ROBUST TRANSITIVE SETS OF TRIANGULAR MAPS

D. CARRASCO-OLIVERA, B. SAN MARTÍN AND C. VIDAL

Abstract. In [BM] it was proved that discontinuous triangular maps with specific expanding hypotheses carry hyperbolic periodic orbits. Here we define a class of discontinuous triangular maps with contracting hypotheses and prove that they carry robustly transitivity sets with respect to a suitable weight topology.

1. Introduction

By a *triangular map* we mean a transformation

$$R : Dom(R) \subset X \times Y \to X \times Y$$

with domain $Dom(R) = A \times Y$, $A \subset X$, having the form

$$R(x, y) = (f(x), g(x, y))$$

for some $f : A \subset X \to X$ (throughout called the *base map*) and $g : Dom(R) \to Y$. We shall be interested in the case when both X and Y are the unit interval $I = [0, 1]$ and, in such a case, $\Sigma = X \times Y$ is the unit square. We say that R is *everywhere defined* if $A = X$.

Ccontinuous everywhere defined triangular maps have been considered in the literature. For instance, in 1989, A.N. Sharkovsky (at the European Conference on Iteration Theory, ECIT'89, Batschuns, Austria) posed the problem of extending properties of the topological entropy from continuous interval maps to continuous triangular maps. For a chronological list of authors whom contributed to this and

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other interesting problems, see [S], [K1], [K2], [DJ], [BS], [D], [BGM], [FPS1], [FPS2], [V].

On the other hand, discontinuous (or not everywhere defined) triangular maps have been considered by some authors. A pionering work in this direction is [G] (see also [GW] or [ABS]) where smooth triangular maps defined everywhere except in a single vertical line were studied. In particular, these authors obtained robustly transitive sets under certain expanding conditions including their famous lower bound $\sqrt{2}$ for the derivative of the base map f. We can also mention [Ro] where a contracting condition was imposed and where a two-parameter persistence of robustly transitivity was obtained instead. The mixed case of discontinous triangular maps not everywhere defined was considered in [BM]. In that paper it were imposed certain expanding hypotheses in order to obtain the existence of periodic orbits. This last conclusion was recently improved in [Re] where existence of homoclinic orbits associated to hyperbolic periodic orbits was proved instead. Further information about triangular maps and their relationship with vector fields can be found in [B], [LP], [Ro], [PR], [BLMP].

In this paper we introduce a family $\mathcal{T}$ of C^1 but not everywhere defined triangular maps in Σ exhibiting contracting properties in the spirit of [Ro]. We also define a weight topology in $\mathcal{T}$ and select a proper subset $\tilde{\mathcal{T}}$ of $\mathcal{T}$. It is proved that there is an open neighborhood of $\tilde{\mathcal{T}}$ in $\mathcal{T}$ with respect to that topology all of whose elements have transitive maximal invariant set.

The organization of this paper is as follows. In Section 2 we introduce the class of triangular maps we shall be interested in and present the statement of our main result. In Section 3 we prove exponential growth of derivatives for the triangular maps under consideration. Finally in the last section we will prove our result.

2. Triangular maps and Statement of the Main Theorem.

In this section we introduce the class of triangular maps we shall be interested in.

2.1. **Triangular maps.**

Recall $\Sigma = [0,1] \times [0,1]$ denotes the unit close square and $U \subset \mathbb{R}^2$ an open set containing Σ.

Denote by $p = (x, y) = (x_p, y_p)$ the natural coordinate system in U.

Fix two real numbers a, b with $0 < a < b < 1$.

Put

$$L_0 = \{(x,y) : 0 \leq x \leq 1, y = 0\};\ L_a = \{(x,y) : 0 \leq x \leq 1, y = a\};$$

$$L_b = \{(x,y) : 0 \leq x \leq 1, y = b\} \text{ and } L_1 = \{(x,y) : 0 \leq x \leq 1, y = 1\}.$$

We call a closed subset $H \subseteq \Sigma$ (resp. $V \subseteq \Sigma$) a *horizontal* (resp. *vertical*) *band* if it is bounded by two disjoint continuous curves connecting the vertical (resp. horizontal) slides of Σ, $\{(0, y) : 0 \leq y \leq 1\}$ (resp. L_0) and $\{(1, y) : 0 \leq y \leq 1\}$ (resp. L_1). See Figure 1.

Given a map R, we denote by Dom(R) the domain of R.

A *curve* c in U is the image of a C^1 injective map $c : Dom(c) \subset \mathbb{R} \to U$ with $Dom(c)$ being a compact interval. We often identify c with its image set. A curve c is *horizontal* if it is the graph of a C^1 map $h : [0,1] \to U \cap \mathbb{R}$, i.e., $c = \{(x, h(x)) : x \in [0,1]\} \subset U$.

Let M be a differentiable m dimensional manifold, $m > 0$. A foliation $\mathscr{F}$ of dimension n, $0 < n < m$, is a decomposition of M in n-dimensional submanifolds, called leaves of the foliation. The foliation $\mathscr{F}$ is C^k, $k \geq 0$, if the holonomy map defined in transversal cross-sections is a C^k map. If $k = 0$ we said that the $\mathscr{F}$ is a *continuous foliation.*

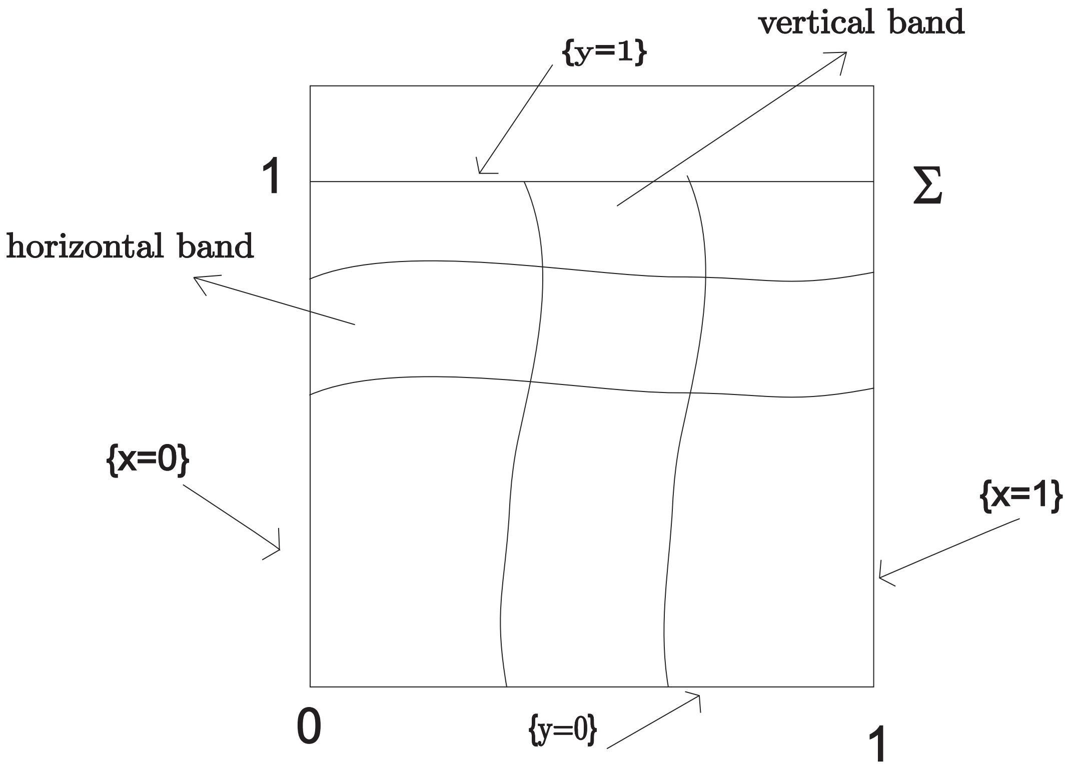

Figure 1. Shape of vertical and horizontal band.

Definition 2.1. *A continuous foliation $\mathscr{F}$ on U is called horizontal if its leaves are horizontal curves and the curves L_0, L_a, L_b, L_1 are leaves of $\mathscr{F}$.*

It follows from the definition above that the leaves of a horizontal foliation $\mathscr{F}$ are horizontal curves hence differentiable ones. In particular, for all leaf L, the tangent space T_pL is well defined for all $p \in L$.

Let H_{y_0,y_1} denote the horizontal band $[0,1] \times [y_0, y_1]$.

Definition 2.2. *Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ be a map and $\mathscr{F}$ be a continuous foliation on U. We say that R* preserves *$\mathscr{F}$ if for every leaf L of $\mathscr{F}$ contained in $H_{0,a} \cup H_{b,1}$ there is a leaf $\tilde{L}$ of $\mathscr{F}$ such that $R(L) \subset \tilde{L}$). In this case we say that $\mathscr{F}$ is contracting if there are a constant $C > 0$ and $0 < \lambda < 1$ such that*

$$\| DR^n(p) \cdot v \| \leq C \cdot \lambda^n,$$

for all $n \in \mathbb{N}$, $p \in L$, $L \in \mathscr{F}$ and $v \in T_pL$.

Now we can define triangular map.

Definition 2.3. ***(Triangular map).*** *A map $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ is called triangular if it preserves and contrae a horizontal foliation $\mathscr{F}$. Moreover $R(L_0) \subset ([0,1) \times \{0\})$, $R(L_a) \subset U \backslash \Sigma$, $R(L_b) \subset U \backslash \Sigma$ and $R(L_1) = \{(c_0, 0)\}$ for some $c_0 \in (0,1)$.*

Notation. Let $\mathcal{T}$ denote the set of all the triangular maps R.

2.2. **Quasi-hyperbolic triangular maps.** The Quasi-hyperbolicity will be defined through cone fields in U: Denote by TU the tangent bundle of U. Given $p \in U$ and $\gamma > 0$, we denote by $C^\gamma(p)$ the vertical cone with inclination γ, i.e.,

$$C^\gamma(p) = \{v \in T_pU : v = (u, w); |u| \leq \gamma \cdot |w|\}.$$

A *cone field* in U is a continuous map $C^\gamma : p \in U \mapsto C^\gamma(p) \subset T_pU$, where $C^\gamma(p)$ is a cone with constant inclination γ on T_pU. Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ be any differentiable map. A cone field C^γ is called *R-invariant* if $DR(C^\gamma(p)\backslash\{0\}) \subset int(C^\gamma(R(p)))$ for all $p \in H_{0,a} \cup H_{b,1}$. Moreover, C^γ is called *transversal* to a horizontal foliation $\mathscr{F}$ on U if $T_pL \cap C^\gamma(p) = \{0\}$, $\forall p \in L$ and $\forall L \in \mathscr{F}$.

Now we can to define the class of triangular map satisfying contracting hypotheses.

Definition 2.4. ***(Quasi-hyperbolic triangular map).*** *Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ be a triangular map with associated horizontal foliation $\mathscr{F}$. For two maps $\alpha, \tilde{\alpha} : \mathcal{T} \to (1, \infty)$ and numbers K_0, K_1, ν and μ such that $K_0 > 0$, $K_1 > 0$ and $1 < \nu \leq \mu$, we say that R is $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi hyperbolic if*

(H1) *R is a C^1-diffeomorphism in $H_{0,a} \cup (H_{b,1} \setminus \{y = 1\})$.*
(H2) *ν, μ, $\alpha = \alpha(R)$ and $\tilde{\alpha} = \tilde{\alpha}(R)$ satisfy: $\nu \cdot \mu^{\frac{1-\tilde{\alpha}}{\alpha}} > 1$.*
(H3) $y_{R(p)} \leq K_0 \cdot \mid y_p - 1 \mid^\alpha, \qquad \forall p \in H_{b,1}$.

