A GENERALIZATION OF EXPANSIVITY

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ABSTRACT. We study dynamical systems for which at most n orbits can accompany a given arbitrary orbit. For simplicity we call them *n-expansive* (or *positively n-expansive* if positive orbits are considered instead). We prove that these systems can satisfy properties of expansive systems or not. For instance, unlike positively expansive maps [3], positively n-expansive homeomorphisms may exist on certain infinite compact metric spaces. We also prove that a map (resp. bijective map) is positively n-expansive (resp. n-expansive) if and only if it is so outside finitely many points. Finally, we prove that a homeomorphism on a compact metric space is n-expansive if and only if it is so outside finitely many orbits. These last results extends previous ones about expansive systems [2],[11],[12].

1. **Introduction.** In classical terms, a discrete dynamical system on a metric space is *expansive* if every orbit can be accompanied by just one orbit up to some prefixed radius. This concept originally introduced for bijective maps [10] has been generalized to *positively expansiveness* in which positive orbits are considered instead [5]. Further generalizations are the *pointwise expansiveness* (with the above radius depending on the point [9]), the *entropy-expansiveness* [1], the *continuum-wise expansiveness* [7], the *measure-expansiveness* (involving Borel probability measures [8]) and their corresponding positive counterparts. However, as far as we know, no one have considered the generalization in which at most n companion orbits are allowed for a certain prefixed positive integer n. For simplicity we call these systems *n-expansive* (or *positively n-expansive* if positive orbits are considered instead). The natural question is whether these systems can satisfy properties of expansive systems or not.

In this paper we shall provide both positive and negative answers for this question. For instance, unlike positively expansive maps [3], we shall exhibit arbitrarily large values of n for which there are infinite compact metric spaces carrying positively n-expansive homeomorphisms. Next, we prove that a map (resp. bijective map) is positively n-expansive (resp. n-expansive) if and only if it is so outside finitely many *points*. Finally, we prove that a homeomorphism on a compact metric space is n-expansive if and only if it is so outside finitely many *orbits*. These last two results extend previous ones for expansive dynamical systems in [2],[11],[12].

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This paper is organized as follows. In Section 2 we present some topological preliminaires. In Section 3 we give the precise definition of n-expansive systems and study their basic properties. In Section 4 we prove our main results using those in sections 2 and 3.

2. **Preliminaries.** In this section we establish some topological preliminaries. Let X a set and n be a nonnegative integer. Denote by $\#A$ the cardinality of A. The set of metrics of X (including ∞-metrics [4]) will be denoted by $\mathbb{M}(X)$. Sometimes we say that $\rho \in \mathbb{M}(X)$ has a certain property whenever its underlying metric space (X, ρ) does. For example, ρ is compact whenever (X, ρ) is, a point a is ρ-isolated in $A \subset X$ if it is isolated in A with respect to the metric space (X, ρ), etc.. The closure operation in (X, ρ) will be denoted by $Cl_\rho(\cdot)$. A map $f : X \to X$ is a *ρ-homeomorphism* if it is a homeomorphism of the metric space (X, ρ). If $x \in X$ and $\delta > 0$ we denote by $B^\rho[x, \delta]$ the closed δ-ball around x (or $B[x, \delta]$ if there is no confusion).

Given $\rho \in \mathbb{M}(X)$ and $A \subset X$ we say that ρ is *n-discrete on A* if there is $\delta > 0$ such that $\#(B[x, \delta] \cap A) \leq n$ for all $x \in A$. Equivalently, if there is $\delta > 0$ such that $\#(B[x, \delta] \cap A) \leq n$ for all $x \in X$. When necessary we emphasize δ by saying that *ρ is n-discrete on A with constant δ*. We say that ρ is n-discrete if it is n-discrete on X. Clearly ρ is n-discrete on A if and only if the *restricted metric $\rho/A \in \mathbb{M}(A)$* defined by $\rho/A(a, b) = \rho(a, b)$ for $a, b \in A$ is n-discrete.

Evidently, there are no 0-discrete metrics and the 1-discrete metrics are precisely the discrete ones. Since every n-discrete metric is m-discrete for $n \leq m$ one has that every discrete metric is n-discrete. There are however n-discrete metrics which are not discrete. Moreover, we have the following example ([1]).

