Instituto Nacional de Matemática Pura e Aplicada

Multi-Period Risk Management and Portfolio Optimization: Case Studies of BM&F-BOVESPA Assets

Autor: **Leandro Lyra Braga Dognini**

Advisor: **Jorge Passamani Zubelli**

Rio de Janeiro
August 23, 2015

ATA DE DEFESA DE DISSERTAÇÃO PARA CONCESSÃO DO GRAU DE MESTRE PROFISSIONAL EM MÉTODOS MATEMÁTICOS EM FINANÇAS

PROGRAMA: **MESTRADO PROFISSIONAL EM MÉTODOS MATEMÁTICOS EM FINANÇAS**

CANDIDATO: **LEANDRO LYRA BRAGA DOGNINI**

BANCA EXAMINADORA		
	NOME	INSTITUIÇÃO
1	**JORGE PASSAMANI ZUBELLI** – Orientador	IMPA
2	HUBERT MARIE LACOIN	IMPA
3	JUAN PABLO GAMA TORRES	IMPA
4	JUAN PABLO CAJAHUANCA LUNA	UFRJ
5	CARLOS HEITOR CAMPANI - Suplente	UFRJ

Título da Dissertação: **"Multi-Period Risk Management and Portfolio Optimization: Case Studies of BM&F-Bovespa Assets"**

Data: 25/09/2015 Hora: 14:00 Sala: 224

O candidato fez uma exposição de uma hora sobre o resultado da sua dissertação seguida de arguição realizada pelos membros da Banca Examinadora que o considerou:

[X] APROVADO

[ ] REPROVADO

Instituto Nacional de Matemática Pura e Aplicada IMPA

ASSOCIAÇÃO INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA
CONFERE COM O ORIGINAL
Certifico a autenticidade desta fotocópia
Rio, 08/12/15
ISABEL TREIGER CHERQUES
CHEFE DA DIVISÃO DE ENSINO
IMPA

A presente Ata foi lavrada e assinada pelos membros da Banca, na ordem acima determinada, e pelo candidato.

BANCA EXAMINADORA	**JORGE PASSAMANI ZUBELLI**	
	HUBERT MARIE LACOIN	
	JUAN PABLO GAMA TORRES	
	JUAN PABLO CAJAHUANCA LUNA	
CANDIDATO	**LEANDRO LYRA BRAGA DOGNINI**	

impa SELO DE AUTENTICIDADE
INSTITUTO NACIONAL DE MATEMÁTICA PURA E APLICADA
7154

À minha família.

Agradecimentos

Agradeço aos meus pais, Odilmar e Deusa, e à minha irmã Alessandra, por terem sempre estado ao meu lado, me apoiando em cada passo que eu decidi dar.

Agradeço à Cíntia Fernanda, por todos estes anos que você foi, para mim, fonte de alegria, carinho e motivação.

Agradeço ao IMPA, pelas oportunidades que me foram dadas e que me permitiram chegar até aqui. Em especial, ao Professor Jorge Zubelli, que sempre me permitiu abraçar os desafios que me fascinavam.

Agradeço à ANBIMA, pela confiança investida neste trabalho.

Resumo

Esta dissertação analisa medidas de risco e alocação de riqueza sob uma perspectiva multi-período. Nós propomos um modelo geral de alocação de riqueza e apresentamos as medidas clássicas de risco multi-período sob esta perspectiva, quando as séries de preços dos ativos são modeladas por processos GARCH ou Brownianos Geométricos. Em seguida é proposta uma medida alternativa de risco multi-período, denominada "Multi-period Relative Value-at-Risk" (MRVaR), visando corrigir propriedades indesejadas de medidas de risco baseadas em valores absolutos dos fluxos de caixa. Nós então apresentamos uma cota analítica para o MRVaR e estudamos as consequências do uso desta como medida de risco para otimização de portfólio. Por fim, é feito um estudo numérico do erro cometido ao se utilizar a cota como medida de risco e desenvolvemos um estudo de caso, utilizando ações da BM&FBovespa, para validar a metodologia proposta.

Palavras-chave: medidas de risco, risco multi-período, otimização de portfólio, Multi-period Relative Value-at-Risk.

Abstract

This dissertation analyzes risk measures and wealth allocation under a multi-period perspective. We propose a general wealth allocation model and present the classic multi-period risk measures under this perspective, when asset prices time series are modeled by GARCH processes or Geometric Brownian Motions. We further propose an alternative multi-period risk measure, called the Multi-period Relative Value-at-Risk (MRVaR), aiming to correct some undesired features of measures based on absolute values of cash-flows. We then present an analytic bound for the MRVaR and study the consequences of using the bound itself as a risk metric for portfolio optimization. Finally, we study numerically the error we incur when using the bound as risk measure and develop a case study, using assets from BM&FBovespa, to validate the proposed methodology.

Key words: risk measures, multi-period risk, portfolio optimization, Multi-period Relative Value-at-Risk.

Contents

Introduction and Overview

Defining multi-period strategies in a scenario whose characteristics evolve in time demands an accurate and consistent measure of the risk present in each possible option. The main difference from a single period scenario to a multi-period one is that in the latter the risk measure must incorporate the information structure contained at the evolution of the system characteristics.

Among all risk measures the Multi-period Average Value-at-Risk, and similarly Multi-period Value-at-Risk, emerges as one of the most prominent alternatives to risk analysis in financial applications. In order to model the evolution of assets returns, representing the information structure of the system, GARCH (Generalized Autoregressive Conditional Heteroskedasticity) models appear as a well established theory in the literature.

The main drawback of using such multi-period risk measures is the lack of an analytical expression for its calculation. In this case Monte-Carlo methods are necessary to calculate the measure, increasing the computational cost and sometimes making such approach unfeasible for certain applications.

The present work proposes an alternative risk measure, called Multi-period Relative Value-at-Risk (MRVaR), based on the Multi-period Value-at-Risk, in order to correct undesired features of such risk functional, as further explained in Chapter 2. We then present an analytic lower bound for the proposed multi-period risk measure when using GARCH models for the return series, and we study the error incurred when one uses such analytic bound itself as a risk metric.

Chapter 1 establishes the theoretical basis for the definition of the multi-period risk measures and GARCH models. Chapter 2 proposes a model for wealth allocation, proposes the alternative risk measure (MRVaR) and develops the expression of the analytic bounds for such risk measures. Chapter 3 introduces the agent's wealth allocation problem using as risk metric the analytic bound presented in Chapter 2. Chapter 4 analyses the error incurred when using the approach implemented for the wealth allocation problem in Chapter 3. Chapter 5 applies the analytic lower bound for the MRVaR as risk metric in a case study of assets from BM&FBovespa, in order to compare the effectiveness of such method against a classical approach.

Chapter 1

Literature Review

Investment activities are intrinsically related to uncertainty, and the ultimate goal of every investor is to obtain the best possible return from its allocations. Stock markets are a genuine example of such situation, and risk plays a major role in this case. There are two main problems to be faced when trying to deal with such markets. The first one is what is the appropriate model for the prices time series? The second, how to characterize risk of an investment strategy?

Stochastic calculus and consequently stochastic differential equations (SDE's), with the pioneering work done by K. Itô, appear as a possible solution in the first case. Another possible approach comes with the development of Autoregressive Conditional Heteroskedasticity models with the work done by R. F. Engle[4] and T. Bollerslev[2]. The second problem was addressed trough the development of functionals that incorporate desirable properties related to risk. As examples of such risk measures we have the Value-at-Risk, and extensions like the Conditional Value-at-Risk and the Multi-period Conditional Value-At-Risk, developed in works conducted by R. T. Rockafellar, S. Uryasev [12] and others.

1.1 The Brownian Motion

The first description of a Brownian Motion was done in the context of physics, being motivated, for example, through the observation of apparently random trajectories described by particles of pollen in water by the botanist Robert Brown. A rigorous mathematical description was latter developed and finds applications on several fields, ranging from diffusion theory to financial modeling. The approach presented in the next sections follows closely the work done by R. Korn and E. Korn[6], and also by S. E. Shreve [10, 11].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The following definitions we'll be useful in order to define the desired process.

Definition 1. *A family $\{\mathcal{F}_t\}_{t\in I}$ of sub-σ-algebras of $\mathcal{F}$ indexed by an ordered set I with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$, $s, t \in I$, is called a filtration.*

A filtration is used to represent the amount of information available until time t, and so if a random variable X_t is $\mathcal{F}_t$-measurable, we are able to determine its value from the information available at time t. We generally take $I = (0, +\infty)$ or $I = (0, T)$.

Definition 2. *A set $\{(X_t, \mathcal{F}_t)\}_{t\in I}$ consisting of a filtration and a family of $\mathbb{R}^n$-valued random variables with X_t being $\mathcal{F}_t$-measurable is called a stochastic process with filtration $\{\mathcal{F}_t\}_{t\in I}$.*

Every process $\{X_t\}_{t \in I}$ has associated to it a canonical, or natural, filtration given by $\{\mathcal{F}_t^X\}_{t \in I}$, where $\mathcal{F}_t^X = \sigma\{X_s \mid s \leq t, s \in I\}$.

Definition 3. *The real-valued stochastic process $\{W_t\}_{t \geq 0}$ with continuous sample paths and*

1. W_0*=0,* $\mathbb{P}$*-a.s.*
2. $W_t - W_s \sim N(0, t-s)$ *for* $0 \leq s < t$.
3. $W_t - W_s$ *independent of* $W_u - W_r$ *for* $0 \leq r \leq u \leq s < t$.

is called the one-dimensional Brownian motion.

