*

Proof. Let Y C^k close to X. We define $\Pi_{out,Y} = \Pi^+_{out,Y}$

By Taylor expansion near $q_0(Y) = (0, 0)$,

$$\Pi_{out,Y}(\bar{x}, \bar{y}) = \Pi_{out}(q_0(Y)) + D\Pi_{out,Y}.(\bar{x}, \bar{y}) + \mathscr{O}_{out,Y}(\bar{x}, \bar{y}),$$

where

$$D\Pi_{out,Y}(q_0(Y)) = \begin{bmatrix} a_Y & b_Y \\ c_Y & d_Y \end{bmatrix}$$

with $a_Y, b_Y, c_Y, d_Y \in \mathbb{R}$ and $\lim_{(\bar{x},\bar{y})\to(0,0)} \frac{\Theta_{out,Y}(\bar{x},\bar{y})}{\|(\bar{x},\bar{y})\|} = 0$.

As $D\Pi_{out,Y}(q_1(Y)) \cdot e_1 = (a_Y, c_Y)$ and $D\Pi_{out,Y}(q_1(Y)) \cdot -e_2 = (-b_Y, -d_Y)$, by assumption about orientation of $D\Pi_{out,Y}$, we have that $a_Y > 0$ b_Y and d_Y are negatives and $c_Y = 0$. Also we have that $\Pi_{out,Y}(q_1(Y)) = (x_0(Y), y_0(Y))$. Then

$$\begin{aligned} \Pi_{out,Y}(\bar{x}, \bar{y}) &= (x_0(Y) + a_Y \cdot \bar{x} + b_Y \cdot \bar{y} + \Theta_{1,Y}(\bar{x}, \bar{y}), \\ &\quad\ y_0(Y) + d_Y \cdot \bar{y} + \Theta_{2,Y}\bar{x}, \bar{y}), \end{aligned}$$

where $\Theta_{out,Y} = (\Theta_{1,Y}, \Theta_{2,Y})$.

Note that $\Theta_{2,Y}(\bar{x}, 0) = 0$.

Remember that

$$\Pi^+_{loc,Y}(x, y) = (axy^{\alpha_0(Y)+\beta_0(Y)\beta_1(Y)}, by^{\beta_0(Y)\alpha_1(Y)}) \text{ if } y > 0;$$

Then

$$R_Y(x, y) = (x_0(Y) + a_Y \cdot \bar{x} + b_Y \cdot \bar{y} + \Theta_{1,Y}(\bar{x}, \bar{y}), y_0(Y) + d_Y \cdot \bar{y} + \Theta_{2,Y}(\bar{x}, \bar{y})),$$

where,

$$\begin{aligned} \bar{x} &= axy^{\alpha_0(Y)+\beta_0(Y)\beta_1(Y)} \\ \bar{y} &= by^{\beta_0(Y)\alpha_1(Y)}. \end{aligned}$$

Note that $H_Y(x,y) = x_0(Y) + a_Y \cdot \bar{x} + b_Y \cdot \bar{y} + \Theta_{1,Y}(\bar{x},\bar{y})$ and $f_Y(y) = y_0(Y) + d_Y \cdot \bar{y} + \Theta_{2,Y}(\bar{x},\bar{y})$ therefore,

$$\begin{aligned} f_Y(y) &= y_0(Y) + d_Y \cdot b y^{\beta_0(Y)\alpha_1(Y)} + \Theta_{2,Y}(\bar{x},\bar{y}) \\ &= y_0(Y) + y^{\beta_0(Y)\alpha_1(Y)}\Big(d_Y \cdot b + \frac{\Theta_{2,Y}(\bar{x},\bar{y})}{d_y b}\Big) \\ &= y_0(Y) + y^{\beta_0(Y)\alpha_1(Y)} H_g^1(y), \end{aligned}$$

where $H_g^1(y) = d_Y b + \frac{\Theta_{2,Y}(\bar{x},\bar{y})}{d_y b}$. Note on the one hand that

$$f_y'(y) = y^{\beta_0(Y)\alpha_1(Y)-1}\Big(\beta_0(Y)\alpha_1(Y) H_g^1(y) + \frac{DH_g^1(y)}{y}\Big)$$