(H4) *There are* $0 < \gamma < \frac{1}{2}$ *and an invariant cone field* C^γ *in* U *transverse to* $\mathscr{F}$ *such that:*

(H4-a) $\forall p \in H_{b,1}, \forall v \in C^\gamma(p)$*;*

$$\| DR(p) \cdot v \| \geq K_1 \cdot \mid y_p - 1 \mid^{\tilde{\alpha}-1} \cdot \| v \| \, .$$

(H4-b) $\forall p \in H_{0,a}, \forall v \in C^\gamma(p)$

$$\nu \cdot \| v \| \leq \| DR(p) \cdot v \| \leq \mu \cdot \| v \| \, .$$

Notation. Let $\tilde{\mathcal{T}}$ denote the set of all the maps R which are $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolics.

Figure 2 displays the essential features of the map $R \in \tilde{\mathcal{T}}$.

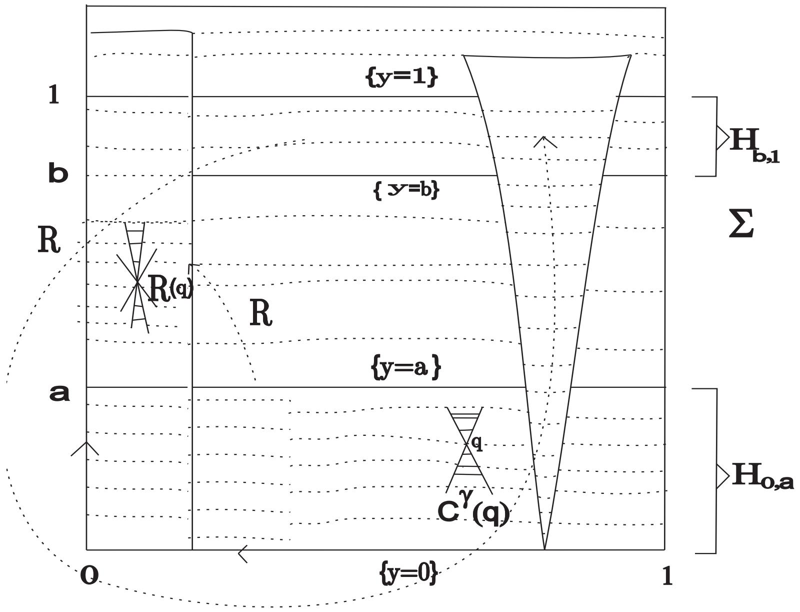

FIGURE 2. Shape of R.

2.3. **Schwarzian derivative.**

Definition 2.5. *Let* $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ *be a* C^3 *map. The Schwarzian derivative of* f *at* $x \in dom(f)$ *with* $Df(x) \neq 0$ *is defined as*

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2} \cdot \Big(\frac{D^2 f(x)}{Df(x)} \Big)^2 .$$

We say that f has negative Schwarzian derivative if $Sf(x) < 0$ for all $x \in Dom(f)$ such that $Df(x) \neq 0$.

From the definition 2.5, the following formula for the Schwarzian derivative of the composition of two C^3 maps follows immediately by the chain rule,

$$S(g \circ f)(x) = Sg(f(x)) \cdot \mid Df(x) \mid^2 + Sf(x).$$

Hence, the Schwarzian derivative of the iterates of f is given by

$$Sf^n(x) = \sum_{i=0}^{n-1} Sf(f^i(x)) \cdot \mid Df^i(x) \mid^2 .$$

Therefore, if a map has negative Schwarzian derivative, so do all its iterates.

The lemma below (see [MS], pg. 154) is the main analytical property of maps of negative Schwarzian derivative that will be used in the section 3.

Lemma 2.6. ***(Minimum Principle).*** *Let T be a closed interval with end points r, s and $f : T \subset dom(f) \to \mathbb{R}$ be a map with negative Schwarzian derivative. If $Df(x) \neq 0$ for all $x \in T$, then*

$$\mid Df(x) \mid > \min\{\mid Df(r) \mid, \mid Df(s) \mid\}, \qquad \forall x \in (r, s).$$

2.4. **Hypothesis (H).** We impose some regularity on a horizontal foliation $\mathscr{F}$ associate to a triangular map.

To any horizontal foliation $\mathscr{F}$ we can associate the *holonomy map* $\Pi^{\mathscr{F}} : U \to \mathbb{R}$ defined by

$$\Pi^{\mathscr{F}}(p) = \mathscr{F}(p) \cap \{0\} \times \mathbb{R},$$

where $\mathscr{F}(p)$ is the leaf of $\mathscr{F}$ through the point p.

A R-invariant horizontal foliation $\mathscr{F}$ can be used to define a new coordinate system

$$(\bar{x}, \bar{y}) = \varphi(x, y) = (x, \Pi^{\mathscr{F}}(x, y)) \tag{2.1}$$

in a such way that $\bar{R} = \varphi \circ R \circ \varphi^{-1}$ has the following shape:

$$\bar{R}(\bar{x}, \bar{y}) = (g(\bar{x}, \bar{y}), f(\bar{y})) \tag{2.2}$$

holds for some maps $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ and $g : dom(g) \subset \mathbb{R}^2 \to \mathbb{R}$.

Definition 2.7. ***(linear-contracting map).*** *Let $f : [0, a] \cup [b, 1] \subset \mathbb{R} \to \mathbb{R}$ be a C^3 map. We say that its is a linear-contracting if:*

- (h1) *There exists $\rho_f > 1$ such that $f(x) = \rho_f \cdot x$ for all $x \in [0, a]$. Moreover, f is decreasing on $[b, 1]$, $f(1) = 0$ and $Df(x) = 0$ if and only if $x = 1$. Additionally $f(a) > 1$ and $f(b) > 1$.*
- (h2) *f has negative Schwarzian derivative on $[b, 1)$.*

Figure 3 displays the essential features of a linear-contracting map f.

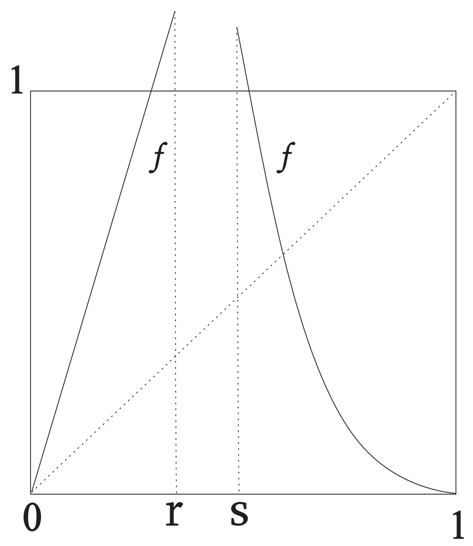

FIGURE 3. Shape of f.

Definition 2.8. ***(Hypothesis (H)).*** *Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ be a triangular map with associate horizontal foliation $\mathscr{F}$ of class C^3. We say that R satisfies (H) if the map f given by (2.2) is linear-contracting and $\Pi^{\mathscr{F}}$ additionally satisfies the property:*

$$\left|\frac{\partial \Pi^{\mathscr{F}}}{\partial x}(x, y)\right| < \frac{1}{2} \text{ and } \left|\frac{\partial \Pi^{\mathscr{F}}}{\partial y}(x, y)\right| > \frac{3}{4}.$$

2.5. Definition of the C^1-topology in $\tilde{\mathcal{T}}$.

Definition 2.9. *In the space of maps $\tilde{\mathcal{T}}$ we consider the C^1-topology which is defined by the metric*

$$\begin{aligned} d_{C^1}(R,\hat{R}) &= \max\Big\{ \parallel R(p)-\hat{R}(p) \parallel, \parallel DR(p)-D\hat{R}(p) \parallel, \\ &\qquad \mid \alpha(R)-\alpha(\hat{R}) \mid, \mid \tilde{\alpha}(R)-\tilde{\alpha}(\hat{R}) \mid : p \in H_{0,a}\cup H_{b,1}\Big\}. \end{aligned}$$

Our main theorem related to the robust transitive of the principal map to consider is the following:

Theorem 2.10. ***(Main Theorem).*** *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there exists a C^1-neighborhood $\mathscr{U} = \mathscr{U}(R_0)$ of R_0 in $\tilde{\mathcal{T}}$ such that for all $R \in \mathscr{U}$, the maximal invariant set,*

$$\bigcap_{n\in\mathbb{Z}} R^n(\Sigma) \tag{2.3}$$

be transitive, *i.e., there is z on it such that $\{R^n(z) : n \in \mathbb{N}\}$ is dense in the maximal invariant set given by (2.3).*

3. Elementary Results

We start with the following lemma.

Lemma 3.1. *Let R_0 be a triangular map with associate horizontal foliation $\mathscr{F}$ of class C^1 such that $\Pi^{\mathscr{F}}$ satisfies $\left|\frac{\partial \Pi^{\mathscr{F}}}{\partial x}(x,y)\right| < \frac{1}{2}$ and $\left|\frac{\partial \Pi^{\mathscr{F}}}{\partial y}(x,y)\right| > \frac{3}{4}$. Let $(\bar{x},\bar{y}) = \varphi(x,y)$ be the coordinates given by (2.1) and f be the map as in (2.2). Then, there exists a constant $\hat{C}_0 = \hat{C}_0(R_0) > 0$ such that for all $i \in \mathbb{N}$, for all $p \in dom(R_0^i)$ and for all $v \in C^{\gamma}(p)$,*

$$\parallel DR_0^i(p)\cdot v \parallel \geq \hat{C}_0 \cdot \mid Df^i(\bar{y}) \mid \cdot \parallel v \parallel, \tag{3.1}$$

where $(\bar{x},\bar{y}) = \varphi(p)$.

Proof. Fix R_0 as in the statement of the lema. Let consider the coordinates $(\bar{x},\bar{y})$ given by (2.1):

$$(\bar{x},\bar{y}) = \varphi(x,y) = (x, \Pi^{\mathscr{F}}(x,y)). \tag{3.2}$$

So, $\bar{R}_0 = \varphi \circ R_0 \circ \varphi^{-1}$ became the map

$$\bar{R}_0 = (g(\bar{x},\bar{y}), f(\bar{y})) \tag{3.3}$$

according (2.2).

We define

$$M = \max\{\| D\varphi(\hat{z}) \| : \hat{z} \in dom(\varphi)\}.$$

and

$$\hat{C}_0 = \frac{1}{2 \cdot M}.$$

To continue we claim

Claim A. For all $\bar{p} = (\bar{x},\bar{y}) \in dom(\bar{R}_0)$ and for all $\bar{v} = (\bar{u},\bar{w}) \in \mathbb{R}^2$ we obtain

$$\| D\bar{R}_0(\bar{p}) \cdot \bar{v} \| \geq | Df(\bar{y}) | \cdot | \bar{w} | .$$

Indeed, fix $\bar{p} = (\bar{x},\bar{y}) \in dom(\bar{R}_0)$ and $\bar{v} = (\bar{u},\bar{w}) \in \mathbb{R}^2$.