By an n-dimensional Brownian motion we mean the $\mathbb{R}^n$-valued process $W(t) = (W_1(t), ..., W_n(t))$ with each component W_t being a Brownian motion. The existence of a Brownian motion as a stochastic process has to be shown, although we shall not do so here. The reader can find a construction based on weak convergence and approximation by random walks in P. Billingsley[1].

1.2 The Itô Integral

We describe in this section a new kind of integral, called Itô Integral, motivated by the fact that the one-dimensional Brownian motion is nowhere differentiable and does not have finite variation over any non-empty open interval, and so the object $\int_0^t X_s(w) dW_s(w)$ cannot be derived from an extension of the Lebesgue-Stieltjes integral.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \in [0,\infty]}$ and a Brownian motion $\{(W_t, \mathcal{F}_t)\}_{t \in [0,\infty]}$.

Definition 4. *A stochastic process $\{X_t\}_{t \in [0,T]}$ is called a simple process if there exists real numbers $0 = t_0 < t_1 < ... < t_p = T, p \in \mathbb{N}$ and bounded random variables $\Phi_i : \Omega \to \mathbb{R}, i = 0, 1, ..., p$ with Φ_0 $\mathcal{F}_0$-measurable and Φ_i $\mathcal{F}_{i-1}$-measurable for $i = 1, ..., p$, such that $X_t(w)$ has the following representation:*

$$X_t(w) = \Phi_0(w) \cdot 1_0(t) + \sum_{i=1}^{p} \Phi_i(w) \cdot 1_{(t_{i-1}, t_i]}(t)$$

for each $w \in \Omega$.

Definition 5. *For a simple process $\{X_t\}_{t \in [0,T]}$ the stochastic integral $I(X)$ for $t \in [0, T]$ is defined by*

$$I_t(X) = \int_0^t X_s dW_s = \sum_{1 \leq i \leq p} \Phi_i (W_{t_i \wedge t} - W_{t_{i-1} \wedge t})$$

So that on each interval where X is constant the increments of the Brownian motion on that interval are multiplied with the corresponding value of Φ_i.

Theorem 1. *Let $X = \{X_t\}_{t \in [0,T]}$ be a simple process. Then we have*

1. $\{(I_t(X))\}_{t \in [0,T]}$ *is a continuous martingale with respect to* $\mathcal{F}_t$
2. $\mathbb{E}\left(\int_0^t X_s dW_s \right)^2 = \mathbb{E}\left(\int_0^t X_s^2 ds \right)$ *for* $t \in [0, T]$

3. $\mathbb{E}\Bigg(\sup_{0\le t\le T} \mid \int_0^t X_s dW_s \mid \Bigg)^2 \le 4\mathbb{E}\Bigg(\int_0^T X_s^2 ds \Bigg)$

Our goal is to extend the defined integral for a broader class of functions, not only simple ones, using density properties and the second statement from the previous theorem, which gives us some sort of norm preserving relation. In order to do so we'll need the following definition

Definition 6. *Let $\{(X_t, \mathcal{F}_t)\}_{t\in[0,\infty)}$ be a stochastic process. This stochastic process will be called progressively measurable if for all $t \ge 0$ the mapping*

$$\begin{aligned} [0,t] \times \Omega &\to \mathbb{R}^n \\ (s,w) &\to X_s(w) \end{aligned}$$

is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^n)$-measurable.

We are now able to state our extension of the defined integral to a broader class of processes, namely to

$$\begin{aligned} L^2[0,T] \quad = \quad & \Big\{ \{(X_t, \mathcal{F}_t)\}_{t\in[0,T]} \text{real-valued stochastic process} \mid \{X_t\}_t \text{ progressively} \\ & \text{measurable, } \mathbb{E}\Bigg(\int_0^T X_t^2 dt \Bigg) < \infty \Big\} \end{aligned}$$

Using the fact that simple processes are dense in $L^2[0,T]$ and that the defined integral acts as an isometry for this class of functions, we are able to obtain a Cauchy sequence on the space of stochastic integrals. It can then be proved that such space is complete and that the obtained limit is independent of the sequence of simple processes that approximates the element of $L^2[0,T]$. We summarize this process in the following theorem

Theorem 2. *There exist a unique linear mapping J from $L^2[0,T]$ into the space of continuous martingales on [0,T] with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying*

1. *If X is a simple process then* $\mathbb{P}\Bigg(J_t(X) = I_t(X) \text{ for all } t \in [0,T] \Bigg) = 1$

2. $\mathbb{E}\Bigg(J_t(X)^2 \Bigg) = \mathbb{E}\Bigg(\int_0^t X_s^2 ds \Bigg)$ *(Itô Isometry)*

We conclude with a formal definition of the integral

Definition 7. *For $X \in L^2[0,T]$ and J as in Theorem 2 we let*

$$\int_0^t X_s dW_s = J_t(X)$$

and call this the stochastic integral or the Itô integral of X with respect to W.

1.3 The Itô Formula and Stochastic Differential Equations (SDE's)

In this section we aim to establish the basic tool for defining and solving Stochastic Differential Equations, the Itô Formula. It is based on the Itô Integral defined on the previous section and we will develop it for a larger class of processes called Itô processes.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,\infty]}$ and a Brownian motion $\{(W_t, \mathcal{F}_t)\}_{t\in[0,\infty]}$.

Definition 8. *Let $\{(W_t, \mathcal{F}_t)\}_{t\in[0,\infty)}$ be an m-dimensional Brownian motion, $m \in \mathbb{N}$. Then*

1. *$\{(X(t), \mathcal{F}_t)\}_{t\in[0,\infty)}$ is called a real-valued Itô process if for all $t \geq 0$ it admits the representation*
$$X(t) = X(0) + \int_0^t K(s)ds + \sum_{j=1}^{m} \int_0^t H_j(s)dW_j(s),\ \mathbb{P} - a.s.,$$
where, X(0) is a $\mathcal{F}_0$-measurable, and $\{K(t)\}_{t\in[0,\infty)}$ and $\{H(t)\}_{t\in[0,\infty)}$ are progressively measurable processes with
$$\int_0^t \mid K(s) \mid ds < \infty,\ \mathbb{P} - a.s.$$
$$\int_0^t H_i^2(s)ds < \infty,\ \mathbb{P} - a.s.$$
for all $t \geq 0$, $i = 1, ..., m$.

2. *An n-dimensional Itô process $X = (X^1, .., X^n)$ consists of a vector with components being real-valued Itô processes.*

The differential notation for the above definition is given by

$$dX_t = K_t dt + H_t dW_t$$

Definition 9. *Let X and Y be two real-valued Itô processes with representations*

$$\begin{aligned} X(t) &= X(0) + \int_0^t K(s)ds + \sum_{j=1}^{m} \int_0^t H_j(s)dW_j(s) \\ Y(t) &= Y(0) + \int_0^t L(s)ds + \sum_{j=1}^{m} \int_0^t M_j(s)dW_j(s) \end{aligned}$$

Then,

$$[X, Y]_t = \sum_{i=1}^{m} \int_0^t H_i(s)M_i(s)ds$$

is called the quadratic covariation of X and Y. In particular, $[X]_t = [X, X]_t$ is called the quadratic variation of X.

For a real-valued Itô process, and Y a real-valued, progressively measurable process we set

$$\int_0^t Y(s)dX(s) = \int_0^t Y(s)K(s)ds + \int_0^t Y(s)H(s)dW(s)$$

if all the integrals on the right-hand side are defined.

Theorem 3. *Let $X(t) = (X_1(t), ..., X_n(t))$ be an n-dimensional Itô process with*

$$X_i(t) = X_i(0) + \int_0^t K_i(s)ds + \sum_{j=1}^{m} \int_0^t H_{ij}(s)dW_j(s), \ \ i = 1, ..., n$$

where $W(t) = (W_1(t), ..., W_m t)$ is an m-dimensional Brownian motion. Further let $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a $C^{1,2}$-function, i.e. f is continuous, continuously differentiable with respect to the first variable and twice continuously differentiable with respect to the last n variables. We then have

$$\begin{aligned} f(t, X_1(t), ..., X_n(t)) &= f(0, X_1(0), ..., X_n(0)) + \int_0^t f_t(s, X_1(s), ..., X_n(s))ds \\ &+ \sum_{i=1}^{n} \int_0^t f_{x_i}(s, X_1(s), ..., X_n(s))dX_i(s) \\ &+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t f_{x_i x_j}(s, X_1(s), ..., X_n(s))d[X_i, X_j](s) \qquad (1.3.1) \end{aligned}$$

We present below a simple example where the above formula can be used to derive a stock price equation.

Example 1. *Applying Equation (1.3.1) in differential form to the case of a one dimensional Brownian motion, a single Itô process and $f(t, X(t)) = \ln(X(t))$ we obtain*

$$d(\ln(X(t))) = \frac{1}{X(t)} dX(t) - \frac{1}{2} \frac{d[X]_t}{X^2(t)}$$

Assuming that $X(t)$ satisfies the following Stochastic Differential Equation (SDE)

$$dX(t) = X(t) \left[(\mu + \frac{\sigma^2}{2})dt + \sigma dW(t) \right]$$

We may conclude that

$$d(\ln(X(t))) = \mu dt + \sigma dW(t)$$

Exponentiating gives us the following final expression

$$X(t) = X(0) \exp(\mu t + \sigma W(t))$$

1.4 Generalized Autoregressive Conditional Heteroskedasticity (GARCH)

Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models are used as an econometric tool to deal with processes in which volatility evolves over time. Such models are able to predict future volatility and serve as a tool for accessing risk and uncertainty on financial models.