Example 2.1. *Every infinite set X carries an n-discrete metric which is not $(n-1)$-discrete.*

Indeed, if $n = 1$ we simply choose ρ as the standard discrete metric $\delta(x, y)$ defined by $\delta(x, y) = 1$ whenever $x \neq y$. Otherwise, we can arrange n disjoint sequences $x_k^1, x_k^2 \cdots, x_k^n$ in X and define ρ by

$$\rho(x, y) = \begin{cases} \frac{1}{4+k} & \text{if} \quad \exists k \in \mathbb{N}, \exists 1 \leq i \neq j \leq n \ : \ (x, y) = (x_k^i, x_k^j), \\ \\ \delta(x, y) & \text{otherwise.} \end{cases}$$

On the one hand, ρ is n-discrete with constant $\delta = 1/4$ since

$$B\left[x, \frac{1}{4}\right] = \begin{cases} \{x_k^1, \cdots, x_k^n\} & \text{if} \quad \exists k \in \mathbb{N}, \exists 1 \leq i \leq n \ : \ x = x_k^i, \\ \\ \{x\} & \text{otherwise} \end{cases}$$

and, on the other, ρ is not $(n-1)$-discrete since for all $\delta > 0$ the set of points x for which $\#B[x, \delta] = n$ is infinite (e.g. take $x = x_k^1$ with k large).

Remark 2.2. *None of the metrics in Example 2.1 can be compact for, otherwise, we could cover X with finitely many balls of radius $\delta = 1/4$ which would imply that X is finite.*

In the sequel we present some basic properties of n-discrete metrics. Clearly if ρ is n-discrete on A, then it is also n-discrete on B for all $B \subset A$. Moreover, if ρ

[1] communicated by professors L. Florit and A. Iusem.

is n-discrete on A and m-discrete on B, then it is $(n+m)$-discrete on $A \cup B$. A better conclusion is obtained when the distance between A and B is positive.

Lemma 2.3. *If ρ is n-discrete on A, m-discrete on B and $\rho(A,B) > 0$, then ρ is $\max\{n,m\}$-discrete on $A \cup B$.*

Proof. Choose $0 < \delta < \frac{\rho(A,B)}{2}$ such that $\#(B[x,\delta] \cap A) \leq n$ (for $x \in A$) and $\#(B[x,\delta] \cap B) \leq m$ (for $x \in B$). If $x \in A$ then $B[x,\delta] \cap B = \emptyset$ because $\delta < \frac{\rho(A,B)}{2}$ so $\#(B[x,\delta] \cap (A \cup B)) = \#(B[x,\delta] \cap A) \leq n \leq \max\{n,m\}$. If $x \in B$ then $B[x,\delta] \cap A = \emptyset$ because $\delta < \frac{\rho(A,B)}{2}$ so $\#(B[x,\delta] \cap (A \cup B)) = \#(B[x,\delta] \cap B) \leq m \leq \max\{n,m\}$. Then, ρ is $\max\{n,m\}$-discrete on $A \cup B$ with constant δ. □

Lemma 2.4. *If ρ is n-discrete on A, then A is ρ-closed and so $\rho(A,B) > 0$ for every ρ-compact subset B with $A \cap B = \emptyset$.*

Proof. We only have to prove the first part of the lemma. By hypothesis there is $\delta > 0$ such that $\#(B[x,\delta] \cap A) \leq n$ for all $x \in A$. Let x_k be a sequence in A converging to some $y \in X$. It follows that there is $k_0 \in \mathbb{N}^+$ such that $x_k \in B[y, \delta/2]$ for all $k \geq k_0$. Triangle inequality implies $\{x_k : k \geq k_0\} \subseteq B[x_{k_0}, \delta] \cap A$ and so $\{x_k : k \geq k_0\}$ is a finite set. As $x_k \to y$ we conclude that $y \in A$ hence A is closed. □

Now we prove that n-discreteness is preserved under addition of finite subsets.