From (3.3) and the definition of Jacobian matrix we obtain

$$D\bar{R}_0(\bar{p}) \cdot \bar{v} = \begin{pmatrix} \frac{\partial g}{\partial \bar{x}}(\bar{p}) \cdot \bar{u} + \frac{\partial g}{\partial \bar{y}}(\bar{p}) \cdot \bar{w} \\ Df(\bar{y}) \cdot \bar{w} \end{pmatrix}.$$

In this equality, we denote by $\tilde{u} = \frac{\partial g}{\partial \bar{x}}(\bar{p}) \cdot \bar{u} + \frac{\partial g}{\partial \bar{y}}(\bar{p}) \cdot \bar{w}$ and $\tilde{w} = Df(\bar{y}) \cdot \bar{w}$. Therefore, considering the maximum norm, we obtain

$$\| D\bar{R}_0(\bar{p}) \cdot \bar{v} \| = \max\{| \tilde{u} |, | \tilde{w} |\}. \tag{3.4}$$

If $| \tilde{u} | \geq | \tilde{w} |$ then of (3.4) and the of the definition of $\tilde{w}$ we have

$$\begin{aligned} \| D\bar{R}_0(\bar{p}) \cdot \bar{v} \| &= \max\{| \tilde{u} |, | \tilde{w} |\} \\ &= | \tilde{u} | \\ &\geq | \tilde{w} | \\ &= | Df(\bar{y}) | \cdot | \bar{w} | . \end{aligned}$$

On the other hand, if $\mid \tilde{u} \mid < \mid \tilde{w} \mid$ then of (3.4) and the of the definition of $\tilde{w}$ we get that

$$\begin{aligned} \| D\bar{R}_0(\bar{p}) \cdot \bar{v} \| &= \max\{\mid \tilde{u} \mid, \mid \tilde{w} \mid\} \\ &= \mid \tilde{w} \mid \\ &= \mid Df(\bar{y}) \mid \cdot \mid \bar{w} \mid . \end{aligned}$$

This proof the Claim A.

Claim B. For all $j \in \mathbb{N}$, for all $\bar{p} = (\bar{x}, \bar{y}) \in dom(\bar{R}_0^j)$ and for all $\bar{v} = (\bar{u}, \bar{w}) \in \mathbb{R}^2$ we have that

$$\| D\bar{R}_0^j(\bar{p}) \cdot \bar{v} \| \geq \mid Df^j(\bar{y}) \mid \cdot \mid \bar{w} \mid . \tag{3.5}$$

Indeed, using induction on j and the Claim A, the proof of (3.5) of Claim B follows.

Now fix $i \in \mathbb{N}$, $p \in dom(R_0^i)$ and $v \in C^\gamma(p)$. We denote by $p = (x, y)$ and $v = (u, w)$.

From (3.2) and definition of Jacobian matrix we obtain

$$D\varphi(p) \cdot v = \begin{pmatrix} u \\ \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \cdot u + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \cdot w \end{pmatrix}.$$

We denote $\bar{u} = u$, $\bar{w} = \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(x, y) \cdot u + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(x, y) \cdot w$ and $\varphi(p) = \bar{p} = (\bar{x}, \bar{y})$.

For other hand, as $\varphi \circ \varphi^{-1} = Id$ then for all $\bar{q} \in dom(\varphi^{-1})$ and for all $\bar{z} \in \mathbb{R}^2$ we have that

$$\begin{aligned} \| \bar{z} \| &= \| D\varphi(\varphi^{-1}(\bar{q})) \cdot D\varphi^{-1}(q) \cdot \bar{z} \| \\ &\leq \| D\varphi(\varphi^{-1}(\bar{q})) \| \cdot \| D\varphi^{-1}(\bar{q}) \cdot \bar{z} \| \end{aligned}$$

therefore we obtain

$$\| D\varphi^{-1}(\bar{q}) \cdot \bar{z} \| \geq \frac{1}{M} \cdot \| \bar{z} \| . \tag{3.6}$$

Remember that $R_0 = \varphi^{-1} \circ \bar{R}_0 \circ \varphi$. Then from Chain Rule Theorem and definition of $\bar{v}$ we obtain

$$\begin{aligned} DR_0^i(p) \cdot v &= D\varphi^{-1}(\bar{R}_0^i(\varphi(p))) \cdot D\bar{R}_0^i(\varphi(p)) \cdot D\varphi(p) \cdot v \\ &= D\varphi^{-1}(\bar{R}_0^i(\varphi(p))) \cdot D\bar{R}_0^i(\varphi(p)) \cdot \bar{v}. \end{aligned} \tag{3.7}$$

From (3.7), the definition of $\bar{p}$, the inequality (3.6) (taking $\bar{q} = \bar{R}_0^i(\varphi(p))$ and $\bar{z} = D\bar{R}_0^i(\bar{p}) \cdot \bar{v}$) and (3.5) of Claim B we get that

$$\begin{aligned} \| DR_0^i(p) \cdot v \| &= \| D\varphi^{-1}(\bar{R}_0^i(\varphi(p))) \cdot D\bar{R}_0^i(\bar{p}) \cdot \bar{v} \| \\ &\geq \frac{1}{M} \cdot \| D\bar{R}_0^i(\bar{p}) \cdot \bar{v} \| \\ &\geq \frac{1}{M} \cdot \mid Df^i(\bar{y}) \mid \cdot \mid \bar{w} \mid . \end{aligned} \tag{3.8}$$

Then using that $\bar{w} = \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \cdot u + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \cdot w$, the bounds for the partial derivatives of $\Pi^{\mathscr{F}}$ and the fact that $0 < \gamma < \frac{1}{2}$, we obtain that

$$\begin{aligned} \mid \bar{w} \mid &= \Big| \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \cdot u + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \cdot w \Big| \\ &= \mid w \mid \cdot \Big| \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \cdot \frac{u}{w} + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \Big| \\ &\geq \mid w \mid \cdot \Big(- \Big| \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \Big| . \Big| \frac{u}{w} \Big| + \Big| \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \Big| \Big) \\ &\geq \mid w \mid \cdot \Big(- \frac{1}{2} \cdot \gamma + \frac{3}{4} \Big) \\ &\geq \mid w \mid \cdot \Big(- \frac{1}{2} . \frac{1}{2} + \frac{3}{4} \Big) \\ &= \frac{1}{2} \cdot \mid w \mid . \end{aligned} \tag{3.9}$$

Therefore, (3.8) and (3.9) imply that

$$\| DR_0^i(p) \cdot v \| \geq \frac{1}{2 \cdot M} \cdot \mid Df^i(\bar{y}) \mid \cdot \mid w \mid$$

By hypothesis, $0 < \gamma < \frac{1}{2} < 1$ and $v \in C^{\gamma}(p)$. So

$$\begin{aligned} \| v \| &= \max\{| u |, | w |\} \\ &\leq \max\{\gamma \cdot | w |, | w |\} \\ &= | w | \cdot \max\{\gamma, 1\} \\ &= | w |, \end{aligned}$$

then this inequality, the preceding estimates and definition of $\hat{C}_0$ imply that

$$\begin{aligned} \| DR_0^i(p) \cdot v \| &\geq \frac{1}{2 \cdot M} \cdot | Df^i(\bar{y}) | \cdot \| v \| \\ &= \hat{C}_0 \cdot | Df^i(\bar{y}) | \cdot \| v \|, \end{aligned}$$

and this proves (3.1) of Lemma 3.1. □

The Minimum Principle given by Lemma 2.6 will be used to find an lower bound, non depending on i, for the derivative $Df^i(x)$ for all x such that $f^i(x)$ is far from 1 and 0.

Lemma 3.2. *Let $f : [0, a] \cup [b, 1] \to \mathbb{R}$ be a linear-contracting map and let $\tilde{c}$ and $\tilde{d}$ be two real numbers with $0 < \tilde{c} < \tilde{d} < 1$. Then there exists a constant $\tilde{C}_0 = \tilde{C}_0(f, \tilde{c}, \tilde{d}) > 0$ such that for all $i \in \mathbb{N}$ and for all $x \in (0, a) \cup (b, 1)$, if $f^i(x) \in [\tilde{c}, \tilde{d}]$, then*

$$|Df^i(x)| \geq \tilde{C}_0. \tag{3.10}$$

Proof. Fix $f : [0, a] \cup [b, 1] \to \mathbb{R}$ and $\tilde{c}$ and $\tilde{d}$ as in lemma.

Define

$$\tilde{C}_0 = \min\{\tilde{c}, 1 - \tilde{d}\}.$$

Let consider a interval $J \subset \mathbb{R}$. We denote by $lenght(J)$ the length of J.

Now, fix $i \in \mathbb{N}$ and $x \in (0, a) \cup (b, 1)$ such that $f^i(x) \in [\tilde{c}, \tilde{d}]$.

Let us consider $I_x = [\xi_0, \xi_1]$ the maximal interval containing x where f^i is defined. For maximality of I_x we have that either $[0, \tilde{c}] \subset$

$f^i([\xi_0, x])$ and $[\tilde{d}, 1] \subset f^i([x, \xi_1])$, or $[\tilde{d}, 1] \subset f^i([\xi_0, x])$ and $[0, \tilde{c}] \subset f^i([x, \xi_1])$.

In both cases, by Mean Valued Theorem there are $\tilde{\xi_0}, \tilde{\xi_1}$ with $\xi_0 < \tilde{\xi_0} < x < \tilde{\xi_1} < \xi_1$ such that

$$
(3.11) \quad | Df^i(\tilde{\xi_0}) | = \frac{lenght(f^i([\xi_0, x]))}{lenght([\xi_0, x])} \geq lenght(f^i([\xi_0, x])) \geq \tilde{C}_0
$$

and

$$
(3.12) \quad | Df^i(\tilde{\xi_1}) | = \frac{lenght(f^i([x, \xi_1]))}{lenght([x, \xi_1])} \geq lenght(f^i([x, \xi_1])) \geq \tilde{C}_0,
$$

Using the Minimum Principle (Lemma 2.6), the definition of f restricted to $[0, a]$, the fact $S(f^i)(y) = S(f^{i-1}|(b, 1) \circ f|(0, a))(y) < 0$ for all $y \in (0, a)$, inequalities (3.11) and (3.12) we obtain either

$$
|Df^i(x)| > \min \left\{ |Df^i(\tilde{\xi_0})|, |Df^i(\tilde{\xi_1})| \right\} \geq \tilde{C}_0,
$$

or

$$
|Df^i(x)| = \beta_f^i > 1 > \tilde{C}_0.
$$

Figure 4 it illustrates the situation for f^i with $i = 2$.

Therefore, the proof follows. □

Lemma 3.3. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H) and let c and d be two real numbers with $0 < c < d < 1$. Then there exists a constant $C_0 = C_0(R_0, c, d) > 0$ such that for every $i \in \mathbb{N}$ and every $p \in dom(R_0^i)$, if $R_0^i(p) \in [0, 1] \times [c, d]$, then*

$$
(3.13) \qquad \| DR_0^i(p) \cdot v \| \geq C_0 \cdot \| v \|
$$

for all $v \in C^\gamma(p)$.

Proof. Fix R_0, c and d as in the statement of the lemma. Also consider the R_0-invariant foliation $\mathscr{F}$ and the coordinates $(\bar{x}, \bar{y})$ defined by $\mathscr{F}$ in (2.1) and $\bar{R}_0$ given by (2.2).

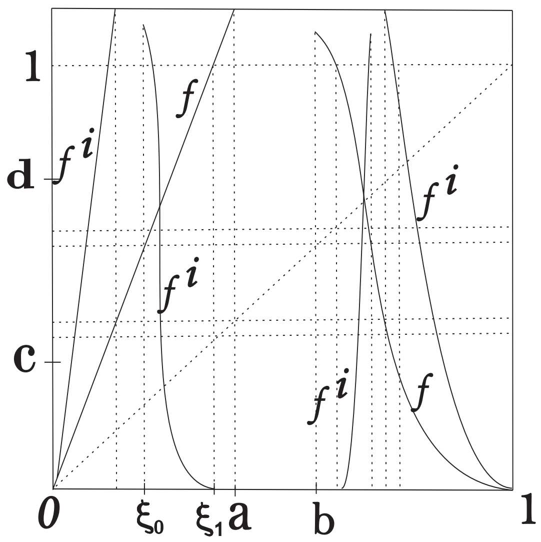

FIGURE 4. Case f^i for $i = 2$.