Regarding the behavior of single assets and portfolios such models are adequate for modeling their volatility and correlation structure. An important feature captured by such models is that volatility tends to cluster in periods of high volatility and low volatility[4]. Such processes are defined below, and are further used to model logarithmic returns of stocks.

Definition 10. *Let ε_t be a real discrete-time stochastic process and $\mathcal{F}_t$ the σ-algebra representing the information available until time t. The GARCH(p,q) process is given by*

$$\varepsilon_t \mid \mathcal{F}_{t-1} \sim N(0, \sigma_t^2) \tag{1.4.1}$$

$$\sigma_t^2 = K + \sum_{i=1}^{p} \gamma_i \sigma_{t-1}^2 + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}^2 \tag{1.4.2}$$

Where $p \geq 0, q > 0, K > 0, \gamma_i \geq 0$ and $\alpha_i \geq 0$.

Notice that the GARCH models allow not only the influence of past sample variances but also of lagged conditional variances. Let $A(x) = \sum_{i=1}^{p} \gamma_i x^{i-1}$ and $B(x) = \sum_{j=1}^{q} \alpha_j x^{j-1}$. The following theorem describes the stationarity of the GARCH process:

Theorem 4. *The GARCH(p,q) process defined by Equations (1.4.1) and (1.4.2) satisfies $\mathbb{E}[\varepsilon_t] = 0$, $\lim_{t\to\infty} var(\varepsilon_t) = K(1-A(1)-B(1))^{-1}$ and $cov(\varepsilon_t, \varepsilon_j) = 0, t \neq s$ if, and only if, $A(1)+B(1) < 1$.*

1.5 Risk Measures

Risk measures were developed aiming to treat uncertainty and quantify risk. Several desirable properties of risk functionals were defined over time so that one could reach a reasonable tool capable of consistently comparing possible investment decisions. The approach presented in the next sections follows the work done by G. C. Pflug[9, 8].

Consider a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{Y}$ a linear space of random variables defined over it. A probability functional is a function $\mathcal{D} : \mathcal{Y} \to \mathbb{R}$. If $Y_1, Y_2 \in \mathcal{Y}$ have te same distribution function, we say that Y_1 is a version of Y_2, and we write $Y_1 \triangleq Y_2$.

Definition 11. *A probability functional $\mathcal{D}$ in a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called version-independent if $Y_1 \triangleq Y_2$ implies $\mathcal{D}(Y_1) = \mathcal{D}(Y_2)$.*

Version-independent functionals are generally defined for families of probability distributions. When one uses such functionals, it can always be assumed w.l.o.g. that the random variables are defined in $[0, 1]$ with Lebesgue measure.

Definition 12. *Given a functional $\mathcal{D} : \mathcal{Y} \to \mathbb{R}$, we say that $\mathcal{D}$ is*

1. *Translation-equivariant if $\mathcal{D}(Y + c) = \mathcal{D}(Y) + c, \forall Y \in \mathcal{Y}, c \in \mathbb{R}$*
2. *Translation-invariant if $\mathcal{D}(Y + c) = \mathcal{D}(Y), \forall Y \in \mathcal{Y}, c \in \mathbb{R}$*

Definition 13. *Given a functional $\mathcal{D} : \mathcal{Y} \to \mathbb{R}$, we say that $\mathcal{D}$ is*

1. *Homogeneous if $\mathcal{D}(cY) = c\mathcal{D}(Y), \forall Y \in \mathcal{Y}, c \in \mathbb{R}$*
2. *Positively homogeneous if $\mathcal{D}(cY) = c\mathcal{D}(Y), \forall Y \in \mathcal{Y}, c \in \mathbb{R}^+$*

Definition 14. *Given a functional $\mathcal{D} : \mathcal{Y} \to \mathbb{R}$, we say that $\mathcal{D}$ is*

1. *Additive if $\mathcal{D}(Y_1 + Y_2) = \mathcal{D}(Y_1) + \mathcal{D}(Y_2), \forall Y_1, Y_2 \in \mathcal{Y}$*
2. *Subadditive if $\mathcal{D}(Y_1 + Y_2) \leq \mathcal{D}(Y_1) + \mathcal{D}(Y_2), \forall Y_1, Y_2 \in \mathcal{Y}$*

3. *Superadditive if* $\mathcal{D}(Y_1 + Y_2) \geq \mathcal{D}(Y_1) + \mathcal{D}(Y_2), \forall Y_1, Y_2 \in \mathcal{Y}$

Definition 15. *Given a functional* $\mathcal{D} : \mathcal{Y} \to \mathbb{R}$*, we say that* $\mathcal{D}$ *is*

1. *Concave if* $\mathcal{D}(\lambda Y_1 + (1-\lambda)Y_2) \geq \lambda\mathcal{D}(Y_1) + (1-\lambda)\mathcal{D}(Y_2), \forall Y_1, Y_2 \in \mathcal{Y}, \lambda \in [0,1]$
2. *Convex if* $\mathcal{D}(\lambda Y_1 + (1-\lambda)Y_2) \leq \lambda\mathcal{D}(Y_1) + (1-\lambda)\mathcal{D}(Y_2), \forall Y_1, Y_2 \in \mathcal{Y}, \lambda \in [0,1]$

Definition 16. *Given a functional* $\mathcal{D} : \mathcal{Y} \to \mathbb{R}$*, we say that* $\mathcal{D}$ *is monotonic if* $Y_1 \leq Y_2$ $\mathbb{P}-a.s.$ *implies* $\mathcal{D}(Y_1) \leq \mathcal{D}Y_2$.

Definition 17. *A function* $\mathcal{R} : \mathcal{Y} \to \mathbb{R}$ *is a risk measure if it is translation-equivariant, concave and monotonic.*

1.5.1 Value-at-Risk

The Value-at-Risk is the most known risk metric for financial applications, and its widespread use is due to its simplicity and objective interpretation. It gives an investor information about the worst case losses of his portfolio, and so serves as a metric for investment optimization. It has all basic characteristics of a risk measure, although it fails when one looks forward for investment diversification, which is implied by concavity of the risk functional.

Definition 18. *The Value-at-Risk of level* $\alpha \in (0,1)$ *corresponds to the* α*-quantile of the profit distribution*

$$\mathrm{VaR}_\alpha(Y) = G^{-1}(\alpha). \tag{1.5.1}$$

Where G *is the profit distribution function.*

Proposition 1. *The following properties are satisfied by the Value-at-Risk:*

1. *Translation-equivariance*
2. *Positive homogeneity*
3. *Monotonicity*

Example 2. *Suppose* $Y \sim N(0,1)$ *and let* $\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} exp(-\frac{-v^2}{2})dv$*. Using Definition 18 we have*

$$\mathrm{VaR}_\alpha(Y) = \Phi^{-1}(\alpha). \tag{1.5.2}$$

Example 3. *Suppose* $Y \sim N(\mu, \sigma^2)$*. Using Equation (1.5.2) and Proposition 1 we have*

$$\mathrm{VaR}_\alpha(Y) = \mu + \sigma\Phi^{-1}(\alpha). \tag{1.5.3}$$

1.5.2 Average Value-at-Risk

The Average Value-at-Risk was developed in order to overcome the VaR lack of concavity, allowing for the risk metric to incorporate the desirable property of portfolio diversification. Objectively, instead of furnishing the α-quantile portfolio return, it gives the average among all returns bellow such threshold.

Definition 19. *The Average Value-at-Risk of level* $\alpha \in (0,1)$ *is defined as*

$$\mathrm{AVaR}_\alpha(X_1) = \frac{1}{\alpha}\int_0^\alpha G^{-1}(u)du. \tag{1.5.4}$$

Using Equation (1.5.1), (1.5.4) can be seen as an average of the VaR_{α^*} for $\alpha^* \in (0, \alpha)$.

Proposition 2. *The following properties are satisfied by the Average Value-at-Risk:*

1. *Translation-equivariance*
2. *Positive homogeneity*
3. *Concavity*
4. *Monotonicity*

Example 4. *Suppose $Y \sim N(0,1)$. By (1.5.4) we have that $\forall \alpha \in (0, \frac{1}{2})$*

$$\begin{aligned} \mathrm{AVaR}_\alpha(Y) &= \frac{1}{\alpha}\int_0^\alpha \Phi^{-1}(\alpha) \\ &= -\frac{1}{\alpha\sqrt{2\pi}} \exp\left[\frac{-1}{2}(\Phi^{-1}(\alpha))^2\right]. \end{aligned} \tag{1.5.5}$$

Example 5. *Suppose $Y \sim N(\mu, \sigma^2)$ and $\alpha \in (0, \frac{1}{2})$. Using Equation (1.5.5) and Proposition 2 we can conclude that*

$$\mathrm{AVaR}_\alpha(X_1) = \mu - \frac{\sigma}{\alpha\sqrt{2\pi}} \exp\left[\frac{-1}{2}(\Phi^{-1}(\alpha))^2\right]. \tag{1.5.6}$$

Remark 1. *Let $Y \sim N(0,1)$. The table below shows the Value-at-Risk and the Average-Value-at-Risk for different values of α:*

α	VaR_α	AVaR_α
0.001	-3.09	-3.37
0.010	-2.32	-2.66
0.050	-1.64	-2.06
0.100	-1.28	-1.75
0.500	0.00	-0.80
0.900	1.28	-0.20

1.6 Conditional Risk Measures

In the case where the probability functionals are version-independent they can be applied to conditional distributions, leading to conditional risk measures. A central conditional risk metric is the Conditional Value-at-Risk, which improves the VaR characteristics and is suitable for applications with GARCH models, as was done by S. Maffra[3]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space and $\mathcal{F}_1$ a σ-algebra contained in $\mathcal{F}$. Let $\mathcal{Y} = L_p(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{Y}_1 = L_p(\Omega, \mathcal{F}_1, \mathbb{P})$, $p \in [1, \infty)$, so that $\mathcal{Y}_1 \subseteq \mathcal{Y}$.