Also we obtain ,

$$f_y'(y) = y^{\beta_0(Y)\alpha_1(Y)-1}\Big(\beta_0(Y)\alpha_1(Y) a_y b + A(y) + B(y)\Big),$$

were

$$A(y) = \frac{\partial\Theta_{2,Y}(\bar{x},\bar{y})}{\partial\bar{x}} \frac{a(\alpha_0(Y) + \beta_0(Y)\beta_1(Y))\bar{x} y^{\alpha_0(Y)+\beta_0(Y)\beta_1(Y)-\beta_0(Y)\alpha_1(Y)}}{(\beta_0(Y)\alpha_1(Y)) a_y b}$$

and

$$B(y) = \frac{\partial\Theta_{2,Y}(\bar{x},\bar{y})}{\partial\bar{y}} \frac{1}{a_Y}.$$

As $\lim_{y\to 0} H_g^1(y) = a_Y b$ then $\lim_{y\to 0} \frac{DH_g^1(y)}{y} = 0$. Therefore $\lim_{y\to 0} f_Y'(y) \to \infty$.
The estimates for $\Pi^{-}_{out,Y}$ is in similar form.
Therefore under iterations of the maps f_Y and f_X we obtain the result required. □

We need to introduce some definition related to cone fields. Denote by $T\Sigma_0$ the tangent bundle of Σ_0. Given $p \in \Sigma_0$ and $\gamma > 0$, we denote by $C_H^\gamma(p)$ the horizontal γ-cone with inclination γ, i.e.,

$$C_H^\gamma(p) = \{v \in T_p\Sigma_0 : v = (u,w); |w| \le \gamma \cdot |u|\}.$$

Also, we denote by $C_V^\gamma(p)$ the vertical γ-cone with inclination γ, i.e.,

$$C_V^\gamma(p) = \{v \in T_p\Sigma_0 : v = (u,w); |u| \le \gamma \cdot |w|\}.$$

A γ-cone field in Σ_0 is a continuous map $C^\gamma : p \in \Sigma_0 \mapsto C^\gamma(p) \subset T_p\Sigma_0$, where $C^\gamma(p)$ is a γ-cone with constant inclination γ on $T_p\Sigma_0$. Let $R : \Sigma_0 \to \Sigma_0$ be any map. A γ-cone field C^γ is called *R-invariant* if $DR(C^\gamma(p) \setminus \{0\}) \subset int(C^\gamma(R(p)))$ for all $p \in \Sigma_0$. A γ-cone field C^γ is called *R-expanding* if there are $C > 0$ and $\lambda > 1$ such that $\| DR^n(p) \cdot v \| \ge C \cdot \lambda^n \cdot \| v \|$, $\forall n \in \mathbb{N}$, $\forall p$ with $R^j(p) \in \Sigma_0$, $0 \le j \le n-1$ and $\forall v \in C^\gamma(p)$. A γ-cone field C^γ is called *transversal* to a foliation $\mathscr{F}$ on Σ_0 if $T_pL \cap C^\gamma(p) = \{0\}$, $\forall p \in L$ and $\forall L \in \mathscr{F}$.

Proposition 2.5. *Let $X \in \mathscr{X}^\infty(M, \partial M)$ and k as above. Then exists γ with $0 < \gamma \leq 1$ such that for all Y C^1-close to X there are invariants γ-cone fields C_H^γ and C_V^γ on Σ_0. Moreover C_H^γ is R_Y^{-1}-expanding and C_V^γ is R_Y-expanding and transversal to the foliation $\mathscr{F}$.*

Proof. Fix $X \in \mathscr{X}^\infty(M, \partial M)$ as in the Statements of Proposition. Then there exists a γ with $0 < \gamma \leq 1$ such that for all Y C^1-close to X there is a horizontal γ-cone field C_H^γ on Σ which is invariant and expanding by R_Y^{-1} (see [2]). □

From now on we fix such γ, C_H^γ and C_V^γ.