From Hypothesis (H), the quotient map f according (2.2) has negative Schwarzian derivative.

Take two real numbers $\tilde{c}, \tilde{d}$, with $0 < \tilde{c} < \tilde{d} < 1$ such that

$$z = (x, y) \in [0, 1] \times [c, d] \Longrightarrow \bar{y} \in [\tilde{c}, \tilde{d}]. \tag{3.14}$$

Take $\tilde{C}_0$ given by Lemma 3.2 applied to f, $\tilde{c}$ and $\tilde{d}$ as above, and take $\hat{C}_0$ given by Lemma 3.1.

Taking

$$C_0 = \hat{C}_0 \cdot \tilde{C}_0$$

we prove that C_0 works. For this we fix $i \in \mathbb{N}$, $p = (x, y) \in dom(R_0^i)$ with $R_0^i(p) \in [0, 1] \times [c, d]$ and $v \in C^\gamma(p)$.

Note that for (3.14), $f^i(\bar{y}) \in [\tilde{c}, \tilde{d}]$, so Lemma 3.2 applied to $\bar{y}$ give us

$$\mid Df^i(\bar{y}) \mid \geq \tilde{C}_0. \tag{3.15}$$

Therefore, from (3.1) of Lemma 3.1, (3.15) and definition of C_0 we obtain

$$\| DR_0^i(p) \cdot v \| \geq C_0 \cdot \| v \|,$$

and the proof follows. $\square$

Lemma 3.4. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H) and let c_1 and d_1 two real numbers with $0 < c_1 < d_1 < 1$. Then there is $C_1 = C_1(R_0, c_1, d_1) > 0$ such that for each $N \in \mathbb{N}$ there exists a C^1-neighborhood $\mathscr{V}_1 = \mathscr{V}_1(R_0, N, c_1, d_1)$ of R_0 in $\tilde{\mathcal{T}}$ such that for all $l \leq N$, for all $R \in \mathscr{V}_1$ and for all $p \in dom(R^l)$, if $R^l(p) \in [0,1] \times [c_1, d_1]$, then*

$$\| DR^l(p) \cdot v \| \geq C_1 \cdot \| v \|, \tag{3.16}$$

for all $v \in C^\gamma(p)$.

Proof. Fix R_0 and c_1 and d_1 as in the statement of the lemma. Take the real numbers c and d such that $0 < c < c_1$ and $d_1 < d < 1$. It follow from the definition of the C^1-topology of $\tilde{\mathcal{T}}$ that for all $j \geq 1$ there is a C^1-neighborhood $\bar{\mathscr{V}}(j)$ of R_0 in $\tilde{\mathcal{T}}$ such that if $R \in \bar{\mathscr{V}}(j)$, $q \in dom(R^j)$ and $R^j(q) \in [0,1] \times [c_1, d_1]$ then $q \in dom(R_0^j)$ and

$$R_0^j(q) \in [0,1] \times [c, d]. \tag{3.17}$$

To see this, we extend the maps of $\tilde{\mathcal{T}}$ to maps as it is shown in the figure 5.

Take C_0 given by Lemma 3.3 applied to R_0, c and d as above. It follows again from the definition of the C^1-topology of $\tilde{\mathcal{T}}$ that for all $i \geq 1$ there is a C^1-neighborhood $\tilde{\bar{\mathscr{V}}}(i)$ of R_0 in $\tilde{\mathcal{T}}$ such that if $R \in \tilde{\bar{\mathscr{V}}}(i)$, $\tilde{q} \in \Sigma$ and $\tilde{v} \in C^\gamma(\tilde{q})$ then

$$\| DR_0^i(\tilde{q}) \cdot \tilde{v} - DR^i(\tilde{q}) \cdot \tilde{v} \| \leq \frac{C_0}{2} \cdot \| \tilde{v} \| \,. \tag{3.18}$$

Define

$$C_1 = \frac{C_0}{2}.$$

Now fix an integer $N \geq 1$. Define

$$\bar{\mathscr{V}} = \bar{\mathscr{V}}(f, N) = \bigcap_{1 \leq j \leq N} \bar{\mathscr{V}}(j)$$

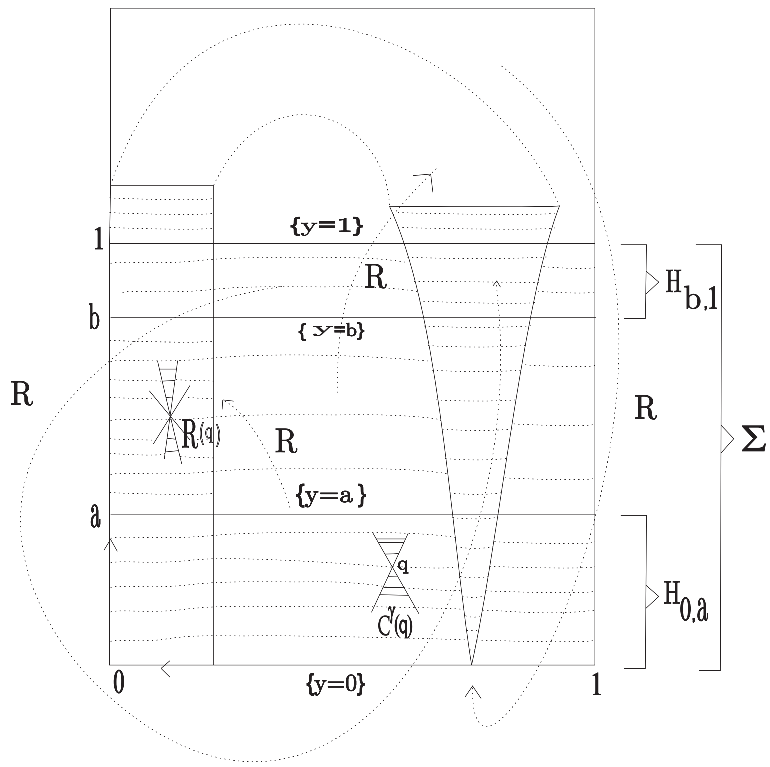

FIGURE 5. Shape of extension of R.

and

$$\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(f, N) = \bigcap_{1 \le i \le N} \tilde{\mathscr{V}}(i).$$

Define

$$\mathscr{V}_1 = \bar{\mathscr{V}} \cap \tilde{\mathscr{V}}.$$

Let us prove that the neighborhood $\mathscr{V}_1$ works. For this we fix an integer $1 \le l \le N$, $R \in \mathscr{V}_1$ and $p \in dom(R^l)$ such that $R^l(p) \in [0,1] \times [c_1, d_1]$ and $v \in C^\gamma(p)$. In particular $R \in \bar{\mathscr{V}}$, then (3.17) implies that $R_0^l(p) \in [0,1] \times [c,d]$ (taking $j = l$), then (3.13) in Lemma 3.3 (taking $i = l$) implies

$$\| DR_0^l(p) \cdot v \| \ge C_0 \cdot \| v \| . \tag{3.19}$$

Moreover, as in particular $R \in \tilde{\mathscr{V}}$, then (3.18) implies (taking $i = l$, $\tilde{q} = p$ and $\tilde{v} = v$)

$$\| DR_0^l(p) \cdot v - DR^l(p) \cdot v \| \leq \frac{C_0}{2} \cdot \| v \| . \tag{3.20}$$

From (3.20), (3.19) and definition of C_1 we obtain

$$\begin{aligned}
\| DR^l(p) \cdot v \| &= \| DR^l(p) \cdot v - DR_0^l(p) \cdot v + DR_0^l(p) \cdot v \| \\
&\geq \| DR_0^l(p) \cdot v \| - \| DR^l(p) \cdot v - DR_0^l(p) \cdot v \| \\
&\geq C_0 \cdot \| v \| - \frac{C_0}{2} \cdot \| v \| \\
&= \frac{C_0}{2} \cdot \| v \| \\
&= C_1 \cdot \| v \|
\end{aligned}$$

and the lemma follows. □

Let $R \in \mathcal{T}$ and $\delta > 0$. Now, we define the sets

$$\begin{aligned}
V(\delta) &= [0,1] \times ([0,a]) \cup [b, 1-\delta]); \\
W_R^k(\delta) &= \big\{p = (x,y) \in V(\delta) : R^i(p) \in V(\delta), i = 0, ..., k-1\big\}, \\
&\quad \forall k \geq 1; \\
\Lambda_R(\delta) &= \bigcap_{k \geq 1} W_R^k(\delta).
\end{aligned}$$

This set is called the R-maximal invariant set on $V(\delta)$.

Definition 3.5. *Let $R \in \mathcal{T}$ and $\delta > 0$. We said that R is* vertically expansive *on $V(\delta)$ if there are a vertical invariant cone field C^γ on $V(\delta)$, a positive constant $C = C(R,\delta) > 0$ and $\lambda = \lambda(R,\delta) > 1$ such that if $n \in \mathbb{N}$ and $p \in dom(R^{n-1})$ satisfy $R^i(p) \in V(\delta)$ for every $i = 0, \cdots, n-1$ then*

$$\| DR^n(p) \cdot v \| > C \cdot \lambda^n \cdot \| v \|$$

for all $v \in C^\gamma(p)$.

Lemma 3.6. *Let R_0 be a triangular map which satisfies the hypothesis (H). Then for each $\delta > 0$, R_0 is vertical expansive on $V(\delta) = [0,1] \times ([0,a]) \cup [b, 1-\delta])$.*

Proof. Fix R_0 as in the statement of the lemma. Let consider the R_0-invariant foliation $\mathscr{F}$. By definition of the $(\bar{x}, \bar{y})$ coordinates, $\bar{R}_0 = \varphi \circ R_0 \circ \varphi^{-1}$ according (2.2) became

$$\bar{R}_0(\bar{x}, \bar{y}) = (g(\bar{x}, \bar{y}), f(\bar{y}))$$

where f has negative Schwarzian derivative by Hypothesis (H).

Also fix $\delta > 0$. Let us consider $\bar{\delta} = \bar{\delta}(\delta) > 0$ in such way that

$$z = (x, y) \notin ([0,1] \times (1-\delta, 1]) \Longrightarrow \bar{y} \notin (1 - \bar{\delta}, 1]. \tag{3.21}$$

From (3.1) of Lemma 3.1, there exists $\hat{C} > 0$ that for all $k \in \mathbb{N}$, for all $p \in dom(R_0^k)$ and for all $v \in C^\gamma(p)$ we get

$$\| DR_0^k(p) \cdot v \| \ \geq \ \hat{C} \cdot \mid Df^k(\bar{y}) \mid \cdot \| v \| \tag{3.22}$$

where $\varphi(p) = (\bar{x}, \bar{y})$.