Definition 20. *A function $\mathcal{R} : \mathcal{Y} \to \mathcal{Y}_1$ is a conditional risk measure if it satisfies the following properties $\mathbb{P}$-a.e.:*

1. *Predictable translation-equivariance: $\mathcal{R}(Y + Y_1) = \mathcal{R}(Y) + Y_1, \forall Y \in \mathcal{Y}, \forall Y_1 \in \mathcal{Y}_1$*
2. *Concavity: $\mathcal{R}(\lambda Y + (1-\lambda)X) \geq \lambda\mathcal{R}(Y) + (1-\lambda)\mathcal{R}(X), \forall Y, X \in \mathcal{Y}$*

3. *Monotonicity:* $X \leq Y$ *implies* $\mathcal{R}(X) \leq \mathcal{R}(Y), \forall Y, X \in \mathcal{Y}$

Definition 21. *The* $AVaR_\alpha(Y \mid \mathcal{F}_1)$ *is defined in* $L_1(\mathcal{F})$ *by:*

$$\mathbb{E}(\mathrm{AVaR}_\alpha(Y \mid \mathcal{F})\chi_B) = \inf(\mathbb{E}(YZ) : 0 \leq Z \leq \frac{1}{\alpha}\chi_B, \mathbb{E}(Z \mid \mathcal{F}) = \chi_B) \tag{1.6.1}$$

Let $\mathbb{G}^1$ be the ensemble of the distribution functions over $\mathbb{R}$ with the Lipschitz metric, i.e.:

$$\mathbb{G}^1 = \left\{ G(\cdot) : \text{G is monotonic, continuous to the right with } \lim_{x\to\infty} G(x) = 1 \text{ and } \lim_{x\to-\infty} G(x) = 0 \right\}$$

$$d_L(G_1, G_2) = \sup\left\{ \mid \int f dG_1 - \int f dG_2 \mid : \sup_u \mid f(u) \mid + \sup_{u \neq v} \mid \frac{f(u) - f(v)}{u - v} \mid \leq 1 \right\}$$

Remark 2. *$(\mathbb{G}^1, d_L)$ is separable. If Y is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_1$ is a subsigma-algebra of $\mathcal{F}$, the conditional probabilities $G(u \mid \mathcal{F}_1) = \mathbb{P}(Y \leq u \mid \mathcal{F}_1)$ are well defined a.e. Consider such functions over the rationals $\mathbb{Q}$. Possibly excluding a measure zero set,all maps $u \to G(u \mid \mathcal{F}_1)$; $u \in \mathbb{Q}$ are monotonic and can be extended to $\mathbb{R}$ by monotonicity and right continuity. In this way the following map can be seen as a random function $\mathcal{F}_1$-measurable taking values on $\mathbb{G}^1$.*

$$G(\cdot \mid \mathcal{F}_1) \tag{1.6.2}$$

So each pair $(Y, \mathcal{F})$ induces a probability measure over $\mathbb{G}^1$.

Definition 22. *Let $\mathcal{R}$ be a risk measure. The conditional risk measure $\mathcal{R} : \mathcal{Y} \to \mathcal{Y}_1$ is called version-independent if $\mathcal{R}(Y \mid \mathcal{F}_1)$ can be written as a function of the distribution induced by $(Y, \mathcal{F}_1)$ over $\mathbb{G}^1$, $\mathbb{P}$-a.s.*

Proposition 3. *The Conditional Average Value-at-Risk is version-independent.*

Proposition 4. *The following properties are a direct consequence of Proposition 2 and the definition of the Conditional Average Value-at-Risk:*

1. *If* $\mathcal{F}_1 = \{\emptyset, \Omega\}$ *then* $\mathrm{AVaR}_\alpha(Y \mid \mathcal{F}_1) = \mathrm{AVaR}(Y)$

2. *For all* $0 \leq \Lambda \leq 1$ $\mathcal{F}_1$*-measurable,*
   $\mathrm{AVaR}_\alpha(\Lambda Y_1 + (1 - \Lambda)Y_2 \mid \mathcal{F}_1) \geq \Lambda \mathrm{AVaR}_\alpha(Y_1 \mid \mathcal{F}_1) + (1 - \Lambda)\mathrm{AVaR}_\alpha(Y_2 \mid \mathcal{F}_1)$

3. *For all* Λ $\mathcal{F}_1$*-measurable, bounded,* $\mathrm{AVaR}_\alpha(\Lambda Y \mid \mathcal{F}_1) = \Lambda \mathrm{AVaR}_\alpha(Y \mid \mathcal{F}_1)$

4. *If* $Y_1 \leq Y_2$ *then* $\mathrm{AVaR}_\alpha(Y_1 \mid \mathcal{F}_1) \leq \mathrm{AVaR}_\alpha(Y_2 \mid \mathcal{F}_1)$

5. *If* $Y \in \mathcal{F}_1$ *then* $\mathrm{AVaR}_\alpha(Y \mid \mathcal{F}_1) = Y$

6. *If* $\mathcal{F}_1 \subset \mathcal{F}_2$ *then,*
   $\mathrm{AVaR}_\alpha(Y \mid \mathcal{F}_1) \leq \mathbb{E}[\mathrm{AVaR}_\alpha(Y \mid \mathcal{F}_2) \mid \mathcal{F}_1] \leq \mathrm{AVaR}_\alpha(\mathbb{E}(Y \mid \mathcal{F}_2) \mid \mathcal{F}_1)$

1.7 Multi-period Risk Measures

Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability space and $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T)$ a filtration in $\mathcal{F}$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Let $\mathcal{Y}$ be the space of the stochastic processes describing the wealth variation of the agent, $Y = (Y_1, ..., Y_T)$, adapted to $\mathcal{F}$, i.e., Y_t is $\mathcal{F}_t$-measurable, $\forall t \in \{1, ..., T\}$. A multi-period probability functional is a map $(\mathcal{Y}, \mathcal{F}) \to \mathbb{R}$.

Definition 23. *A probability functional $\mathcal{R}(Y; \mathcal{F})$ is a multi-period risk measure if it satisfies the following properties:*

1. *Information monotonicity: if $\mathcal{F}_t \subset \mathcal{F}'_t, \forall t$ then $\mathcal{R}(Y; \mathcal{F}) \leq \mathcal{R}(Y; \mathcal{F}')$*

2. *Predictable translation-equivariance: $\mathcal{R}(Y_1, ..., Y_t + C_t, ..., Y_T; \mathcal{F}) = \mathbb{E}(C_t) + \mathcal{R}(Y; \mathcal{F}), \forall C_t$ $\mathcal{F}_{t-1}$-measurable.*

3. *Concavity: $Y \to \mathcal{R}(Y; \mathcal{F})$ is concave.*

4. *Monotonicity: if $Y_t^1 \leq Y_t^2$ $\mathbb{P} - a.e.$ $\forall t$, then $\mathcal{R}(Y^1; \mathcal{F}) \leq \mathcal{R}(Y^2; \mathcal{F})$*

5. *Positive homogeneity*

Definition 24. *Let $Y = (Y_1, ..., Y_T)$ be an integrable stochastic process. For a given sequence of constants $c = (c_1, ..., c_T)$, $\alpha = (\alpha_1, ..., \alpha_T) \in (0, 1)^T$ and a filtration $\mathcal{F} = (\mathcal{F}_0, ..., \mathcal{F}_T)$, we define the Multi-period Average Value-at-Risk by:*

$$\mathrm{MAVaR}_{\alpha,\mathrm{c}}(Y, \mathcal{F}) = \sum_{t=1}^{T} c_t \mathbb{E}[\mathrm{AVaR}_{\alpha_t}(Y_t \mid \mathcal{F}_{t-1})] \tag{1.7.1}$$

Definition 25. *Let $Y = (Y_1, ..., Y_T)$ be an integrable stochastic process. For a given sequence of constants $c = (c_1, ..., c_T)$, probabilities $\alpha = (\alpha_1, ..., \alpha_T)$ and a filtration $\mathcal{F} = (\mathcal{F}_0, ..., \mathcal{F}_T)$, we define the Multi-period Value-at-Risk by:*

$$\mathrm{MVaR}_{\alpha,\mathrm{c}}(Y, \mathcal{F}) = \sum_{t=1}^{T} c_t \mathbb{E}[\mathrm{VaR}_{\alpha_t}(Y_t \mid \mathcal{F}_{t-1})] \tag{1.7.2}$$

Chapter 2

Wealth Allocation and Asset Prices Model

Dealing with financial time series and wealth allocation problems requires an adequate modeling of the price series and of the trading environment. In this chapter we will present a general multi-period model for wealth allocation problems and analyze the risk behavior of allocation strategies based on different risk measures and models for the time series of assets returns.

We notice that classical risk measures allow, in the multi-period case, a misleading increase in risk values caused by wealth growth. In order to solve this problem we propose an alternative risk metric, one that takes into account only inter-period relative changes in wealth.