Proposition 2.5. *If ρ is n-discrete on A, then ρ is n-discrete on $A \cup F$ for all finite $F \subset X$.*

Proof. We can assume that $A \cap F = \emptyset$. As F is finite (hence compact) we can apply Lemma 2.4 to obtain $\rho(A,F) > 0$. As F is finite one has that ρ is 1-discrete on F so ρ is n-discrete on $A \cup F$ by Lemma 2.3. □

For the next result we introduce some basic definitions. Let $f : X \to X$ be a map. We say that $A \subset X$ is *invariant* if $f(A) = A$. If f is bijective and $x \in X$ we denote by $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ the orbit of x. An *isometry* (or *ρ-isometry* to emphasize ρ) is a bijective map f satisfying $\rho(f(x), f(y)) = \rho(x,y)$ for all $x, y \in X$.

The following elementary fact will be useful later one: If f is a ρ-isometry and $a \in X$ satisfies that a is ρ-isolated in $O_f(a)$, then ρ is discrete on $O_f(a)$. Indeed, if ρ were not discrete on $O_f(a)$, then there are integer sequences $n_k \neq m_k$ such that $\rho(f^{n_k}(a), f^{m_k}) \to 0$ as $k \to \infty$. As f is an isometry one has that $\rho(f^{n_k}(a), f^{m_k}(a)) = \rho(a, f^{l_k}(a))$, where $l_k = m_k - m_k$, so $\rho(a, f^{l_k}(a)) \to 0$ for some sequence $l_k \in \mathbb{Z} \setminus \{0\}$ thus a is not ρ-isolated in $O_f(a)$.

Given $d, \rho \in \mathbb{M}(X)$ we write $d \leq \rho$ whenever $d(x,y) \leq \rho(x,y)$ for all $x, y \in X$. We write $\rho \preceq d$ to indicate lower semicontinuity of the map $\rho : X \times X \to [0, \infty]$ with respect to the product metric $d \times d$ in $X \times X$. Equivalently, the following property holds for all sequences x_k, y_k in X and all $\delta > 0$, where $x_k \xrightarrow{d} x$ indicates convergence in (X,d):

$$x_k \xrightarrow{d} x, \quad y_k \xrightarrow{d} y \quad \text{and} \quad y_k \in B^\rho[x_k, \delta] \quad \Longrightarrow \quad y \in B^\rho[x, \delta]. \tag{1}$$

Hereafter we denote by $Fix(f) = \{x \in X : f(x) = x\}$ the set of fixed points of f, and by $Per(f) = \bigcup_{m \in \mathbb{N}^+} Fix(f^m)$ the set of periodic points of f.

The following proposition is inspired on Lemma 2 p. 176 of [12].

Proposition 2.6. *Let $d, \rho \in \mathbb{M}(X)$ be such that d is compact and $d \leq \rho \preceq d$. Let $f : X \to X$ be a map which is simultaneously a d-homeomorphism and a ρ-isometry. If A is an invariant set with countable complement which is n-discrete with respect to ρ and $Per(f) \cap A$ is countable, then ρ is n-discrete on $A \cup O_f(a)$ for all $a \in X$.*

Proof. We can assume $a \notin A$ (otherwise $A \cup O_f(a) = A$) so $\rho(A, a) > 0$ by Lemma 2.4. Since f is a ρ-isometry and A is invariant one has $\rho(A, f^i(a)) = \rho(A, a)$ so $\rho(A, O_f(a)) > 0$. Then, by Lemma 2.3, it suffices to prove that ρ is n-discrete on $O_f(a)$.

Suppose that it is not so. Then, as previously remarked, a is non ρ-isolated in $O_f(a)$. Since $d \leq \rho$ we have that a is also non ρ-isolated in $O_f(a)$. As f is a d-homeomorphism we conclude that $O_f(a)$ is a nonempty ρ-perfect set. As d is compact (and so $\mathrm{F_{II}}$) we obtain that $Cl_d(O_f(a))$ is uncountable. As $X \setminus A$ is countable we conclude that $Cl_d(O_f(a)) \cap A$ is uncountable.