Proposition 2.6. *Let $X \in \mathscr{X}^\infty(M, \partial M)$ and k as above. Then, there exist a neighborhood $\mathscr{V}$ of X in $\mathscr{X}^k(M, \partial M)$ such that: For all $Y \in \mathscr{V}$, R_Y is hyperbolic on Σ_0. More precisely, there are $C_1 = C_1(Y) > 0$ and $\lambda_1 = \lambda_1(Y) > 1$ such that $\forall n \in \mathbb{N}$, $\forall p$ with $R_Y^i(p) \in \Sigma_0$, $-n+1 \leq i \leq 0$ and $\forall v \in C_H^\gamma(p)$ we have*

$$\| DR_Y^{-n}(p) \cdot v \| \geq C_1 \cdot \lambda_1^n \cdot \| v \| \tag{2.7}$$

and there are $C_2 = C_2(Y) > 0$ and $\lambda_2 = \lambda_2(Y) > 1$ such that $\forall n \in \mathbb{N}$, $\forall p$ with $R_Y^i(p) \in \Sigma_0$, $0 \leq i \leq n-1$ and $\forall v \in C_V^\gamma(p)$ we have

$$\| DR_Y^{n}(p) \cdot v \| \geq C_2 \cdot \lambda_2^n \cdot \| v \| \ . \tag{2.8}$$

Proof. Fix $X \in \mathscr{X}^\infty(M, \partial M)$ as in the statement of Proposition.

By Proposition 2.5 we can choice a neighborhood $\mathscr{V}_1$ of X in $\mathscr{X}^k(M, \partial M)$ such that for all $Y \in \mathscr{V}_1$ the γ-cone fields C_H^γ and C_V^γ exist. Moreover, C_H^γ is invariant and expanding by R_Y^{-1} and C_V^γ is R_Y-invariant and transversal to $\mathscr{F}$.

By Proposition 2.4, the one dimensional map g_X is a C^1, belongs to $\mathscr{A}_L$. Then by Proposition 2.4 and 2.3 , there exists a neighborhood $\mathscr{U}$ of g_X in $\mathscr{A}_L$ such that each $g \in \mathscr{U}$ is hyperbolic. We can choice a neighborhood $\mathscr{V}_2$ of X in $\mathscr{X}^k(M, \partial M)$ in a such a way that for all $Y \in \mathscr{V}_2$, g_Y belongs to $\mathscr{U}$ because item (b) in Proposition 2.4. Define $\mathscr{V} = \mathscr{V}_1 \cap \mathscr{V}_2$.

Fix $Y \in \mathscr{V}$. The existence of C_1 and λ_1 (which depend only on Y) satisfying the inequality (2.7) is a consequence of the fact that C_H^γ is R_Y^{-1}-invariant and R_Y^{-1}-expanding.

Now we will prove the remainder of the Proposition 2.6. Indeed, as $g_Y \in \mathscr{U}$ we have that there exists $\lambda = \lambda(Y) > 1$ such that for all $n \in \mathbb{N}$, for all y with $g_Y^i(y) \in [-1/2, 1/2] \backslash \{0\}$, $0 \leq i \leq n-1$ we obtain

$$|Dg_Y^n(y)| \geq \lambda^n. \tag{2.9}$$

Define

$$C_2 = \frac{1}{\max\{\gamma, 1\}} \text{ and } \lambda_2 = \lambda.$$

Fix $n \in \mathbb{N}$, $p = (x, y)$ with $R^i(p) \in \Sigma_0$, $0 \leq i \leq n-1$ and $v \in C_V^\gamma(p)$ with $v = (u, w)$. Then have that $g_Y^i(y) \in [-1/2, 1/2] \backslash \{0\}$, $0 \leq i \leq n-1$.

Moreover, from (2.5) we have

$$DR_Y^n(p)v = (u_n, w_n) = (u_n, \partial_y g_Y^n(y) w).$$

Therefore, from this equality, the inequality (2.9) and definitions of C_2 and λ_2 we get that

$$\begin{aligned}\| DR_Y^n(p)\cdot v \| &= \max\{|u_n|, |\partial_y g_Y^n(y)|\cdot|w|\}\\ &\geq |\partial_y g_Y^n(y)|\cdot|w|\\ &\geq \frac{|\partial_y g_Y^n(y)|}{\max\{\gamma,1\}}\cdot \| v \|\\ &\geq \frac{C}{\max\{\gamma,1\}}\cdot\lambda^n\cdot \| v \|\\ &= C_2\cdot\lambda_2^n\cdot \| v \|,\end{aligned}$$

where $||\cdot||$ denote the maximum norm. This shows (2.8) and finishes the proof. □