2.1 The Wealth Allocation Model

The investment horizon is of T periods and there are M assets on the agent's portfolio. The price of the asset m at time t is given by $p_m(t)$, and the prices at $t = 0$ are known. Let $p(t) = (p_1(t), ..., p_M(t)), \forall t = 0, .., T$, be the price vector at time t. The logarithmic returns are defined as

$$r_m(t) = \ln\left(\frac{p_m(t+1)}{p_m(t)}\right) \approx \frac{p_m(t+1)}{p_m(t)} - 1. \tag{2.1.1}$$

The approximation used is justified by the order of magnitude of the returns. The agent's wealth process is represented by $W(t), \forall t = 0, ..., T$. We define $X(t) = W(t) - W(t-1), \forall t \in \{1, ..., T\}$, as the wealth variation of the agent when going from period $t-1$ to period t. The filtration used is the one generated by the sequence of prices, i.e., $\mathcal{F} = (\mathcal{F}_0, ..., \mathcal{F}_T)$ with $\mathcal{F}_t = \sigma(p(0), ..., p(t)), \forall t \in \{0, ..., T\}$.

The agent decides in $t = 0$ the proportion of wealth that will be allocated at each asset in every subsequent period. Formaly, the decision variables are $s(t) = (s_1(t), ..., s_M(t)) \in [0, 1]^M, \sum_{m=1}^{M} s_m(t) = 1, \forall t \in \{0, ..., T-1\}$.

Using such definitions we can write $\forall t \geq 1$

$$
\begin{aligned}
W(t) &= W(t-1)\sum_{m=1}^{M} s_m(t-1)\frac{p_m(t)}{p_m(t-1)} && (2.1.2)\\
&= W(0)\prod_{j=0}^{j=t-1}\left[1+\sum_{m=1}^{M} s_m(j)\left(\frac{p_m(j+1)}{p_m(j)}-1\right)\right]\\
&\approx W(0)\prod_{j=0}^{j=t-1}\left[1+\sum_{m=1}^{M} s_m(j)r_m(j)\right]. && (2.1.3)
\end{aligned}
$$

Using Equation (2.1.1), Equation (2.1.2) and the definition of $X(t)$ we have

$$
\begin{aligned}
X(t) &= W(t)-W(t-1)\\
&= W(t-1)\sum_{m=1}^{M} s_m(t-1)\left[\frac{p_m(t)}{p_m(t-1)}-1\right]\\
&\approx W(t-1)\sum_{m=1}^{M} s_m(t-1)r_m(t-1). && (2.1.4)
\end{aligned}
$$

2.2 Geometric Brownian Motion

Suppose the price series are modeled with Geometric Brownian Motions, i.e., there are $\mu_m, \sigma_m, \forall m = 1, ..., M$, such that the following stochastic differential equations (SDE's) hold

$$
dp_m(t) = \left(\mu_m + \frac{\sigma_m^2}{2}\right)p_m(t)dt + \sigma_m p_m(t)dB_m(t). \qquad (2.2.1)
$$

The solutions of the SDE's are given by

$$
ln\left(\frac{p_m(t)}{p_m(0)}\right) = \mu_m t + \sigma_m B_m(t), \forall m = 1, ..., M. \qquad (2.2.2)
$$

Using Equation (2.1.1) we obtain the following relation

$$
\begin{aligned}
r_m(t) &= \mu_m + \sigma_m[B_m(t+1) - B_m(t)]\\
&= \mu_m + \sigma_m \Delta B_m(t). && (2.2.3)
\end{aligned}
$$

Let Σ be the covariance matrix of the above Brownian processes. Since we are dealing with Gaussian random variables, the VaR becomes a concave risk functional like the AVaR, and we will adopt it as our standard risk measure from now on. The multi-period risk measure is given by

$$\begin{aligned}
\mathrm{MVaR}^T_{\alpha,c}(X,\mathcal{F}) &= \sum_{t=1}^{T} c_t \mathbb{E}[\mathrm{VaR}_\alpha(X(t) \mid \mathcal{F}_{t-1})] \\
&= \sum_{t=1}^{T} c_t \mathbb{E}\left[\mathrm{VaR}_\alpha\left(W(t-1)\sum_{m=1}^{M} s_m(t-1)r_m(t-1) \mid \mathcal{F}_{t-1}\right)\right] \\
&= \sum_{t=1}^{T} c_t \mathbb{E}\left[W(t-1)\mathrm{VaR}_\alpha\left(\sum_{m=1}^{M} s_m(t-1)r_m(t-1) \mid \mathcal{F}_{t-1}\right)\right] \\
&= \sum_{t=1}^{T} c_t \left[s(t-1)\mu^* + \sqrt{v(t-1)\Sigma v(t-1)^*}\Delta_\alpha\right]\mathbb{E}[W(t-1)]. \quad (2.2.4)
\end{aligned}$$

Where $v(t-1) = (s_1(t-1)\sigma_1, ..., s_M(t-1)\sigma_M), \forall t = 1, ..., T$, and Δ_α stands for the one-period VaR of level α. The penultimate equality comes from the fact that $W(t-1)$ is $\mathcal{F}_{t-1}$-measurable and the last equality comes from the properties of the Value-at-Risk and the following lemma:

Lemma 1. *Let $r \in \mathbb{R}^M$ be a Gaussian vector, $r \sim N(\mu, \Sigma)$, and $s \in \mathbb{R}^M$. Then $s \cdot r \in \mathbb{R}$ is a Gaussian random variable with parameters $(s \cdot \mu, s \cdot \Sigma s)$.*

Using the properties of the Brownian motion we can still develop the term $\mathbb{E}[W(t-1)]$ in the following way

$$\begin{aligned}
\mathbb{E}[W(t-1)] &= \mathbb{E}\left[W(0)\prod_{j=0}^{t-2}\left[1+\sum_{m=1}^{M} s_m(j)r_m(j)\right]\right] \\
&= W(0)\mathbb{E}\left[\prod_{j=0}^{t-2}\left[1+\sum_{m=1}^{M} s_m(j)(\mu_m + \sigma_m \Delta B_m(j))\right]\right] \\
&= W(0)\prod_{j=0}^{t-2}\mathbb{E}\left[1+\sum_{m=1}^{M} s_m(j)(\mu_m + \sigma_m \Delta B_m(j))\right] \\
&= W(0)\prod_{j=0}^{t-2}\left[1+\sum_{m=1}^{M} s_m(j)\mu_m\right], \forall t \geq 2.
\end{aligned}$$

where the next to last equality comes from the independence of the increments of the Brownian motion. To conclude, the risk measure is given by

$$\begin{aligned}
\mathrm{MVaR}^T_{\alpha,c}(X,\mathcal{F}) &= W(0)\left[c_1\left(s(0)\mu^* + \sqrt{v(0)\Sigma v(0)^*}\Delta_\alpha\right) + \sum_{t=2}^{T}\prod_{j=0}^{j=t-2} c_t\left(1+\cdots\right.\right. \\
&\qquad \left.\left.\cdots + s(j)\mu^*\right)\left(s(t-1)\mu^* + \sqrt{v(t-1)\Sigma v(t-1)^*}\Delta_\alpha\right)\right]. \quad (2.2.5)
\end{aligned}$$

Example 6. *Let us calculate the multi-period risk measure in the case where the agent does not change the portfolio composition, i.e., $s(t) = s \in \mathbb{R}^{M+}, \forall t = 0, ..., T-1$. In this case we have $v(t) = v \in \mathbb{R}^{M+}, \forall t = 0, ..., T-1$. In order to simplify the notation define*

$$\Gamma^{\alpha,s}_{\mu,\sigma,\Sigma} = s\mu^* + \sqrt{v(t-1)\Sigma v(t-1)^*}\Delta_\alpha.$$

The scalars $(c_t)_{t=1}^T$ are chosen as $c_t = (\frac{1}{\rho_T})^t, \forall t = 1, ..., T$, where ρ_T is defined so that $\sum_{t=1}^T c_t = 1$. The following lemma will be useful.

Lemma 2. *Using the same definitions, the sequence $(\rho_T)_{T=1}^{\infty}$ is strictly increasing, $\rho_1 = 1$ and $\lim_{T\to\infty}\rho_T = 2$. Also, $\rho_T^{T+1} - 2\rho_T^T + 1 = 0, \forall T \geq 1$.*

Using Equation (2.2.5) we can write

$$\begin{aligned}\mathrm{MVaR}_{\alpha,c}^{T}(X,\mathcal{F}) &= W(0)\frac{\Gamma_{\mu,\sigma,\Sigma}^{\alpha,s}}{\rho_T}\sum_{t=1}^{T}\left[\frac{1+s\cdot\mu}{\rho_T}\right]^{t-1} &(2.2.6)\\ &= W(0)\Gamma_{\mu,\sigma,\Sigma}^{\alpha,s}\frac{(1+s\cdot\mu)^T-\rho_T^T}{(1+s\cdot\mu)\rho_T^T-\rho_T^{T+1}}. &(2.2.7)\end{aligned}$$

Assuming $\mid s\cdot\mu\mid< 1$ the Equation (2.2.7) provides us a long-term risk measure, given by

$$\begin{aligned}\mathrm{MVaR}_{\alpha,c}^{\infty}(X,\mathcal{F}) &= \lim_{T\to\infty}\mathrm{AMVaR}_{\alpha,c}^{T}(X,\mathcal{F})\\ &= \frac{W(0)\Gamma_{\mu,\sigma,\Sigma}^{\alpha,s}}{1-s\cdot\mu}. &(2.2.8)\end{aligned}$$

In order to better comprehend how such risk measure behaves in this situation we can write, using Equation (2.2.6), the ratio between the short-term and long-term measures

$$\frac{\mathrm{MVaR}_{\alpha,c}^{\infty}(X,\mathcal{F})}{\mathrm{MVaR}_{\alpha,c}^{1}(X,\mathcal{F})} = \frac{1}{1-s\cdot\mu}. \qquad (2.2.9)$$

The graphics below show the relation between the short-term and long-term risk measures, for $M = 1$ and different values of μ and σ.