By Singer's and Misiurewicz's Theorems (see [MS]) we have that f is hyperbolic on $[0,a] \cup [b, 1-\bar{\delta}]$ there are positive constants $\bar{C} = \bar{C}(\bar{R}_0, \bar{\delta}) > 0$ and $\bar{\lambda} = \bar{\lambda}(\bar{R}_0, \bar{\delta}) > 1$ such that for all $k \in \mathbb{N}$ and $f^i(\bar{y}) \in [0,a] \cup [b, 1-\bar{\delta}]$, $0 \leq i \leq k-1$ we get that

$$\mid Df^k(\bar{y}) \mid \geq \bar{C} \cdot \bar{\lambda}^k. \tag{3.23}$$

Take $C = \hat{C} \cdot \bar{C}$ and $\lambda = \bar{\lambda}$. Fix $k \in \mathbb{N}$ and p with $R_0^i(p) \in V(s), 0 \leq i \leq k-1$. So, from (3.21) we have that $f^i(\bar{y}) \in [0,a] \cup [b, 1-\bar{\delta}]$, $1 \leq i \leq k-1$ because $R_0^i(p) \in V(s), 0 \leq i \leq k-1$. Note that $(\bar{x}, \bar{y}) = \varphi(p)$ because (3.21).

Take $v \in C^\gamma(p)$, then using (3.22), (3.23) and definition of C and λ we obtain

$$\| DR_0^k(p) \cdot v \| \geq C \cdot \lambda^k \cdot \| v \| .$$

Therefore, from Definition 3.5, the proof follows. □

In the following proposition, we show an easy characterization of vertical expansivity in the terms of the maximal invariant set, like for one dimensional hyperbolic set.

Proposition 3.7. *Let $R \in \mathcal{T}$ and $\delta > 0$. Then, R is vertically expansive on $V(\delta)$ if only if there is a vertical invariant cone field C^γ on $V(\delta)$ such that for every $p \in \Lambda_R(\delta)$ and $\forall v \in C^\gamma(p)$ there exists a positive integer $k = k(p, v)$ such that*

$$\| DR^k(p) \cdot v \| > \| v \| .$$

Proof. If R is vertically expansive on $([0,1] \times [0,a]) \cup ([0,1] \times [b, 1-\delta])$ we take k so that $C \cdot \lambda^k > 1$. So let us prove the reverse implication. So suppose that $\| DR^{k(p,v)}(p)v \| > \| v \|$ for all $p \in \Lambda_R(\delta)$ and $\forall v \in C^\gamma(p)$. Denote by B the set of $v \in C^\gamma(p)$ such that $\| v \| = 1$. By compactness of $\Lambda_R(\delta) \times B$ and continuity of the derivative of R, there exists a finite cover $V_1 \times B_1, \cdots, V_k \times B_k$ of $\Lambda_R(\delta) \times B$ by open sets, integers $n_1, \cdots, n_k$ and number $\lambda_1, \cdots, \lambda_k > 1$, such that $\| DR^{n_i}(p) \cdot v \| > \lambda_i$ for all $(p, v) \in V_i \times B_i$ and every $i = 1, \cdots, k$.

Let consider a neighborhood $V = \bigcup_{j=1}^{k} V_j$. Note that exists n_0 such that if $p \notin V$ then there exists i smaller n_0 with $R^i(p) \notin V(\delta)$. Define $\tilde{n} = \max\{n_j : 0 \le j \le k\}$, $a = \min\{\| DR(p) \cdot v \| : (p, v) \in V(\delta) \times B\}$, $\lambda = \min\{ \sqrt[n_i]{\lambda_i} : 0 < i \le k\}$ and $C = \min\{\frac{a^i}{\lambda^i} : 1 \le i \le \tilde{n}\}$.

Take $n \in \mathbb{N}$ and $p \in dom(R^{n-1})$ such that $R^i(p) \notin [0,1] \times (1-\delta, 1]$ for every $i = 0, \cdots, n-1$. Take $v \in C^\gamma(p)$.

We follows inductively the following alternatives:

a) If $p \notin V$. Then $n \le n_0 \le \tilde{n}$. So

$$\| DR^n(p) \cdot v \| = \| DR^n(p) \cdot \frac{v}{\| v \|} \| \cdot \| v \| > C \cdot \lambda^n \| v \| .$$

b) If $p \in V$. Then, there is i, $1 \le i \le k$ such that $(p, \frac{v}{\|v\|}) \in V_i \times B_i$.

b-1) If $n < n_i$ then $n \leq \tilde{n}$. So

$$\| DR^n(p) \cdot v \| = \| DR^n(p) \cdot \frac{v}{\| v \|} \| \cdot \| v \| > C \cdot \lambda^n \| v \| .$$

b-2) If $n \geq n_i$ then

$$\begin{aligned} \| DR^n(p) \cdot v \| &= \| DR^{n-n_i}(R^{n_i}(p)) \cdot \frac{DR^{n_i}(p) \cdot v}{\| DR^{n_i}(p) \cdot v \|} \| \cdot \\ &\quad \| DR^{n_i}(p) \cdot \frac{v}{\| v \|} \| \cdot \| v \| \\ &\geq \| DR^{n-n_i}(q) \cdot w \| \cdot \lambda^{n_i} \cdot \| v \| \end{aligned}$$

where $q = R^{n_i}(p)$ and $w = \frac{DR^{n_i}(p) \cdot v}{\|DR^{n_i}(p) \cdot v\|}$. Then the proof follows recursively. □

Remark 3.8. *It is clear from definition that Vertical expansiveness is an C^1-open property. In our case, Vertical expansiveness is equivalent to hyperbolicity because the existence of an invariant contracting foliation.*

Lemma 3.9. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there are $\delta_2 = \delta_2(R_0) > 0$ and a constant $C_2 = C_2(R_0) > 0$ satisfying the following property: for each $\delta < \delta_2$, there are $\lambda_2 = \lambda_2(R_0, \delta) > 1$ and a C^1-neighborhood $\mathscr{V}_2 = \mathscr{V}_2(R_0, \delta)$ of R_0 in $\tilde{\mathcal{T}}$ such that for all $R \in \mathscr{V}_2$, for all $k \in \mathbb{N}$ and for all $p \in dom(R^k)$ with $p, R(p), ..., R^{k-1}(p) \in V(\delta)$ but $R^k(p) \in [0,1] \times [1 - \delta_2, 1]$, then*

$$\| DR^k(p) \cdot v \| \geq C_2 \cdot \lambda_2^k \cdot \| v \|, \tag{3.24}$$

for all $v \in C^\gamma(p)$.

Proof. Let consider R_0 as in statement of the Lemma. Choose $\delta_2 > 0$, c_1, d_1 with $0 < c_1 < d_1 < 1$ and a C^1-neighborhood $\bar{\mathscr{V}_2}$ of R_0 in $\tilde{\mathcal{T}}$ in such way that if $R \in \bar{\mathscr{V}_2}$ and $p \in dom(R)$ satisfy $R(p) \in [0,1] \times [1-\delta_2, 1]$ then $p \in [0,1] \times [c_1, d_1]$.

Let C_1 be as in Lemma 3.4 applied to R_0, c_1 and d_1 chosen above. Let consider $\tilde{C}_1 < \inf\{\frac{\|DR_0(p) \cdot v\|}{\|v\|} : p \in [0,1] \times [c_1, d_1], v \in C^\gamma(p)\}$

. Shrinking $\bar{\mathscr{V}}_2$, we can suppose that for all $R \in \bar{\mathscr{V}}_2$, for all $p \in [0,1] \times [c_1, d_1]$ and for all $v \in C^\gamma(p)$, $\| DR(p) \cdot v \| > \tilde{C}_1 \cdot \| v \|$.

Define

$$C_2 = \min\{1, \frac{C_1.\tilde{C}_1}{2}\}.$$

Now fix δ, $0 < \delta < \delta_2$. For such a δ we shall take $\mathscr{V}_2$ and λ_2 as follows:

From Lemma 3.6, R_0 is vertical expansive on $V(\delta) = [0,1] \times ([0,a] \cup [b, 1-\delta])$, so we can find a C^1-neighborhood $\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(R_0, \delta)$ of R_0 in $\tilde{\mathscr{T}}$ and constants $\tilde{C} = \tilde{C}(R_0, \delta)$, $\tilde{\lambda} = \tilde{\lambda}(R_0, \delta)$, with $\tilde{C} > 0$ and $\lambda > \tilde{\lambda} > 1$ such that if $R \in \tilde{\mathscr{V}}$ and p satisfy $R^i(p) \in V(\delta)$, $0 \le i \le k-1$ and $v \in C^\gamma(p)$, then

$$\| DR^k(p) \cdot v \| \ge \tilde{C} \cdot \tilde{\lambda}^k \cdot \| v \|, \tag{3.25}$$

because this is an open property (see Remark 3.8).

From (3.25) we can find $K = K(R_0, \delta) \in \mathbb{N}$ and $\hat{\lambda}_2 = \hat{\lambda}_2(R_0, \delta)$, $\tilde{\lambda} > \hat{\lambda}_2 > 1$ such that if $k \ge K$, $g \in \tilde{\mathscr{V}}$ and p satisfy $R^i(p) \in V(\delta)$, $0 \le i \le k-1$ and $v \in C^\gamma(p)$, then

$$\| DR^k(p) \cdot v \| \ge \hat{\lambda}_2^k \cdot \| v \| . \tag{3.26}$$

(Just take $K = \min\{k : \tilde{C} \cdot \tilde{\lambda}^k > 1\}$ and $\hat{\lambda}_2$ such that $1 < \hat{\lambda}_2 < \min\{\tilde{\lambda} \cdot \tilde{C}^{\frac{1}{K}}, \tilde{\lambda}\}$).

Let $\mathscr{V}_1$ be the C^1-neighborhood of R_0 in $\tilde{\mathscr{T}}$ given by Lemma 3.4 for this K.

Let us consider $\lambda_2 = \lambda_2(R_0, \delta)$, $1 < \lambda_2 < \hat{\lambda}_2$ such that

$$\lambda_2^K < 2. \tag{3.27}$$

We show that Lemma work with $\mathscr{V}_2 = \mathscr{V}_2(R_o, \delta) = \bar{\mathscr{V}}_2 \cap \tilde{\mathscr{V}} \cap \mathscr{V}_1$ and λ_2 as was chosen.

Fix $R \in \mathscr{V}_2$, $k \in \mathbb{N}$ and $p \in dom(R^k)$ with $p, R(p), ..., R^{k-1}(p) \in V(\delta)$ but $R^k(p) \in [0,1] \times [1-\delta_2, 1]$. Also fix $v \in C^\gamma(p)$.

If $k \geq K$, then (3.26), the definition of λ_2 and C_2 imply that

$$\| DR^k(p) \cdot v \| \geq \hat{\lambda}_2^k \cdot \| v \| \geq C_2 \cdot \lambda_2^k \cdot \| v \| . \tag{3.28}$$

To the case when $k < K$, first we observe that $R^{k-1}(p)$ belong to $[0,1] \times [c_1, d_1]$ because $R^k(p) \in [0,1] \times [1-\delta_2, 1]$ and by hypothesis $R \in \mathscr{V}_2$. Then, definition of $\tilde{C}_1$ for one side and (3.16) of Lemma 3.4 imply that

$$\begin{aligned} \| DR^k(p) \cdot v \| &= \| DR(R^{k-1}(p)) \cdot DR^{k-1}(p) \cdot v \| \\ &\geq \tilde{C}_1 \cdot \| DR^{k-1}(p) \cdot v \| \\ &\geq \tilde{C}_1 \cdot C_1 \cdot \| v \| . \end{aligned}$$

So,

$$\| DR^k(p) \cdot v \| > C_2 \cdot \lambda_2^k \tag{3.29}$$

because (3.27) and the definitions of C_2 and λ_2.