Choose $x \in Cl_d(O_f(a)) \cap A$. Then, there is a sequence $l_k \in \mathbb{Z}$ such that

$$f^{l_k}(a) \xrightarrow{d} x. \tag{2}$$

Let $\delta > 0$ be such that ρ is n-discrete on A with constant δ. Since ρ is not n-discrete on $O_f(a)$ we can arrange different integers $N_1, \cdots, N_{n+1}$ satisfying

$$f^{N_j}(a) \in B^\rho[f^{N_1}(a), \delta], \qquad \forall j \in \{1, \cdots, n+1\}.$$

On the other hand, f is a ρ-isometry so the above inclusions yield

$$f^{N_j}(f^{l_k}(a)) \in B^\rho[f^{N_1}(f^{l_k}(a)), \delta], \qquad \forall j \in \{1, \cdots, n+1\}, \quad \forall k \in \mathbb{N}.$$

By taking limit as $k \to \infty$ in the above inclusion, keeping j fixed and applying (1) and (2) to obtain

$$f^{N_j}(x) \in B^\rho[f^{N_1}(x), \delta], \qquad \forall j \in \{1, \cdots, n+1\}.$$

Now observe that $f^{N_j}(x) \in A$ for all $j \in \{1, \cdots, n+1\}$ because A is invariant. Therefore,

$$\{f^{N_1}(x), \cdots, f^{N_{n+1}}(x)\} \subset B^\rho[f^{N_1}(x), \delta] \cap A.$$

But $\#(B^\rho[f^{N_1}(x), \delta] \cap A) \leq n$ by the choice of δ so the above inclusion implies $f^{N_j}(x) = f^{N_r}(x)$ for some different indexes $j, r \in \{1, \cdots, n+1\}$. As the integers $N_1, \cdots, N_{n+1}$ are different we conclute that $x \in Per(f)$ and so $x \in Per(f) \cap A$. Therefore,

$$Cl_d(O_f(a)) \cap A \subset Per(f) \cap A.$$

As $Cl_d(O_f(a)) \cap A$ is uncountable we conclude that $Per(f) \cap A$ also is thus we get a contradiction. This proves the result. □

Corollary 2.7. *Let $d, \rho \in \mathbb{M}(X)$ be such that d is compact and $d \leq \rho \preceq d$. Let $f : X \to X$ be a map which is simultaneously a d-homeomorphism and a ρ-isometry. If $Per(f)$ is countable and there are $a_1, \cdots, a_l \in X$ such that ρ is n-discrete on $X \setminus \bigcup_{i=1}^{l} O_f(a_i)$, then ρ is n-discrete.*

Proof. Define the invariant sets $A_j = X \setminus \bigcup_{i=j}^{l} O_f(a_i)$ for $1 \leq j \leq l$. As $X \setminus A_j = \bigcup_{i=j}^{l} O_f(a_i)$ one has that A_j has countable complement for all $1 \leq j \leq l$. On the other hand, ρ is n-discrete on A_1 by hypothesis and $Per(f) \cap A_1$ is countable (since $Per(f)$ is) so ρ is n-discrete on $A_2 = A_1 \cup O_f(a_1)$ by Proposition 2.6. By the same reasons if ρ is n-discrete on A_j, then ρ also is on $A_{j+1} = A_j \cup O_f(a_i)$. Then, the result follows by inducción. □

3. **n-expansive systems.** In this section we define and study the class of n-expansive systems briefly mentioned in the Introduction. To motivate the definition we recall the classical concepts of expansive and positively expansive systems [5], [10]. Let (X, d) be a metric space and $A \subset X$. A map $f : X \to X$ is *positively expansive on A* if there is $\delta > 0$ such that for every $x, y \in A$ with $x \neq y$ there is $i \in \mathbb{N}$ such that $d(f^i(x), f^i(y)) > \delta$, or, equivalently, if $\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{N}\} = \{x\}$ for all $x \in A$. On the other hand, a bijective map $f : X \to X$ is *expansive on A* if there is $\delta > 0$ such that $\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{Z}\} = \{x\}$ for all $x \in A$. If $A = X$ we say that f is *positively expansive* or *expansive* respectively. These definitions suggest the following ones implicitely mentioned in [8].

Definition 3.1. *Given $n \in \mathbb{N}^+$ a bijective map (resp. map) f is* n-expansive *(resp.* positively n-expansive*) on A if there is $\delta > 0$ such that*

$$\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{Z}\} \quad (\textit{resp. } \{y \in A : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{N}\})$$

has at most n elements, $\forall x \in A$. If case $A = X$ we say that f is n-expansive *(resp.* positively n-expansive*).*

Clearly the 1-expansive bijective maps are precisely the expansive ones (which in turn are n-expansive for all $n \in \mathbb{N}^+$). It is also clear that every n-expansive bijective map is pointwise expansive in the sense of [9].