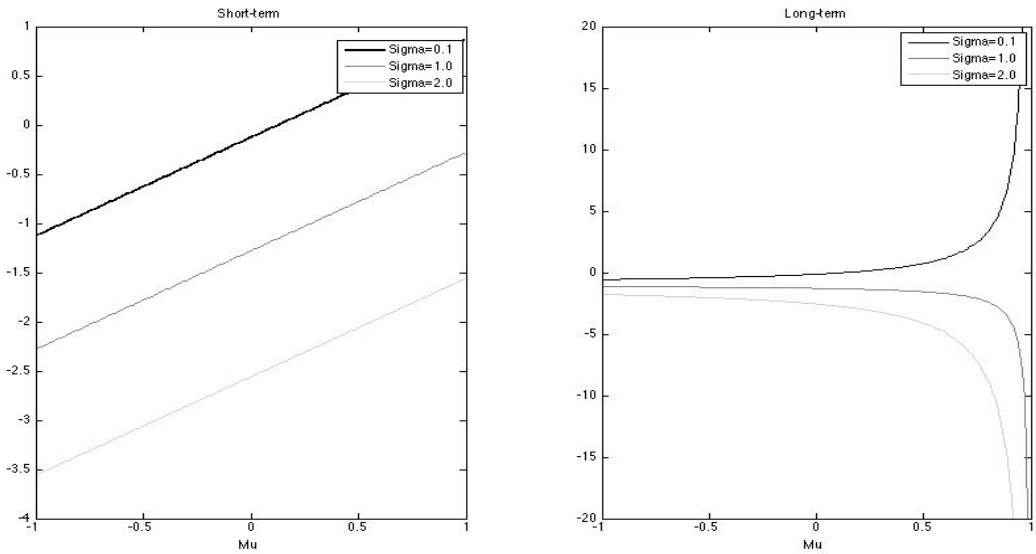

Figure 2.1: Risk measures on the short and long term.

Notice that for values of μ near 1 the long-term risk measure grows rapidly, and its sign depends on the short-term risk measure. Although Equation (2.1.1) assumes small returns, the graphics above indicate an undesired effect that is reinforced by returns of higher magnitude.

Example 7. *Let us use the results obtained in Example 6 to illustrate numerically how the risk measure acts in this case. Suppose the agent must decide between assets A and B to invest his R\$100,00. The parameters $\mu_A, \sigma_A, \mu_B, \sigma_B$ are known. The risk present in each option is calculated based on the short-term and long-term risk measures, having the VaR of level 10% as the basic risk measure, so that $\Delta_\alpha = -1.28$. The short-term risk measure is given by*

$$\mathrm{MVaR}^1_{\alpha,c}(X, \mathcal{F}) = 100(\mu - 1.28\sigma).$$

We shall calculate the risk measures in the following cases:

	μ_A, σ_A	μ_B, σ_B
1	0.2,0.117	0.1,0.039
2	0.1,0.234	0.05,0.195
3	0.05,0.195	-0.05,0.117
4	-0.05,0.117	-0.1,0.078

The results are given by the following table:

	Short-term (A)	Short-term (B)	Long-term (A)	Long-term (B)
1	5	5	6.25	5.55
2	-20	-20	-22.22	-21.05
3	-20	-20	-21.05	-19.04
4	-20	-20	-19.04	-18.18

Suppose the objective of the agent is to maximize his expected wealth, subject to a certain short-term (or long-term) risk threshold. The analysis for each one of the four cases is presented below:

1. *In the first case, the risk measure from asset A is greater than, or equal to, the one of asset B, both in the short and long term. The agent chooses asset A to make his investment. This pattern always happens when the short-term risk measure is positive.*

2. *In the second and fourth cases, the short-term risk measures agree, but for the long-term asset A becomes riskier than asset B. Depending on the long-term threshold the agent chooses to invest in B instead of A. This pattern always happens when the shod-term risk measure is negative and both μ's are positive or negative.*

3. *In the third case, the short-term risk measures agree, but for the long-term asset A becomes riskier than asset B. Depending on the long-term threshold the agent chooses to invest in B instead of A. Notice that the two previous assertions contrast with the fact that $\mu_A > 0 > \mu_B$! This pattern always happens when the short-term risk measure is negative and $\mu_A > 0 > \mu_B$.*

The results of the previous example show that the classic risk measures may present some pathological behavior when used to calculate optimal allocations. Such pathological cases occur because the value of the wealth process $(W(t))_{t=0}^T$ has a strong influence on the risk measure, as can be observed when analyzing the following term of Equation (2.2.4)

$$\mathbb{E}\left[W(t-1)\mathrm{VaR}_\alpha\left(\sum_{m=1}^{M} s_m(t-1)r_m(t-1) \mid \mathcal{F}_{t-1}\right)\right].$$

Such influence can be better understood in the particular case where the assets follow a Geometric Brownian Motion. Notice that in Equation (2.2.6) such influence becomes more clear due to the following relation

$$\mathbb{E}[W(t)] = W(0)(1 + s\mu^*)^t.$$

When $s\mu^* > 0$, this factor grows with t, but in the case of a GBM, if $\Gamma^{\alpha,s}_{\mu,\sigma,\Sigma}$ is negative (or $\mathrm{VaR}_\alpha(\sum_{m=1}^{M} s_m(t-1)r_m(t-1) \mid \mathcal{F}_{t-1}) \leq 0$ in the general case), this will imply that the portfolio is becoming riskier. But this effect is taking into account not only worsening conditions for volatilities and mean returns (in the case of a GBM the conditional volatilities and mean returns are constant), but also the wealth growth is being interpreted as a factor that increases risk.

Suppose that the risk restriction to which the agent is subject to makes reference to an absolute fall on the wealth level, like the loss of R$1,00 in the case where R$100,00 is being invested. In this case the risk measure MAVaR is adequate to the kind of restriction faced by the agent. Now suppose that the restriction makes reference to relative falls on the wealth level, like the fall of 1% over the portfolio value. In this case the MAVaR becomes less appropriate, because if the portfolio value rises to R$1000,00, the increase of the risk measure from R$1,00 to R$10,00 should not be seen as a restriction.

We propose the following solution to this problem. Define an alternative multi-period risk metric that consider relative, and not absolute, inter-period variations on wealth. Such definition is given by:

Definition 26. *Following the notation from Section 2.2, we define the Multi-period Relative Value-at-Risk by*

$$\mathrm{MRVaR}^T_{\alpha,c}(X,\mathcal{F}) = \sum_{t=1}^{T} c_t \mathbb{E}\left[\frac{\mathrm{VaR}_\alpha(X(t) \mid \mathcal{F}_{t-1})}{W(t-1)}\right]. \tag{2.2.10}$$

Example 8. *Revisiting the case of the Brownian Geometric Motion using the new risk measure we obtain*

$$\begin{aligned}
\mathrm{MRVaR}^T_{\alpha,c}(X,\mathcal{F}) &= \sum_{t=1}^{T} c_t \mathbb{E}\left[\frac{\mathrm{VaR}_\alpha(X(t) \mid \mathcal{F}_{t-1})}{W(t-1)}\right] \\
&= \sum_{t=1}^{T} c_t \mathbb{E}\left[\frac{W(t-1)}{W(t-1)}\mathrm{VaR}_\alpha\left(\sum_{m=1}^{M} s_m(t-1)r_m(t-1) \mid \mathcal{F}_{t-1}\right)\right] \\
&= \sum_{t=1}^{T} c_t \left[s(t-1)\mu^* + \sqrt{v(t-1)\Sigma v(t-1)^*}\Delta_\alpha\right]. \qquad (2.2.11)
\end{aligned}$$

If the agent will not shift his portfolio we get

$$\begin{aligned}
\mathrm{MRVaR}^T_{\alpha,c}(X,\mathcal{F}) &= \sum_{t=1}^{T} c_t \Gamma^{\alpha,s}_{\mu,\sigma,\Sigma} \\
&= \Gamma^{\alpha,s}_{\mu,\sigma,\Sigma} \sum_{t=1}^{T} c_t \\
&= \Gamma^{\alpha,s}_{\mu,\sigma,\Sigma}. \qquad (2.2.12)
\end{aligned}$$

where we used the notation from Example 6 and assumed $\sum_{t=1}^{T} c_t = 1$. *Notice that the risk measure is independent of the number of periods, what is a direct consequence from the fact that the conditional means and volatilities do not change over time.*