Finally, from (3.28) and (3.29) the lemma follows. □

Lemma 3.10. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) and let C such that $0 < C \leq 1$. Then there are a C^1-neighborhood $\mathscr{V}_3 = \mathscr{V}_3(R_0, C)$ of R_0 in $\tilde{\mathcal{T}}$ and constants $\delta_3 = \delta_3(R_0, C) > 0$, $\lambda_3 = \lambda_3(R_0, C) > 1$ and $L = L(R_0, C) \in \mathbb{N}$ with $C \cdot \lambda_3^L > 1$ such that for each $R \in \mathscr{V}_3$, for each $p \in [0,1] \times [1-\delta_3, 1)$ there exists an integer $l = l(R, p) > L$ such that $R^j(p) \in [0,1] \times [0, a]$ for $j = 1, ..., l-1$ and*

$$\| DR^l(p) \cdot v \| \geq \lambda_3^l \cdot \| v \| \tag{3.30}$$

for all $v \in C^\gamma(p)$.

Proof. Fix R_0, $K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha}$ and $0 < C < 1$ as in statement of the lemma. For every $\eta > 0$ we consider the C^1-neighborhood for R_0 of size η in $\tilde{\mathcal{T}}$, that is

$$\mathscr{V}_\eta = \{R : d_{C^1}(R_0, R) < \eta\}.$$

By (H2) we have $\nu \cdot \mu^{\frac{1-\tilde{\alpha}(R_0)}{\alpha(R_0)}} > 1$. Then there is $\hat{\lambda} = \hat{\lambda}(R_0) > 1$ and $\eta > 0$ small such that for all $R \in \mathscr{V}_\eta$

$$\nu \cdot \mu^{\frac{1-\tilde{\alpha}(R)}{\alpha(R)}} > \hat{\lambda} > 1. \tag{3.31}$$

As $R_0(\{y = 1\}) \subseteq \{y = 0\}$ we can choose $0 < \hat{\delta} = \hat{\delta}(R_0) < 1$ such that $R_0(p) \in [0,1] \times [0, \frac{a}{2}]$ for all $p \in [0,1] \times (1-\hat{\delta}, 1)$. Shrinking η again, we can assume that $R(p) \in [0,1] \times [0,a]$ for all $p \in [0,1] \times (1-\hat{\delta}, 1)$ and for all $R \in \mathscr{V}_\eta$.

For $R \in \mathscr{V}_\eta$ and $p \in [0,1] \times [1-\hat{\delta}, 1)$ we define

$$l(R,p) = \inf\{j \geq 1 : R^j(R(p)) \notin H_{0,a}\}. \tag{3.32}$$

To choose λ_3 we need to make some estimates. Let consider $R \in \mathscr{V}_\eta$ and $p \in [0,1] \times [1-\hat{\delta}, 1)$ and $v \in C^\gamma(p)$. By definition of l in (3.32) we have that $R(p), ..., R^{l-1}(p) \in [0,1] \times [0,a]$ and

$$y_{R^{l+1}(p)} > a. \tag{3.33}$$

Claim. For all $R \in \tilde{\mathcal{T}}$, for all $q \in \Sigma = [0,1] \times [0,1]$ and $n \in \mathbb{N}$ with q, $R(q)$, $R^2(q), ..., R^{n-1}(q) \in H_{0,a}$ then $y_{R^n(q)} \leq \mu^n \cdot y_q$.

Indeed, fix $R \in \tilde{\mathcal{T}}$, $q \in \Sigma$ and $n \in \mathbb{N}$ such that q, $R(q)$, $R^2(q), ...,$ $R^{n-1}(q) \in H_{0,a}$.

By definition we have

$$y_{R^n(q)} = (\Pi_y \circ R^n)(q)$$

were Π_y is the projection over the second variable y. Define for $t \in [0,1]$ the real valued map $h(t) = y_{R^n(x_q,t)}$, then

$$h(y_q) = y_{R^n(x_q,y_q)}.$$

By Mean Valued Theorem we have

$$y_{R^n(x_q,y_q)} = h'(\xi).y_q \tag{3.34}$$

for some ξ because $y_{R^n(x_q,0)} = 0$.

But

$$h'(\xi) = D(\Pi_y \circ R^n)(x_q, \xi) \cdot (0,1).$$

By (H4-b) because $(0,1) \in C^\gamma(x,y)$, $\forall (x,y) \in H_{0,a}$ with $R(x,y)$, ..., $R^{n-1}(x,y) \in H_{0,a}$, the cone C^γ is invariant and the fact that $\| \Pi_y \| = 1$ we get

$$\begin{aligned} | h'(\xi) | &\leq \| \Pi_y \| \cdot \| DR^n(x_q, \xi).(0,1) \| \\ &\leq \mu^n \end{aligned}$$

and then (3.34) applies and proves the claim.

Moreover, for all $R \in \mathscr{V}_\eta$ and all $p \in [0,1] \times [1-\hat{\delta}, 1)$, the above Claim (for $q = R(p)$ and $n = l$) implies that

$$\begin{aligned} y_{R^{l+1}(p)} &= y_{R^l(R(p))} \\ &\leq \mu^l \cdot y_{R(p)}. \end{aligned}$$

From this inequality and (3.33) we get

$$y_{R(p)} > a \cdot \mu^{-l}. \tag{3.35}$$

But (H3) says

$$y_{R(p)} \leq K_0 \cdot | y_p - 1 |^{\alpha(R)} . \tag{3.36}$$

Note that by definition of the neighborhood $\mathscr{V}_\eta$,

$$\alpha(R_0) - \eta < \alpha_R < \alpha(R_0) + \eta$$

and

$$\tilde{\alpha}(R_0) - \eta < \tilde{\alpha}(R) < \tilde{\alpha}(R_0) + \eta.$$

By definition of the neighborhood $\mathscr{V}_\eta$, as $p \in [0,1] \times [1-\hat{\delta}, 1)$, (3.35) and (3.36) we obtain

$$K_0 \cdot (\hat{\delta})^{\alpha(R_0)-\eta} \geq a \cdot \mu^{-l}.$$

Therefore,

$$l \geq \frac{log(a) - log(K_0) - (\alpha(R_0) - \eta) \cdot log(\hat{\delta})}{log(\mu)} = L(\hat{\delta}). \tag{3.37}$$

Note that $L(\hat{\delta}) \to \infty$ as $\hat{\delta} \to 0$.

Also, from (3.35) and (3.36) we have

$$
\begin{aligned}
\mid y_p - 1 \mid^{\tilde{\alpha}(R)-1} &\geq \left(\frac{y_{R(p)}}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \\
&\geq \left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \cdot \left(\mu^{\frac{1-\tilde{\alpha}(R)}{\alpha(R)}}\right)^l . \qquad (3.38)
\end{aligned}
$$

From the Chain Rule, (H4-a) and (H4-b) we get

$$
\begin{aligned}
\| DR^l(p)\cdot v \| &= \| DR^{l-1}(R(p))\cdot DR(p)v \| \\
&\geq \nu^{l-1}\cdot \| DR(p)\cdot v \| \\
&\geq \nu^{l-1}\cdot K_1 \cdot \mid y_p - 1 \mid^{\tilde{\alpha}(R)-1} \cdot \| v \|, \qquad (3.39)
\end{aligned}
$$

for all $v \in C^{\gamma}(p)$.

Moreover, by C^1 proximity we have

$$
\frac{\tilde{\alpha}(R_0)-1-\eta}{\alpha(R_0)+\eta} < \frac{\tilde{\alpha}(R)-1}{\alpha(R)} < \frac{\tilde{\alpha}(R_0)-1+\eta}{\alpha(R_0)-\eta}. \qquad (3.40)
$$

Using successively (3.39), (3.38), (3.31) and (3.40) we obtain that for all $v \in C^{\gamma}(p)$,

$$
\begin{aligned}
\| DR^l(p)\cdot v \| &\geq \nu^{l-1}\cdot K_1 \cdot \left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \cdot \left(\mu^{\frac{1-\tilde{\alpha}_R}{\alpha_R}}\right)^l \cdot \| v \| \\
&= \frac{K_1}{\nu}\cdot \left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \cdot \left(\nu\mu^{\frac{1-\tilde{\alpha}(R)}{\alpha(R)}}\right)^l \cdot \| v \| \\
&\geq \frac{K_1}{\nu}\cdot \left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \cdot \hat{\lambda}^l \cdot \| v \| \\
&= C(R_0)\cdot \hat{\lambda}^l \cdot \| v \| \qquad (3.41)
\end{aligned}
$$

where $C(R_0) = \min\left\{ \frac{K_1}{\nu}\cdot\left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R_0)-1-\eta}{\alpha(R_0)+\eta}}, \frac{K_1}{\nu}\cdot\left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R_0)-1+\eta}{\alpha(R_0)-\eta}} \right\}$.

Now, fix $L_0 \in \mathbb{N}$ such that $C(R_0)\cdot\hat{\lambda}^{L_0} > 1$. Also take λ_3 such that

$$
1 < \lambda_3 < min\{(C(R_0))^{\frac{1}{L_0}}\cdot\hat{\lambda}, \hat{\lambda}\}.
$$

By (3.37) we take $\hat{\delta}$ such that for all $p \in [0,1] \times [1-\hat{\delta}, 1)$ and $R \in \mathscr{V}_\eta$ we have $L(\hat{\delta}) > L_0$. Therefore, $l = l(R,p) \geq L_0$.

So, using inequality (3.41) and definition of λ_3 we obtain

$$\begin{aligned} \| DR^l(p) \cdot v \| &\geq C(R_0) \cdot \hat{\lambda}^{L_0} \cdot \hat{\lambda}^{l-L_0} \cdot \| v \| \\ &\geq \lambda_3^{L_0} \cdot \lambda_3^{l-L_0} \cdot \| v \| \\ &= \lambda_3{}^l \cdot \| v \|, \end{aligned}$$

for all $v \in C^\gamma(p)$.

Finally take $L = L(f, C)$ with $C \cdot \lambda_3^L > 1$. Shrinking $\hat{\delta}$ in such a way $L(\hat{\delta}) > L$. This implies that for all $p \in [0,1] \times [1-\delta, 1)$ and for all $R \in \mathscr{V}_\eta$, $l = l(R,p) > L$.

The lemma works with $\mathscr{V}_3 = \mathscr{V}_\eta$, λ_3, L and $\delta_3 = \hat{\delta}$ as before. This ends the proof. $\square$

Proposition 3.11. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there are C^1-neighborhood $\mathscr{V}_4 = \mathscr{V}_4(R_0)$ of R_0 in $\tilde{\mathcal{T}}$ and constants $C_4 = C_4(R_0) > 0$, $\delta_4 = \delta_4(R_0) > 0$ and $\lambda_4 = \lambda_4(R_0) > 1$ satisfying the following properties: If $k \in \mathbb{N}$, $R \in \mathscr{V}_4$, $p \in dom(R^k)$ are such that $R^k(p) \in [0,1] \times (1-\delta_4, 1]$ and $v \in C^\gamma(p)$ then*

$$\| DR^k(p) \cdot v \| \geq C_4 \cdot \lambda_4^k \cdot \| v \| . \tag{3.42}$$

Moreover, if $p \in [0,1] \times (1-\delta_4, 1)$ then

$$\| DR^k(p) \cdot v \| \geq \lambda_4^k \cdot \| v \| . \tag{3.43}$$

Proof. Fix R_0 as in lemma. Let us consider $C_2 > 0$ and δ_2 given in Lemma 3.9 applied for R_0.