Proposition 2.7. *Let $X \in \mathscr{X}^\infty(M,\partial M)$ and k as above. Then, there exist a neighborhood $\mathscr{W}$ of X in $\mathscr{X}^k(M,\partial M)$ such that: For all $Y \in \mathscr{W}$, for all smooth curve ζ tangent to C_V^γ with $\zeta\cap(\bigcap_{i=0}^{\infty} R_Y^{-i}(\Sigma_0)) \neq \emptyset$ there exist an integer $n = n(R_Y,\zeta)$ and a smooth curve $\hat\zeta$ contained in ζ such that R_Y^n is continuous on $\hat\zeta$ and $\Pi^{\mathscr{F}}(R_Y^n(\hat\zeta)) = [-1/2,1/2]$.*

Proof. Fix $X \in \mathscr{X}^\infty(M,\partial M)$ as in the statement of Proposition 2.7. Note that X has associate an one dimensional map g_X. From Proposition 2.4 (see statements (a)) we get that g_X is a C^1 map, $g_X \in \mathscr{A}_L$ and g_X has derivative bigger $\sqrt{2}$. Then by proposition 2.2, there exists a $d_{\mathscr{A}_L}$-neighborhood $\mathscr{U}$ of g_X in $\mathscr{A}_L$ such that for all $g\in\mathscr{U}$ and for all interval $J\subset I$ there exist $n = n(g,J)\geq 0$ such that $g^n(J) = [-1/2,1/2]$. We can choice a neighborhood $\mathscr{W}$ of X in $\mathscr{X}^k(M,\partial M)$ in a such away that for all $Y\in\mathscr{W}$, g_Y belongs to $\mathscr{U}$ because item (b) in Proposition 2.4.

Fix $Y\in\mathscr{W}$ and a curve ζ tangent to C_V^γ with $\zeta\cap(\bigcap_{i=0}^{\infty} R_Y^{-i}(\Sigma_0)) \neq \emptyset$ and define $J = \Pi^{\mathscr{F}}(\zeta)$. Therefore there exist an integer $n = n(g_Y,J)$ such that $g_Y^n(J) = [-1/2,1/2]$. So, we obtain $\Pi^{\mathscr{F}}(R_Y^n(\hat\zeta)) = g_Y^n\Big(\Pi^{\mathscr{F}}(\hat\zeta)\Big) = g_Y^n(\hat J) = [-1/2,1/2]$. The Proposition follows. □

Now, we will prove that for all $Y\in\mathscr{V}\cap\mathscr{W}$, Λ_Y is a transitive set, where $\mathscr{V}$ is given in Proposition 2.6 and $\mathscr{W}$ is given in Proposition 2.7. So, we need to prove that the maximal invariant set $\bigcap_{n\in\mathbb{Z}} R_Y^n(\Sigma_0)$ given in (2.6) is a transitive for R_Y.

Claim A: For all $p\in\bigcap_{n\geq 0} R^{-n}(\Sigma_0)$ the stable leaf $L = \mathscr{F}(p)\in\mathscr{F}$ is accumulate by hyperbolic periodic points of saddle type, i.e. every neighborhood of L contains a hyperbolic periodic point of saddle type.

Indeed, let U a neighborhood of L. We can take U in a such way that $U = (\Pi^{\mathscr{F}})^{-1}(\Pi^{\mathscr{F}}(U))$. Take a small curve $\zeta\subset U$ through p and tangent to C_V^γ. From Proposition 2.7 there are $\hat\zeta\subseteq\zeta$ and $n\in\mathbb{N}$ such that $R_Y^i(\hat\zeta)\subseteq\Sigma_0$ $\forall 0\leq i\leq n-1$

and $R_Y^n(\hat{\zeta})$ meets all leaf in $\mathscr{F}$. Thus, for $\hat{J} = \Pi^{\mathscr{F}}(\hat{\zeta})$ we have that $\hat{J} \subset g_Y^n(\hat{J})$. Then there is $y \in \hat{J}$ such that $g_Y^n(y) = y$ and so $L(y) = (\Pi^{\mathscr{F}})^{-1}(y) \subset U$ is a periodic leaf of $\mathscr{F}$. This implies that there exists a periodic point of R_Y belonging to $L(y) \subset U$. By Proposition 2.6 this periodic point is hyperbolic of saddle type. This proves the claim A.

Claim B: The hyperbolic periodic points of saddle type of R_Y are dense in $\bigcap_{n\in\mathbb{Z}} R_Y^n(\Sigma_0)$.