In the sequel we introduce two useful operators. For every $f : X \to X$ and $d \in \mathbb{M}(X)$ we define the pull-back $f_*(d)(x, y) = d(f(x), f(y))$ (clearly $f_*(d) \in \mathbb{M}(X)$ if and only if f is 1-1). Using it we can define the operator $\mathcal{L}_f^+ : \mathbb{M}(X) \to \mathbb{M}(X)$ by

$$\mathcal{L}_f^+(d) = \sup_{i \in \mathbb{N}} f_*^i(d), \qquad \forall d \in \mathbb{M}(X).$$

If f is bijective we can define $\mathcal{L}_f : \mathbb{M}(X) \to \mathbb{M}(X)$ by

$$\mathcal{L}_f(d) = \sup_{i \in \mathbb{Z}} f_*^i(d), \qquad \forall d \in \mathbb{M}(X).$$

Lemma 3.2. *If f is bijective, then $d \leq \mathcal{L}_f(d)$ and f is a $\mathcal{L}_f(d)$-isometry. If, in addition, f is a d-homeomorphism, then $\mathcal{L}_f(d) \preceq d$.*

Proof. The first inequality is evident. As

$$f_*(\mathcal{L}_f(d))(x, y) = \sup_{i \in \mathbb{Z}} d(f^{i+1}(x), f^{i+1}(y)) = \sup_{i \in \mathbb{Z}} d(f^i(x), f^i(y)) = \mathcal{L}_f(d)(x, y)$$

for all $x, y \in X$ one has $f_*(\mathcal{L}_f(d)) = \mathcal{L}_f(d)$ hence f is an $\mathcal{L}_f(d)$-isometry. Now we prove $\mathcal{L}_f(d) \preceq d$ whenever f is a d-homeomorphism. Suppose that $x_k \xrightarrow{d} x$, $y_k \xrightarrow{d} y$ and $\mathcal{L}_f(d)(x_k, y_k) \leq \delta$ for all $k \in \mathbb{N}$. Fixing $i \in \mathbb{Z}$ the latter inequality implies $d(f^i(x_k), f^i(y_k)) \leq \delta$ for all k. As f is a d-homeomorphism one can take the limit as $k \to \infty$ in the last inequality to obtain $d(f^i(x), f^i(y)) \leq \delta$. As $i \in \mathbb{Z}$ is arbitrary we obtain $\mathcal{L}_f(d)(x, y) \leq \delta$ which together with (2) implies the result. □

These operators give the link between discreteness and expansiveness by the following result. Hereafter we shall write f is (positively) n-expansive (on A) *with respect to d* in order to emphazise the metric d in Definition 3.1.

Lemma 3.3. *The following properties hold for all $f : X \to X$, $A \subset X$ and $d \in \mathbb{M}(X)$:*

1. *f is positively n-expansive on A with respect to d if and only if $\mathcal{L}_f^+(d)$ is n-discrete on A.*

2. *If f is bijective, f is n-expansive on A with respect to d if and only if $\mathcal{L}_f(d)$ is n-discrete on A.*

Proof. Clearly for all $x \in X$ and $\delta > 0$ one has

$$B^{\mathcal{L}_f^+(d)}[x,\delta] \cap A = \{y \in A : d(f^i(x), f^i(y)) \leq \delta, \quad \forall i \in \mathbb{N}\},$$

so

$$\#(B^{\mathcal{L}_f^+(d)}[x,\delta] \cap A) \leq n \quad \iff \quad \#(\{y \in A : d(f^i(x), f^i(y)) \leq \delta, \quad \forall i \in \mathbb{N}\}) \leq n$$

which proves the equivalence (1). The proof of the equivalence (2) is analogous. □