Take $C_4 = \min\{1, C_2\}$. Applying Lemma 3.10 for R_0 and $C = C_4$ we obtain a C^1-neighborhood $\mathscr{V}_3$ and the real numbers δ_3 and λ_3 and an integer L. Choose δ_4 such that $0 < \delta_4 = \frac{1}{2} \cdot \min\{\delta_2, \delta_3\}$. Take λ_2 and $\mathscr{V}_2$ given by Lemma 3.9 applied to $\delta = \delta_4$. Let us consider $\mathscr{V}_4 = \mathscr{V}_2 \cap \mathscr{V}_3$

and choose λ_4 in a such way that $1 < \lambda_4 < \min\{C_2^{\frac{1}{L}} \cdot \lambda_3, \lambda_2\}$. Note that $C_2^{\frac{1}{L}} \cdot \lambda_3 > 1$ because $C_2 > C = C_4$.

Now we prove that the proposition works with $\mathscr{V}_4$, C_4, δ_4 and λ_4 chosen above.

Fix $R \in \mathscr{V}_4$, $k \in \mathbb{N}$, $p \in dom(R^k)$ such that $R^k(p) \in [0,1]\times(1-\delta_4, 1]$ and $v \in C^\gamma(p)$.

We decompose the orbit $\{R^i(p)\}_{i=0}^k$ in several blocks as follows:

$\{p = p_1, R(p_1), ..., R^{k_1-1}(p_1)\}, \{q_1 = R^{k_1}(p_1), R(q_1), ..., R^{l_1-1}(q_1)\}$, $\{p_2 = R^{l_1}(q_1), R(p_2), ..., R^{k_2-1}(p_2)\}, \{q_2 = R^{k_2}(p_2), R(y_2), ..., R^{l_2-1}(q_2)\}$, $\dots, \{p_m = R^{l_{m-1}}(q_{m-1}), R(p_m), ..., R^{k_m}(p_m) = q_m = R^k(p)\}$, where k_1 is the first integer such that $R^{k_1}(p_1) \in (1-\delta_4, 1)$, $l_1 \geq L$ is given by the conclusion of Lemma 3.10 applied to q_1, k_2 is the first integer that $R^{k_2}(p_2) \in (1-\delta_4, 1)$ and so on.

Notice that $k_1 + l_1 + \cdots + k_{m-1} + l_{m-1} + k_m = k$.

Using the Chain Rule Theorem, (3.24) of Lemma 3.9, (3.30) of Lemma 3.10, and considering the definitions of C_4 and λ_4 we obtain

$$\begin{aligned}
\| DR^k(p)v \| &= \| DR^{k_m}(p_m) \cdot DR^{l_{m-1}}(q_{m-1}) \dots DR^{k_3}(p_3) \cdot \\
&\quad DR^{l_2}(q_2) \cdot DR^{k_2}(p_2) \cdot DR^{l_1}(q_1) \cdot Dg^{k_1}(p_1)v \| \\
&\geq \Big((C_2 \cdot \lambda_2^{k_m}) \cdot \lambda_3^{l_{m-1}}\Big) \dots \Big((C_2 \cdot \lambda_2^{k_3}) \cdot \lambda_3^{l_2}\Big) \cdot \\
&\quad \Big((C_2 \cdot \lambda_2^{k_2}) \cdot \lambda_3^{l_1}\Big) \cdot (C_2 \cdot \lambda_2^{k_1}) \cdot \| v \| \\
&\geq (\lambda_2^{k_m} \cdot \lambda_4^{l_{m-1}}) \cdots (\lambda_2^{k_3} \cdot \lambda_4^{l_2}) \cdot (\lambda_2^{k_2} \cdot \lambda_4^{l_1}) \cdot \\
&\quad (C_2 \cdot \lambda_2^{k_1}) \cdot \| v \| \\
&= \lambda_2^{k_2+\cdots+k_m} \cdots \lambda_4^{l_1+\cdots+l_{m-1}} \cdot (C_2 \cdot \lambda_2^{k_1}) \cdot \| v \| \qquad (3.44) \\
&\geq C_4 \cdot \lambda_4^k \cdot \| v \|,
\end{aligned}$$

this proves (3.42) of Proposition 3.11.

For finish the proof note that if $p \in [0,1] \times (1-\delta_4, 1)$ then in the decomposition of the orbit $\{R^i(p)\}_{i=0}^k$ given above, k_1 does not exist, so in (3.44), the expression $C_2 \cdot \lambda_2^{k_1}$ does not appears. Therefore, (3.43) of Proposition 3.11 follows. This concludes the proof. $\square$

Corollary 3.12. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there exists a C^1-neighborhood $\mathscr{V}_5$ of R_0 en $\tilde{\mathcal{T}}$ such that for each $R \in \mathscr{V}_5$ and for each $\delta > 0$, R is vertically expansive on $[0,1] \times ([0,a] \cup [b, 1-\delta])$.*

Proof. Fix R_0 as in the statement of the lemma. Let us consider δ_4 and the C^1-neighborhood $\mathscr{V}_4$ given by Proposition 3.11. Because $R_0 \in \tilde{\mathcal{T}}$ which satisfies (H) implies that R_0 is vertically expansive on $[0,1] \times ([0,a] \cup [b, 1-\delta_4])$ (see Lemma 3.6). From Remark 3.8, we can find a C^1-neighborhood $\tilde{\mathscr{V}_4} = \tilde{\mathscr{V}_4}(R_0)$ of R_0 in $\tilde{\mathcal{T}}$ such that all $R \in \tilde{\mathscr{V}_4}$ is vertically expansive on $[0,1] \times ([0,a] \cup [b, 1-\delta_4])$. Define $\mathscr{V}_5 = \mathscr{V}_4 \cap \tilde{\mathscr{V}_4}$. Now take $R \in \mathscr{V}_5$ and δ with $0 < \delta < \delta_4$. We will prove that R is vertical expansive on $[0,1] \times [b, 1-\delta]$. In order to do this, let consider p in the maximal R-invariant set in $[0,1] \times ([0,a] \cup [b, 1-\delta])$ and $v \in C^{\gamma}(p)$. Then, we have that either $\forall k > 1, R^k(p) \notin [0,1] \times (1-\delta_4, 1]$ or for some $k_1 > 1, R^{k_1}(p) \in [0,1] \times (1-\delta_4, 1]$. In the first case, taking k big enough we have that $\| DR^k(p) \cdot v \| > \| v \|$ because $R \in \tilde{\mathscr{V}_4}$. In the other case, by (3.42) of Proposition 3.11, $\| DR^{k_1}(p) \cdot v \| \geq C_4 \cdot \lambda_4^{k_1} \cdot \| v \|$.

Now applying the same argument to $R^{k_1}(p)$, we have two alternatives: there is k big enough such that $\| DR^{k_1+k}(p) \cdot v \| > \| v \|$ or there is k_2 such that $R^{k_2}(R^{k_1}(p)) \in [0,1] \times (1-\delta_4, 1)$ in a such case by (3.43) of Proposition 3.11 we have that

$$\| DR^{k_2+k_1}(p) \cdot v \| \geq C_4 \cdot \lambda_4^{k_1} \cdot \lambda_4^{k_2} \cdot \| v \| .$$

Inductively, we obtain that for some $k = k(R, \delta, n, p, v) \in \mathbb{N}$ big enough such that $\| DR^k(p) v \| > \| v \|$.

Therefore, from Proposition 3.7 the proof follows. $\square$

4. Proof of the Main Theorem

In this section we prove the Main Theorem (Theorem 2.10).

Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$. A point $p \in H_{0,a} \cup H_{b,1}$ is *periodic* for R if there is $n \geq 1$ such that $R^j(p) \in H_{0,a} \cup H_{b,1}$ for all $0 \leq j \leq n-1$ and $R^n(p) = p$. The *ω-limit set* of a point $p \in \bigcap_{i=0}^{\infty} R^{-i}(\Sigma)$ is the set

$$\{q \in U : q = \lim_{k\to\infty} R^{n_k}(p) \, \text{for some sequence} \, n_k \to \infty\}.$$

The *basin* of a periodic point p is the set of points whose $\omega-$limit set contains p. We say that a periodic point p of period n is *sink* if its basin contain an open set.

Given a curve ζ in Σ we denote by *length*(ζ) the length of a curve ζ. We say that γ is *tangent* to the cone field C^γ if $T_p\zeta$ is contained in $C^\gamma(p)$ for all $p \in \zeta$.

If $\mathscr{F}$ is a continuous foliation on U and $A \subset U$ then the *saturated* of A for $\mathscr{F}$ it is union of leaves of $\mathscr{F}$ which pass through points of A and will denoted by $[A]$.

Theorem 4.1. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there exists a C^1-neighborhood $\mathscr{V} = \mathscr{V}(R_0)$ of R_0 in $\tilde{\mathcal{T}}$ such that for all $R \in \mathscr{V}$, the maximal R-invariant set contained in $\Sigma = [0,1] \times [0,1]$, $\Lambda_R = \bigcap_{i=0}^{\infty} R^{-i}(\Sigma)$, don't contain a curve tangent to C^γ.*

Proof. Fix R_0 as in statement of the theorem. Let us consider $\mathscr{V}_4$, δ_4 and λ_4 given by Proposition 3.7. Let us consider the C^1-neighborhood $\mathscr{V}_4$ of R_0 in $\tilde{\mathcal{T}}$ given by Corollary 3.12. Take $\mathscr{V} = \mathscr{V}_4 \cap \mathscr{V}_5$. Now fix $R \in \mathscr{V}$.

Suppose, by contradiction, $\Lambda_R = \bigcap_{i=0}^{\infty} R^{-i}(\Sigma)$, has a curve ζ tangent to C^γ.

We split the proof in some steps. Let consider the invariant foliation $\mathscr{F}$ associated to the triangular map R. Also, we denote by g the quotient map induced by $\mathscr{F}$.

Step 1. R has no sinks. Indeed, Corollary 3.12 and the fact that the foliation $\mathscr{F}$ is contracting imply that all the periodic points p are saddle like hyperbolic point(see Remark 3.8). Therefore, R has no sinks.

Step 2. For all $m \neq n$, $[R^m(\zeta)] \cap [R^n(\zeta)]$ has no interior. Indeed, if there are integers $m \neq n$ such that $[R^m(\zeta)] \cap [R^n(\zeta)]$ has non empty interior. Now we denote by $J = \Pi^{\mathscr{F}}(\zeta)$ and it is clear that J is a homterval for g (i.e., $g^n|J$ is a homeomorphism for all $n \in \mathbb{N}$). Therefore $g^m(J) \cap g^n(J)$ has non empty interior, then by standard arguments (see [G], Lemma A, pag. 142) we obtain that g has a sink. Furthermore as the foliation $\mathscr{F}$ is contracting we obtain that R has a sink. This is a contradiction with step 1.

Therefore, the sequence of horizontal bands $\{[R^n(\zeta)]\}_{n=0}^{\infty}$ are pairwise disjoint and can not accumulate a sink, i.e. ζ is a "wandering curve". From this it follows that

Step 3.

$$lengh(R^n(\zeta)) \to 0 \text{ as } n \to +\infty. \tag{4.1}$$

Indeed, suppose that (4.1) is not valid , then there exists a real number $\beta > 0$ such that

$$lenght(R^n(\zeta)) > \beta.$$

for infinitely n.

Let $\Sigma^2 = \Sigma \times \Sigma$. Consider the compact $K \subset \Sigma^2$ defined by

$$K = \Big\{(p, q) \in \Sigma^2 : p \in \Sigma, q \in C^\gamma(p) \cap \Sigma \text{ and } \mid y_p - y_q \mid \geq \gamma \cdot \beta\Big\}.$$

If β is small then K is a non empty compact set.