Indeed, take a point $z \in \bigcap_{n\in\mathbb{Z}} R_Y^n(\Sigma_0)$ and take a neighborhood V of z. Take an integer n large enough such that $L = \mathscr{F}(R_Y^{-n}(z))$ the leaf that contains $R_Y^{-n}(z)$ is applied by R_Y^n into V. So the same applies to a small horizontal band U around the leaf L. By Claim A there exists a periodic point of saddle type in U. Therefore the orbit of this periodic point visits the neighborhood V and the claim B follows.

To finish the proof of the transitivity of R_Y we will use the classical Birkhoff's criterium to prove transitivity: for all $p, q \in \bigcap_{n\in\mathbb{Z}} R_Y^n(\Sigma_0)$ and $\varepsilon > 0$ there are $z \in \bigcap_{n\in\mathbb{Z}} R_Y^n(\Sigma_0)$ and $n_z \in \mathbb{N}$ such that $d(z,p) < \varepsilon$ and $d(R_Y^{n_z}(z), q) < \varepsilon$. Indeed, fix p, q and ϵ. By the above claim B we can assume that p and q are hyperbolic periodic points of saddle type. Fix a curve γ in $W^u(p)$ contained in Σ. We can assume that γ intersects to the leaf $\mathscr{F}(q)$ transversely in some point z^* by Proposition 2.7. Since the positive (resp. negative) orbit of z^* is asymptotic to q (resp. p) we have $z^* \in \bigcap_{n\in\mathbb{Z}} R_Y^n(\Sigma_0)$. By taking the negative orbit of z^* we have some $n_1^* \in \mathbb{N}$ such that

$$d(R_Y^{-n_1^*}(z^*), p) < \varepsilon.$$

By taking the positive orbit of z^* we have some $n_2^* \in \mathbb{N}$ such that

$$d(R_Y^{n_2^*}(z^*), q) < \varepsilon.$$

Then $z = R_Y^{-n_1^*}(z^*)$ and $n_z = n_1^* + n_2^*$ works.

Therefore, the proof of Fist step, Second step and Third step follows. Therefore Λ is C^k-robust transitive set. The proof follows.

■

The following lemma is as in [8].

Lemma 2.8. *$F_{\sigma_0} = E^{ss}_{\sigma_0}$, $F^c_{\sigma_0} = E^{cu}_{\sigma_0}$, $E^{ss}_{\sigma_1} = F^s_{\sigma_1}$ and $F^c_{\sigma_1} = E^c_{\sigma_1}$.*

Proof. This a consequence of the uniqueness of the dominates splitting $T_\Lambda M = F^s_\Lambda \oplus F^C_\Lambda$. □

Remark 2.9. *As $F_{\sigma_0} \not\subset T_{\sigma_0}\partial M$ then by the invariance of $T\partial M$ and Lemma 2.8 we have that $F^c_{\sigma_0} = T_{\sigma_0}\partial M$.*

Lemma 2.10. *The invariant transitive set Λ does not has dominates splitting.*

Proof. By contradiction we suppose that Λ has a dominated splitting $T_\Lambda M = F^s_\Lambda \oplus F^C_\Lambda$. Take $x \in \gamma \subset W^u(\sigma_0) \cap W^s(\sigma_1)$ close to σ_0 then $F^s_x \not\subset T_x\partial M$ by continuity of the splitting $T_\Lambda M = F^s_\Lambda \oplus F^C_\Lambda$ and Remark 2.9. Moreover note that $W^u(\sigma_0) \subset \partial M$ and $W^s(\sigma_1) \subset \partial M$. We take $v_x \in T_x\partial M \backslash F^s_x$ with $v_x \neq 0$. Using the fact that the splitting $T_\Lambda M = F^s_\Lambda \oplus F^c_\Lambda$ is dominated and $F^s_x \not\subset T_x\partial M$ we have that angle $\angle(DX_t(x) \cdot v_x, F^c_{X_t(x)})$ goes to 0 exponentially as $t \to \infty$ (see also Remark 2.2 [8]). Moreover, as $DX_t(x) \cdot v_x \in T_{X_t(x)}\partial M$, then using the continuity of the dominates splitting and $\omega(x) = \sigma_1$ we get $\angle T_{\sigma_1}\partial M, F^c_{\sigma_1}) = 0$ and so $F^c_{\sigma_1} = T_{\sigma_1}\partial M$ because that $dim(T_{\sigma_1}\partial M) = dim(F^c_{\sigma_1})$ and this is a contradiction with $F^c_{\sigma_1} = E^c_{\sigma_1}$ (see Lemma 2.8). The proof follows. □

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