As a first application of the above equivalence we shall exhibit non-trivial examples of positively n-expansive maps. More precisely, we prove that every bijective map $f : X \to X$ with at least n non-periodic points ($n \geq 2$) carries a metric ρ making it continuous positively n-expansive but not positively $(n-1)$-expansive. Indeed, by hypothesis there are $x^1, \cdots, x^n \in X$ such that $f^i(x^j) \neq f^k(x^j)$, for all $1 \leq j \leq n$ and $i \neq k \in \mathbb{N}$, and $f^i(x^j) \neq f^i(x^k)$ for all $i \in \mathbb{N}$ and $1 \leq j \neq k \leq n$. Define the sequences $x_k^1, \cdots, x_k^n$ in X by $x_k^i = f^k(x^i)$ for $1 \leq i \leq n$ and $k \in \mathbb{N}$. Clearly these sequences are disjoint thus they induce a metric ρ in X which is n-discrete but not $(n-1)$-discrete as in Example 2.1. On the other hand, a straightforward computation yields $\mathcal{L}_f^+(\rho) = \rho$ thus f is continuous (in fact Lipschitz) for ρ. Since ρ is n-discrete and $\rho = \mathcal{L}_f^+(\rho)$ one has that $\mathcal{L}_f^+(\rho)$ is n-discrete so f is positively n-expansive by Lemma 3.3. Since ρ is not $(n-1)$-discrete and $\rho = \mathcal{L}_f^+(\rho)$ the same lemma implies that f is not positively $(n-1)$-expansive.

Notice however that none of the above metrics is compact (see for instance Remark 2.2). This fact leads the question as to whether a bijective map can carry a compact metric making it positively n-expansive but not positively $(n-1)$-expansive. Indeed, the following result gives a partial positive answer for this question.

Proposition 3.4. *For every $k \in \mathbb{N}^+$ there is a homeomorphism f_k of a compact metric space (X_k, ρ_k) which is positively 2^k-expansive but not positively $(2^k - 1)$-expansive.*

Proof. To start with we recall that a *Denjoy map* of the circle S^1 is a nontransitive homeomorphism of S^1 with irrational rotation number. As is well known [6] every Denjoy map h exhibits a unique minimal set E_h which is also a Cantor set.

Hereafter we fix the standard Riemannian metric l of S^1. We shall prove that h/E_h is positively 2-expansive with respect to l/E_h. Let α be half of the length of the largest interval I in the complement $S^1 \setminus E_h$ and $0 < \delta < \alpha$.

We claim that $Int(B^{\mathcal{L}_h^+(l)}[x,\delta]) \cap E_h = \emptyset$ for all $x \in E_h$. Otherwise, there is some $z \in Int(B^{\mathcal{L}_h^+(l)}[x,\delta]) \cap E_h$. Pick $w \in \partial I$ (thus $w \in E_h$). Since E_h is minimal there is a sequence $n_k \to \infty$ such that $h^{-n_k}(w) \to z$. Now, the interval sequence $\{h^{-n}(I) : n \in \mathbb{N}\}$ is disjoint so we have that the length of the intervals $h^{-n_k}(I) \to 0$ as $k \to \infty$. It turns out that there is some integer k such that $h^{-n_k}(I) \subset B^{\mathcal{L}_h^+(l)}[x,\delta]$. From this and the fact that $h(B^{\mathcal{L}_h^+(l)}[x,\delta]) \subset B^{\mathcal{L}_h^+(l)}[h(x),\delta]$ one sees that $I \subset B^{\mathcal{L}_h^+(l)}[h^{n_k}(x),\delta]$ which is clearly absurd because the length of I is greather than $\alpha > 2\delta$. This contradiction proves the claim.

Since $B^{\mathcal{L}_h^+(l)}[x,\delta]$ reduces to closed interval (possibly trivial) the claim implies that $B^{\mathcal{L}_h^+(l)}[x,\delta] \cap E_h$ consists of at most two points. It follows that $\mathcal{L}_h^+(l)$ is 2-discrete on E_h (with constant δ), so, h/E_h is positively 2-expansive with respect to

l/E_h by Lemma 3.3. Since there are no positively expansive homeomorphisms on infinite compact metric spaces (e.g. [3]) one sees that h/E_h cannot be positively expansive with respect to l/E_h. Taking $X_1 = E_h$, $\rho_1 = l/E_h$ and $f_1 = h/E_h$ we obtain the result for $k = 1$. To obtain the result for $k \geq 2$ we shall proceed according to the following straightforward construction.