Define $H : \Sigma^2 \to [0, +\infty)$ by

$$H(p, q) = \mid \Pi^{\mathscr{F}}(p) - \Pi^{\mathscr{F}}(q) \mid .$$

As $H(p,q) = 0$ if only if $\Pi^{\mathscr{F}}(p) = \Pi^{\mathscr{F}}(q)$, then $H(p,q) > 0$ for all $(p,q) \in K$ because $\mid y_p - y_q \mid \geq \frac{\beta}{\gamma}$. The continuity of $H(\cdot,\cdot)$ and the compactness of K imply that there exists a real number $\theta > 0$ such that

$$H(p,q) \geq \theta \tag{4.2}$$

for all $(p,q) \in K$.

From step 2 we obtain that $lenght(\Pi^{\mathscr{F}}(R^n(\zeta))) \to 0$ as $n \to \infty$.

Now, let us consider n be the natural number which satisfies

$$lenght(\Pi^{\mathscr{F}}(R^n(\zeta))) < \frac{\theta}{2} \tag{4.3}$$

and

$$lenght(R^n(\zeta)) > \beta. \tag{4.4}$$

Moreover we claim that there exists $p, q \in R^n(\zeta)$ such that $(p,q) \in K$. Indeed, let $p, q \in R^n(\zeta)$ such that the length of the curve $\overline{\zeta}$, parameterized in a such way that $D\overline{\zeta}(y) = (v(y), 1)$, is contained in $R^n(\zeta)$ between p and q is equal to β (see (4.4) given above).

In other hand as $\gamma < 1$ we have

$$\begin{aligned} lenght(\overline{\zeta}) &= \int_{y_q}^{y_p} \|D\overline{\zeta}(y)\| dy \\ &\leq \gamma \cdot \mid y_p - y_q \mid, \end{aligned}$$

and this implies that $\mid y_p - y_q \mid \geq \gamma \cdot \beta$, i,e., $(p,q) \in K$.

So using (4.2) we obtain that

$$H(p,q) \geq \theta. \tag{4.5}$$

Also notice that

$$H(p,q) \leq lenght(\Pi^{\mathscr{F}}(R^n(\zeta))). \tag{4.6}$$

Therefore using (4.3), (4.5) and (4.6) we obtain

$$\theta \leq H(p,q) \leq lenght(\Pi^{\mathscr{F}}(R^n(\zeta))) < \frac{\theta}{2}$$

and this is a contradiction. The proof of (4.1) of the step 3 follows.

Step 4. $\Pi^{\mathscr{F}}(R^n(\zeta))$ accumulate to 1 . Indeed, suppose that $\Pi^{\mathscr{F}}(R^n(\zeta))$ not accumulate to 1, then there exists $\tilde{\theta} > 0$ such that $\Pi^{\mathscr{F}}(R^n(\zeta)) \subset [0, 1 - \tilde{\theta}]$ for all $n \in \mathbb{N}$. Therefore as the foliation $\mathscr{F}$ is C^0 we can choose $\theta > 0$ such that $R^n(\zeta) \cap [0,1] \times (1-\theta, 1] = \emptyset$. Applying Corollary 3.12 after a reparametrization of the curve ζ we obtain that there are $C > 0$ and $\lambda > 1$ such that for all $t \in [0,1]$, we have

$$\begin{aligned} lengh(R^n(\zeta)) &= \int_0^1 \parallel \frac{dR^n(\zeta(t))}{dt} \parallel dt \\ &= \int_0^1 \parallel DR^n(\zeta(t)) \cdot \frac{d\zeta(t)}{dt} \parallel dt \\ &\geq C \cdot \lambda^n \cdot \int_0^1 \parallel \frac{d\zeta(t)}{dt} \parallel dt \\ &= C \cdot \lambda^n . lenght(\zeta). \end{aligned}$$

Then $lengh(R^n(\zeta)) \to \infty$ as $n \to \infty$ in contradiction with (4.1) of the step 3. Therefore, $\Pi^{\mathscr{F}}(R^n(\zeta))$ accumulate to 1.

Next, we argue in order to arrive a contradiction. Let us consider $0 < \eta < \delta_4$ and an integer n_0 in a such way that $\forall n \geq n_0$ $lengh(R^n(\zeta)) < \delta_4 - \eta$. So, if for $n \geq n_0$ and $R^n(\zeta) \cap [0,1] \times (1-\eta, 1) \neq \emptyset$ then $R^n(\zeta) \subset [0,1] \times (1-\delta_4, 1)$.

As $\Pi^{\mathscr{F}}([R^n(\zeta)])$ accumulate to 1, there is a sequence n_k such that $R^n_k(\zeta) \subset [0,1] \times (1-\delta_4, 1)$. We can apply (3.43) of Proposition 3.11, after a reparametrization, to obtain that

$$lengh(R^{n_k}(\zeta)) \geq \lambda_2^{n_k - n_0} \cdot lengh(R^{n_0}(\zeta)).$$

As $n_k \to \infty$ we have that

$$lengh(R^{n_k}(\zeta)) \to \infty$$

and so we get a contradiction with (4.1) of the step 3. The proof follows. $\square$

Corollary 4.2. *Let us consider $\mathscr{V}$ the C^1-neighborhood given by Theorem 4.1. Then for all $R \in \mathscr{V}$ and for all curve ζ tangent to C^γ such that $\zeta \cap (\bigcap_{i=0}^{\infty} R^{-i}(\Sigma)) \neq \emptyset$, there exists $n = n(R,\zeta)$ such that $\Pi^{\mathscr{F}}(R^n(\zeta)) \supseteq [0,1]$, where $\Pi^{\mathscr{F}}$ is the projection along the invariant foliation $\mathscr{F}$.*

Proof. We fix $R \in \mathscr{V}$ and a curve ζ tangent to C^γ such that $\zeta \cap (\bigcap_{i=0}^{\infty} R^{-i}(\Sigma)) \neq \emptyset$. Then Theorem 4.1 implies that $\bigcap_{i=0}^{\infty} R^{-i}(\Sigma)$ don't contain the curve ζ. Therefore there are a curve $\bar{\zeta} \subseteq \zeta$ with $\bar{\zeta}(0) = p$ and $\bar{\zeta}(1) = q$ for some $p, q \in \Sigma$ with $p \in \zeta \cap (\bigcap_{i=0}^{\infty} R^{-i}(\Sigma)) \neq \emptyset$ and a integer $n_0 = n_0(\bar{\zeta})$ such that $R^{n_0}(q) \in [0,1] \times \{0, a, b, 1\}$. But $R(\{y = 1\}) \subset \{y = 0\}$ and $\{y = 0\}$ is preserved by R so there is $n > n_0$ such that $\Pi^{\mathscr{F}}(R^n(\zeta)) \supseteq [0,1]$. Therefore, from this the proof follows. $\square$

Finally, we prove the main result using some facts proved in [CMS1].

Proof. **(Theorem 2.10).** Fix R_0 as in Theorem 2.10. Take the neighborhoods $\mathscr{V}_5$ and $\mathscr{V}$ given by Corollary 3.12 and Theorem 4.1, respectively. Define $\mathscr{U} = \mathscr{V}_5 \cap \mathscr{V}$. Now, fix $R \in \mathscr{U}$ and also fix the invariant foliation $\mathscr{F}$ given by hypothesis (because R is in particular a triangular map).

Claim A: For all $p \in \bigcap_{n \geq 0} R^{-n}(\Sigma)$ the stable leaf $L = \mathscr{F}(p) \in \mathscr{F}$ is accumulate by hyperbolic periodic points of saddle type, i.e. every neighborhood of L contains a hyperbolic periodic point of saddle type.

Indeed, let U a neighborhood of L. We can take U in a such way that $U = (\Pi^{\mathscr{F}})^{-1}(\Pi^{\mathscr{F}}(U))$. Take a small curve $\zeta \subset U$ through p and tangent to C_V^γ. From Corollary 4.2 the existence of a curve $\overline{\zeta} \subseteq \zeta$ and $n \in \mathbb{N}$ such that $R^i(\overline{\gamma}) \subseteq H_{0,a} \cup H_{b,1}$ $\forall 0 \leq i \leq n-1$ and $R^n(\overline{\gamma})$ meets all leaf in $\mathscr{F}$.

Let $H(\overline{\zeta})$ be the horizontal band in Σ consisting of saturating $\overline{\zeta}$ by the foliation $\mathscr{F}$. By the property of $\overline{\zeta}$ and $n = n(\overline{\zeta})$ above we have that $R^n(H(\overline{\zeta}))$ crosses $H(\overline{\zeta})$ in a hyperbolic way. Then, by standard index arguments (see [N]) there is a periodic point of R on $R^n(H(\overline{\zeta})) \cap H(\overline{\zeta})$. By taking ζ close to L, the band $H(\overline{\zeta})$ remain close to L and then we have that such a point belongs to U. Such a periodic point is hyperbolic saddle by Corollary 3.12 and the fact that the foliation $\mathscr{F}$ is R-contracting (see Remark 3.8). This proves our Claim A.

Claim B: The hyperbolic periodic points of saddle type of R are dense in $\bigcap_{n\in\mathbb{Z}} R^n(\Sigma)$.

Indeed, take a point $z \in \bigcap_{n\in\mathbb{Z}} R^n(\Sigma)$ and take a neighborhood V of z. Take an integer n large enough such that $L = \mathscr{F}(R^{-n}(z))$ the leaf that contains $R^{-n}(z)$ is applied by R^n into V. So the same applies to a small horizontal band U around the leaf L. By Claim A there exists a periodic point of saddle type in U. Therefore the orbit of this periodic point visits the neighborhood V. This proves our Claim B.

To finish the proof of the transitivity of R (i.e., the invariant maximal set given by (2.3) is transitive) we will use the classical Birkhoff's criterium to prove transitivity: for all $p, q \in \bigcap_{n\in\mathbb{Z}} R^n(\Sigma)$ and $\varepsilon > 0$ there are $z \in \bigcap_{n\in\mathbb{Z}} R^n(\Sigma)$ and $n_z \in \mathbb{N}$ such that $d(z,p) < \varepsilon$ and $d(R^{n_z}(z), q) < \varepsilon$. Indeed, fix p, q and ϵ. By the above claim B we can assume that p and q are hyperbolic periodic points of saddle type. Fix a curve γ in $W^u(p)$ contained in Σ. We can assume that γ intersects to the leaf $\mathscr{F}(q)$ transversely in some point z^* by Corollary 4.2. Since the positive (resp. negative) orbit of z^* is asymptotic to q (resp. p) we have $z^* \in \bigcap_{n\in\mathbb{Z}} R^n(\Sigma)$. By taking the negative orbit of z^* we have some $n_1^* \in \mathbb{N}$ such that

$$d(R^{-n_1^*}(z^*), p) < \varepsilon.$$

By taking the positive orbit of z^* we have some $n_2^* \in \mathbb{N}$ such that

$$d(R^{n_2^*}(z^*), q) < \varepsilon.$$

Then $z = R^{-n_1^*}(z^*)$ and $n_z = n_1^* + n_2^*$ works

This finish the proof of the Theorem 2.10. □

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D., Carrasco-Olivera
Departamento de Matemáticas
Universidad del Bío Bío
Casilla 5-C, Región del Bío Bío, Concepción, Chile
e-mail: dcarrasc@ubiobio.cl

B., San Martin
Departamento de Matemáticas
Universidad Católica del Norte
Casilla 1280, Antofagasta, Chile
e-mail: sanmarti@ucn.cl

C., Vidal
Departamento de Matemáticas
Universidad del Bío Bío
Casilla 5-C, Región del Bío Bío, Concepción, Chile
e-mail: clvidal@ubiobio.cl