Take copies E_1, E_2 of E_h and recall the map

$$\max\{\cdot,\cdot\} : \mathbb{M}(E_1) \times \mathbb{M}(E_2) \to \mathbb{M}(E_1 \times E_2)$$

defined by

$$\max\{d_1, d_2\}(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $E_1 \times E_2$. One clearly sees that

$$B^{\max\{d_1,d_2\}}[x, \delta] = B^{d_1}[x_1, \delta] \times B^{d_2}[x_2, \delta], \qquad \forall x \in E_1 \times E_2, \forall \delta > 0.$$

Afterward, take copies h_1, h_2 of h/E_h and define the product $h_1 \times h_2 : E_1 \times E_2 \to E_1 \times E_2$, $(h_1 \times h_2)(x) = (h_1(x_1), h_2(x_2))$. It turns out that

$$(h_1 \times h_2)_*(\max\{d_1, d_2\}) = \max\{h_{1*}(d_1), h_{2*}(d_2)\}$$

so

$$\mathcal{L}^+_{h_1 \times h_2}(\max\{d_1, d_2\}) = \max\{\mathcal{L}^+_{h_1}(d_1), \mathcal{L}^+_{h_2}(d_2)\}$$

thus

$$B^{\mathcal{L}^+_{h_1\times h_2}(\max\{d_1,d_2\})}[x, \delta] = B^{\mathcal{L}^+_{h_1}(d_1)}[x_1, \delta] \times B^{\mathcal{L}^+_{h_2}(d_2)}[x_2, \delta].$$

Finally, take copies d_1, d_2 of the metric l/E_h each one in E_1, E_2 respectively. As h_i is positively 2-expansive with respect to d_i one has that $\mathcal{L}^+_{h_i}(d_i)$ is 2-discrete for $i = 1, 2$. We can choose the same constant for $i = 1, 2$ (δ say) thus,

$$\#(B^{\mathcal{L}^+_{h_1\times h_2}(\max\{d_1,d_2\})}[x, \delta]) = \#(B^{\mathcal{L}^+_{h_1}(d_1)}[x_1, \delta]) \cdot \#(B^{\mathcal{L}^+_{h_2}(d_2)}[x_2, \delta]) \leq 2^2 \qquad (3)$$

for all $x \in E_1 \times E_2$.

Now, consider the compact metric space $(E_1 \times E_2, \max\{d_1, d_2\})$. It follows from (3) and Lemma 3.3 that $h_1 \times h_2$ (which is clearly a homeomorphism) is positively 2^2-expansive map with respect to $\max\{d_1, d_2\}$. On the other hand, one can see that $\#(B^{\mathcal{L}^+_{h_1\times h_2}(\max\{d_1,d_2\})}[x, \delta]) = 2^2$ for infinitely many x's and arbitrarily small δ thus $h_1 \times h_2$ cannot be positively $2^2 - 1$-expansive. Taking $X_2 = E_1 \times E_2$, $\rho_2 = \max\{d_1, d_2\}$ and $f_2 = h_1 \times h_2$ we obtain the result for $k = 2$.

By repeating this argument we obtain the result for arbitrary $k \in \mathbb{N}^+$ taking $X_2 = E_1 \times \cdots \times E_k$, $\rho_k = \max\{d_1, \cdots, d_k\}$ and $f_k = h_1 \times \cdots \times h_k$. □

As a second application of the equivalence in Lemma 3.3 we establish the following lemma which is well-known among expansive systems (e.g. Lemma 1 in [12]).

Lemma 3.5. *If a homeomorphism f of a metric space (X, d) is n-expansive on A, then $Per(f) \cap A$ is countable.*

Proof. It follows from the hypothesis and Lemma 3.3 that there is $\delta > 0$ such that $\#(B^{\mathcal{L}_f(d)}[x, \delta] \cap A) \leq n$ for all $x \in X$.

First we prove that f^m is n-expansive on A, $\forall m \in \mathbb{N}^+$. Observe that f is continuous since d is compact so there is $\epsilon > 0$ such that $d(x, y) \leq \epsilon$ implies $d(f^i(x), f^i(y)) \leq \delta$ for all integer $-m \leq i \leq m$. Then, $B^{\mathcal{L}_{f^m}(d)}[x, \epsilon] \subset B^{\mathcal{L}_f(d)}[x, \delta]$ for all $x \in X$, so, $\#(B^{\mathcal{L}_{f^m}(d)}[x, \epsilon] \cap A) \leq \#(B^{\mathcal{L}_f(d)}[x, \delta] \cap A) \leq n$ for all $x \in A$. Therefore, $\mathcal{L}_{f^m}(d)$ is n-discrete on A (with constant ϵ) which implies that f^m is n-expansive on A by Lemma 3.3. This proves the assertion.

Since $Per(f) = \bigcup_{m\in\mathbb{N}^+} Fix(f^m)$ by the previous assertion we only have to prove that $Fix(f) \cap A$ is finite whenever f is n-expansive on A. To prove it suppose that there is an infinite sequence of fixed points $x_k \in A$. Since d is compact one can assume that $x_n \overset{d}{\to} x$ for some $x \in X$. On the other hand, one clearly has $\mathcal{L}_f(d) = d$ in $Fix(f)$ thus, by the triangle inequality on x, there is $n_0 \in \mathbb{N}$ such that $x_n \in B^{\mathcal{L}_f(d)}[x_{n_0}, \delta]$ for all $n \geq n_0$. Thus, $\#(B^{\mathcal{L}_f(d)}[x_{n_0}, \delta] \cap A) = \infty$ which contradicts the choice of δ above. This ends the proof. □

4. **The results.** In this section we state and prove our main results. The first one establishes that there are arbitrarily large values of n for which there are infinite compact metric spaces carrying positively n-expansive homeomorphisms. As is well known, this is not true in the positively expansive case (see for instance [3]).

Theorem 4.1. *For every $k \in \mathbb{N}^+$ there is an infinite metric space (X_k, ρ_k) carrying positively 2^k-expansive homeomorphisms.*

Proof. Take X_k, ρ_k and f_k as in Proposition 3.4. As f_k is not positively $(2^k - 1)$-expansive one has that X_k is infinite. □

Our second result generalizes the one in [2].

Theorem 4.2. *A map (resp. bijective map) of a metric space (X, d) is positively n-expansive (resp. n-expansive) if and only if it is positively n-expansive (resp. n-expansive) on $X \setminus F$ for some finite subset F.*

Proof. Obviously we only have to prove the if part. We do it in the positively n-expansive case as the n-expansive case follows analogously. Suppose that a map f of X is positively n-expansive on $X \setminus F$ for some finite subset F. Then, $\mathcal{L}_f^+(d)$ is n-discrete on $A = X \setminus F$ by Lemma 3.3. Since F is finite Proposition 2.5 implies that $\mathcal{L}_f^+(d)$ is n-discrete so f is positively n-expansive by Lemma 3.3. □

Finally we state our last result which extends a well-known property of expansive homeomorphisms (c.f. [11],[12]).

Theorem 4.3. *A homemomorphism f of a compact metric space (X, d) is n-expansive if and only if it is n-expansive on $X \setminus \bigcup_{i=1}^{l} O_f(a_i)$ for some $a_1, \cdots, a_l \in X$.*

Proof. We only have to prove the if part. By hypothesis f is a d-homeomorphism so f is an $\mathcal{L}_f(d)$-isometry and $d \leq \mathcal{L}_f(d) \preceq d$ by Lemma 3.2. Since f is n-expansive on $A = X \setminus \bigcup_{i=1}^{l} O_f(a_i)$ one has that $Per(f) \cap A$ is countable by Lemma 3.5. As $X \setminus A = \bigcup_{i=1}^{l} O_f(a_i)$ is clearly countable we conclude that $Per(f)$ is countable. On the other hand, f is n-expansive on $X \setminus \bigcup_{i=1}^{l} O_f(a_i)$ so $\mathcal{L}_f(d)$ is n-discrete on $X \setminus \bigcup_{i=1}^{l} O_f(a_i)$ by Lemma 3.3. Then, $\mathcal{L}_f(d)$ is n-discrete by Corollary 2.7 and so f is n-expansive by Lemma 3.3. □

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