2.3 The GARCH(1,1) Case

Suppose the logarithmic returns satisfy the following equations $\forall m = 1, ..., M$

$$r_m(t) = \mu_m + \sigma_m(t) z_m(t) \tag{2.3.1}$$

and

$$\sigma_m(t)^2 = k_m + \gamma_m \sigma_m(t-1)^2 + \alpha_m \sigma_m(t-1)^2 z_m(t-1)^2 \tag{2.3.2}$$

where the parameters $\mu_m, k_m, \gamma_m, \alpha_m$ and $\sigma_m(0)$ are known $\forall m = 1, ..., M$, and $\mathcal{F} = (\mathcal{F}_0, ..., \mathcal{F}_T)$ is the filtration representing the information available until a certain time period. The random variables $(z_m(t) \mid \mathcal{F}_t)_{t=0}^{T-1}$ are i.i.d standard Gaussians and the covariance matrix of $(z_m(t) \mid \mathcal{F}_t)_{m=1}^{M}, \forall t = 1, ..., T-1$, is given by Σ. The multi-period risk measure is given by

$$\begin{aligned}
\mathrm{MVaR}^T_{\alpha,c}(X, \mathcal{F}) &= \sum_{t=1}^{T} c_t \mathbb{E}[\mathrm{VaR}_\alpha(X(t) \mid \mathcal{F}_{t-1})] \\
&= \sum_{t=1}^{T} c_t \mathbb{E}\left[\mathrm{VaR}_\alpha\left(W(t-1)\sum_{m=1}^{M} s_m(t-1) r_m(t-1) \mid \mathcal{F}_{t-1}\right)\right] \\
&= \sum_{t=1}^{T} c_t \mathbb{E}\left[W(t-1)\mathrm{VaR}_\alpha\left(\sum_{m=1}^{M} s_m(t-1) r_m(t-1) \mid \mathcal{F}_{t-1}\right)\right] \\
&= \sum_{t=1}^{T} c_t \mathbb{E}\left[W(t-1)\left(s(t-1)\mu^* + \sqrt{v(t-1)\Sigma v(t-1)^*}\Delta_\alpha\right)\right] \qquad (2.3.3)
\end{aligned}$$

where $v(t-1) = (s_1(t-1)\sigma_1(t-1), ..., s_M(t-1)\sigma_M(t-1)), \forall t = 1, ..., T$. The analytical solution for Equation (2.3.3) is overly complex, and the computation of such measure can be done by Monte-Carlo methods. Calculating the MRVaR

$$\begin{aligned}
\mathrm{MRVaR}^T_{\alpha,c}(X, \mathcal{F}) &= \sum_{t=1}^{T} c_t \mathbb{E}\left[\frac{\mathrm{VaR}_\alpha(X(t) \mid \mathcal{F}_{t-1})}{W(t-1)}\right] \\
&= \sum_{t=1}^{T} c_t \mathbb{E}\left[\mathrm{VaR}_\alpha\left(\sum_{m=1}^{M} s_m(t-1) r_m(t-1) \mid \mathcal{F}_{t-1}\right)\right] \\
&= \sum_{t=1}^{T} c_t \left[s(t-1)\mu^* + \Delta_\alpha \mathbb{E}\left(\sqrt{v(t-1)\Sigma v(t-1)^*}\right)\right]. \qquad (2.3.4)
\end{aligned}$$

Equation (2.3.2) allows us to write the following relation $\forall m = 1, ..., M$

$$\sigma_m(t)^2 = k_m\left[1 + \sum_{j=1}^{t-1}\prod_{i=t-j}^{t-1}\beta_m(i)\right] + \sigma_m(0)^2\prod_{i=0}^{t-1}\beta_m(i). \tag{2.3.5}$$

where $\beta_m(t) = \gamma_m + \alpha_m z_m(t)^2, \forall t$. Suppose that the innovations are independent, i.e., $\Sigma = Id$. Then we can rewrite Equation (2.3.4) as

$$\begin{aligned}
\mathrm{MRVaR}^T_{\alpha,c}(X,\mathcal{F}) &= \sum_{t=1}^{T} c_t \left[s(t-1)\mu^* + \Delta_\alpha \mathbb{E}\left(\sqrt{v(t-1)\Sigma v(t-1)^*} \right) \right] \qquad (2.3.6) \\
&= \sum_{t=1}^{T} c_t \left[s(t-1)\mu^* + \Delta_\alpha \mathbb{E}\left(\sqrt{\sum_{m=1}^{M} s_m(t-1)^2 \sigma_m(t-1)^2} \right) \right] \\
&\geq \sum_{t=1}^{T} c_t \left[s(t-1)\mu^* + \Delta_\alpha \left(\sum_{m=1}^{M} s_m(t-1)^2 \mathbb{E}\left(\sigma_m(t-1)^2 \right) \right)^{\frac{1}{2}} \right]. \qquad (2.3.7)
\end{aligned}$$

where the last step comes from Jensen's inequality and the fact that $\Delta_\alpha \leq 0, \forall \alpha \leq 0.5$. From Equation (2.3.5) we obtain

$$\begin{aligned}
\mathbb{E}[\sigma_m(t)^2] &= k_m \left[1 + \sum_{j=1}^{t-1} \prod_{i=t-j}^{t-1} \mathbb{E}[\beta_m(i)] \right] + \sigma_m(0)^2 \prod_{i=0}^{t-1} \mathbb{E}[\beta_m(i)] \\
&= k_m \left[1 + \sum_{j=1}^{t-1} \prod_{i=t-j}^{t-1} \mathbb{E}\left[\mathbb{E}\left(\beta_m(i) \mid \mathcal{F}_i \right) \right] \right] + \sigma_m(0)^2 \prod_{i=0}^{t-1} \mathbb{E}\left[\mathbb{E}\left(\beta_m(i) \mid \mathcal{F}_i \right) \right] \\
&= k_m \left[1 + \sum_{j=1}^{t-1} \prod_{i=t-j}^{t-1} (\gamma_m + \alpha_m) \right] + \sigma_m(0)^2 \prod_{i=0}^{t-1} (\gamma_m + \alpha_m) \\
&= k_m \frac{1 - (\gamma_m + \alpha_m)^t}{1 - \gamma_m - \alpha_m} + \sigma_m(0)^2 (\gamma_m + \alpha_m)^t. \qquad (2.3.8)
\end{aligned}$$

Using Equation (2.3.8) we can rewrite Equation (2.3.7) as

$$\begin{aligned}
\mathrm{MRVaR}^T_{\alpha,c}(X,\mathcal{F}) \geq \sum_{t=1}^{T} c_t \Bigg[& s(t-1)\mu^* \\
& + \Delta_\alpha \left(\sum_{m=1}^{M} s_m(t-1)^2 \left[k_m \frac{1 - (\gamma_m + \alpha_m)^{t-1}}{1 - \gamma_m - \alpha_m} \right.\right. \\
& \left.\left. + \sigma_m(0)^2 (\gamma_m + \alpha_m)^{t-1} \right] \right)^{\frac{1}{2}} \Bigg]. \qquad (2.3.9)
\end{aligned}$$

We can still obtain an upper bound for the $MRVaR$ using the fact that $\sqrt{x} \geq \rho x, \forall x \in (0, \frac{1}{\rho^2})$. Suppose that the term under the radical has order of magnitude $\ll \frac{1}{\rho^2}$, or is bounded by such value. We may then apply the first inequality to Equation (2.3.6) and use Equation (2.3.8) to obtain

$$\begin{aligned}
\mathrm{MRVaR}^T_{\alpha,c}(X,\mathcal{F}) &= \sum_{t=1}^{T} c_t \left[s(t-1)\mu^* + \Delta_\alpha \mathbb{E}\left(\sqrt{v(t-1)\Sigma v(t-1)^*} \right) \right] \\
&\leq \sum_{t=1}^{T} c_t \left[s(t-1)\mu^* + \rho\Delta_\alpha \mathbb{E}\left(v(t-1)\Sigma v(t-1)^* \right) \right] \\
&= \sum_{t=1}^{T} c_t \Bigg[s(t-1)\mu^* \\
&\quad + \rho\Delta_\alpha \Bigg(\sum_{m=1}^{M} s_m(t-1)^2 \Bigg[k_m \frac{1-(\gamma_m+\alpha_m)^{t-1}}{1-\gamma_m-\alpha_m} \\
&\quad + \sigma_m(0)^2 (\gamma_m+\alpha_m)^{t-1} \Bigg] \Bigg) \Bigg]. \qquad (2.3.10)
\end{aligned}$$

Example 9. *Let us do the calculation of the lower bound in Equation 2.3.9 in the case where there is an unique asset on the portfolio, so that* $s(t) = 1, \forall t$. *For* $k, \gamma, \alpha, \sigma_0 \in \mathbb{R}^+, \alpha + \gamma = \theta < 1, t \geq 1,$ *define*

$$\begin{aligned}
f(k,\theta,\sigma_0,t) &= k\frac{1-\theta^{t-1}}{1-\theta} + {\sigma_0}^2\theta^{t-1} \\
&= {\sigma_0}^2 + (1-\theta^{t-1})(\frac{k}{1-\theta} - {\sigma_0}^2).
\end{aligned}$$

We can then rewrite Equation (2.3.9) and Equation (2.3.10) as

$$\sum_{t=1}^{T} c_t \left[\mu + \rho\Delta_\alpha f(k,\theta,\sigma_0,t) \right] \geq \mathrm{MRVaR}^T_{\alpha,c}(X,\mathcal{F}) \geq \sum_{t=1}^{T} c_t \left[\mu + \Delta_\alpha f(k,\theta,\sigma_0,t)^{\frac{1}{2}} \right]. \qquad (2.3.11)$$

The calibration of ρ *can be done by observing the general behavior of the assets returns and volatilities, but in general upper bounds have little importance when dealing with risk and we usually will arrive in poor estimates. The terms* $\mu + \Delta_\alpha f(k,\theta,\sigma_0,t)^{\frac{1}{2}}$ *are lower bounds for the expected value of the relative risk measure in period* t. *So the analysis of such sequence tells how the risk evolves over time. The following relation tells how the risk bound behaves on the long term*

$$\lim_{t\to\infty} \left(\mu + \Delta_\alpha f(k,\theta,\sigma_0,t)^{\frac{1}{2}} \right) = \mu + \Delta_\alpha \left(\frac{k}{1-\theta} \right)^{\frac{1}{2}}.$$

Let us do the calculation of such terms, with level $\alpha = 10\%$, *for the following assets:*

	μ	σ_0	θ	k
Asset 1	*0.1*	*0.2*	*0.7*	*0.003*
Asset 2	*0.1*	*0.2*	*0.85*	*0.0015*
Asset 3	*0.1*	*0.2*	*0.7*	*0.027*
Asset 4	*0.1*	*0.2*	*0.85*	*0.0135*

The results are present on the graphic below:

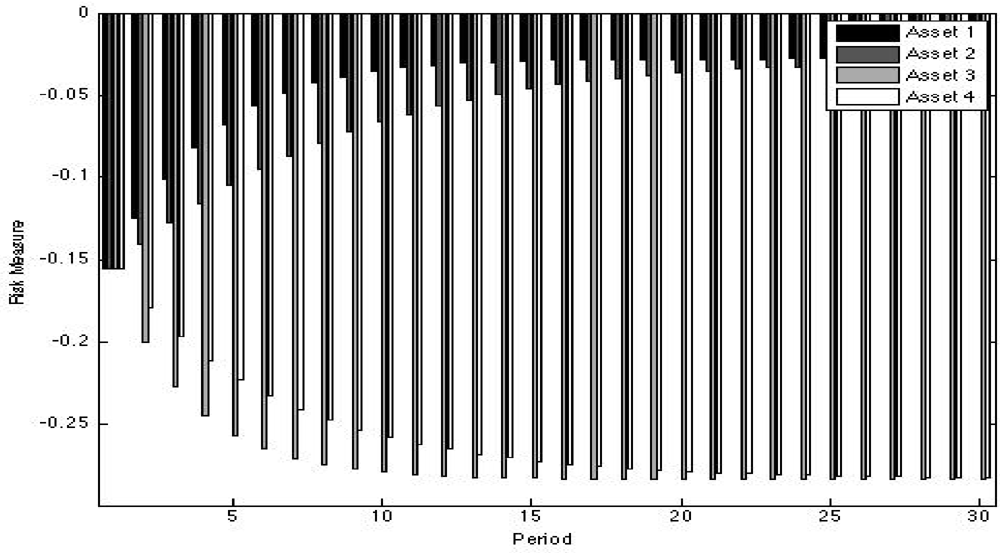

Figure 2.2: Effects of a GARCH(1,1) parameters over the dynamics of the risk measure bound

The short-term risk bound is the same for all assets, and the long-term one coincides for assets 1,2 and 3,4. Parameter θ sets the speed of convergence of the risk bound to its long-term value, and such convergence in this case is always monotonic.

2.4 The GARCH(2,2) Case

Suppose the logarithmic returns satisfy the following equations $\forall m \in \{1, ..., M\}$

$$r_m(t) = \mu_m + \sigma_m(t) z_m(t) \tag{2.4.1}$$

and

$$\sigma_m(t)^2 = k_m + \sum_{j=1}^{2} \sigma_m(t-j)^2 (\gamma_m(j) + \alpha_m(j) z_m(t-j)^2) \tag{2.4.2}$$

where the parameters $\mu_m, k_m, \gamma_m(1), \gamma_m(2), \alpha_m(1), \alpha_m(2), \sigma_m(-1)$ and $\sigma_m(0)$ are known $\forall m \in \{1, ..., M\}$. For simplicity we assume $z_m(-1) = 1, \forall m \in \{1, ..., M\}$. The random variables $(z_m(t) \mid \mathcal{F}_t)_{t=0}^{T-1}$ are i.i.d standard Gaussians and the correlation matrix of $(z_m(t) \mid \mathcal{F}_t)_{m=1}^{M}, \forall t = 1, ..., T-1$, is given by Σ. The multiperiod risk measure is given by Equation (2.3.3). The recurrence from Equation (2.4.2) allows us to write $\forall t \in \{2, ..., T-1\}$

$$\begin{aligned}
\mathbb{E}[\sigma_m(t)^2] &= \mathbb{E}\Bigg[k_m + \sum_{j=1}^{2} \sigma_m(t-j)^2 (\gamma_m(j) + \alpha_m(j) z_m(t-j)^2)\Bigg] \\
&= \mathbb{E}\Bigg[k_m + \sum_{j=1}^{2} \mathbb{E}\Big[\sigma_m(t-j)^2 (\gamma_m(j) + \alpha_m(j) z_m(t-j)^2) \mid \mathbb{F}_{t-j}\Big]\Bigg] \\
&= \mathbb{E}\Bigg[k_m + \sum_{j=1}^{2} \sigma_m(t-j)^2 \Big(\gamma_m(j) + \alpha_m(j) \mathbb{E}\Big[z_m(t-j)^2 \mid \mathbb{F}_{t-j}\Big]\Big)\Bigg] \\
&= \mathbb{E}\Bigg[k_m + \sum_{j=1}^{2} \sigma_m(t-j)^2 \Big(\gamma_m(j) + \alpha_m(j)\Big)\Bigg] \\
&= k_m + \sum_{j=1}^{2} \mathbb{E}[\sigma_m(t-j)^2] \Big(\gamma_m(j) + \alpha_m(j)\Big).
\end{aligned} \tag{2.4.3}$$

Notice that such equation remains valid for t=1, since we are assuming $z_m(-1) = 1, \forall m$. This second order linear recurrence gives us the following expression for $\mathbb{E}[\sigma(t, m)^2]$

$$\begin{aligned}
\mathbb{E}[\sigma_m(t)^2] &= \frac{\lambda_{m+}^{t+1} - \lambda_{m-}^{t+1}}{\lambda_{m+} - \lambda_{m-}} \sigma_m(0)^2 + \frac{\lambda_{m+}\lambda_{m-}^{t+1} - \lambda_{m+}^{t+1}\lambda_{m-}}{\lambda_{m+} - \lambda_{m-}} \sigma_m(-1)^2 \\
&\quad - \frac{k_m}{1 - (\theta_m(1) + \theta_m(2))} \Bigg(\frac{\lambda_{m+}^{t+1} - \lambda_{m-}^{t+1} + \lambda_{m+}\lambda_{m-}^{t+1} - \lambda_{m+}^{t+1}\lambda_{m-}}{\lambda_{m+} - \lambda_{m-}} - 1 \Bigg).
\end{aligned} \tag{2.4.4}$$

where $\theta_m(j) = \gamma_m(j) + \alpha_m(j), j \in \{1, 2\}$, and

$$\lambda_{m+} = \frac{\theta_m(1) + \sqrt{\theta_m(1)^2 + 4\theta_m(2)}}{2},$$

$$\lambda_{m-} = \frac{\theta_m(1) - \sqrt{\theta_m(1)^2 + 4\theta_m(2)}}{2}.$$

Using the results from Equation (2.4.4) in Equation (2.3.7) and Equation (2.3.10) we obtain the bounds for the MRVaR.

Example 10. *Let us do the calculus of the bound from Equations (2.3.7) and (2.3.10) in the case where there is only one asset in the portfolio, so that* $s(t) = 1, \forall t$. *For* $k, \gamma_1, \gamma_2, \alpha_1, \alpha_2, \sigma_0, \sigma_{-1} \in \mathbb{R}^+, \theta_1 = \gamma_1 + \alpha_1, \theta_2 = \gamma_2 + \alpha_2$ *and* $\theta_1 + \theta_2 < 1$, *define*

$$\begin{aligned} f(k, \theta_1, \theta_2, \sigma_0, \sigma_{-1}, t) &= \frac{\lambda_+^t - \lambda_-^t}{\lambda_+ - \lambda_-}{\sigma_0}^2 + \frac{\lambda_+\lambda_-^t - \lambda_+^t\lambda_-}{\lambda_+ - \lambda_-}{\sigma_{-1}}^2 + \cdots \\ &\cdots + \frac{k}{1-(\theta_1+\theta_2)}\left(1 - \frac{\lambda_+^t - \lambda_-^t + \lambda_+\lambda_-^t - \lambda_+^t\lambda_-}{\lambda_+ - \lambda_-}\right). \end{aligned}$$

where $\lambda_+ = \frac{\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}}{2}, \lambda_- = \frac{\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}}{2}$. *Using this definition we can rewrite the inequalities as*

$$\begin{aligned} \mathrm{MRVaR}_{\alpha,c}^T(X, \mathcal{F}) &\leq \sum_{t=1}^T c_t \Big[\mu + \rho\Delta_\alpha f(k, \theta_1, \theta_2, \sigma_0, \sigma_{-1}, t)\Big], \\ \mathrm{MRVaR}_{\alpha,c}^T(X, \mathcal{F}) &\geq \sum_{t=1}^T c_t \Big[\mu + \Delta_\alpha f(k, \theta_1, \theta_2, \sigma_0, \sigma_{-1}, t)^{\frac{1}{2}}\Big]. \end{aligned}$$

Following the same analysis as in Example 8, notice that $\theta_1, \theta_2 > 0, \theta_1 + \theta_2 < 1$ *imply* $\lambda_+, \lambda_- \in (-1, 1)$, *and this allows us to calculate the long-term behavior of the risk bound, which is given by*

$$\lim_{t\to\infty}\left(\mu + \Delta_\alpha f(k, \theta, \sigma_0, t)^{\frac{1}{2}}\right) = \mu + \Delta_\alpha\left(\frac{k}{1-(\theta_1+\theta_2)}\right)^{\frac{1}{2}}.$$

Let us calculate such factors for $\alpha = 10\%$ *and the following assets:*

	μ	k	θ_1	θ_2	σ_0	σ_{-1}
Asset 1	*0.01*	*0.0004*	*0.05*	*0.9*	*0.15*	*0.05*
Asset 2	*0.01*	*0.0004*	*0.05*	*0.65*	*0.15*	*0.05*
Asset 3	*0.01*	*0.0004*	*0.475*	*0.475*	*0.15*	*0.05*
Asset 4	*0.01*	*0.0004*	*0.9*	*0.05*	*0.15*	*0.05*
Asset 5	*0.01*	*0.0004*	*0.05*	*0.9*	*0.15*	*0.25*
Asset 6	*0.01*	*0.004*	*0.05*	*0.9*	*0.15*	*0.05*

The results are presented on the figure below:

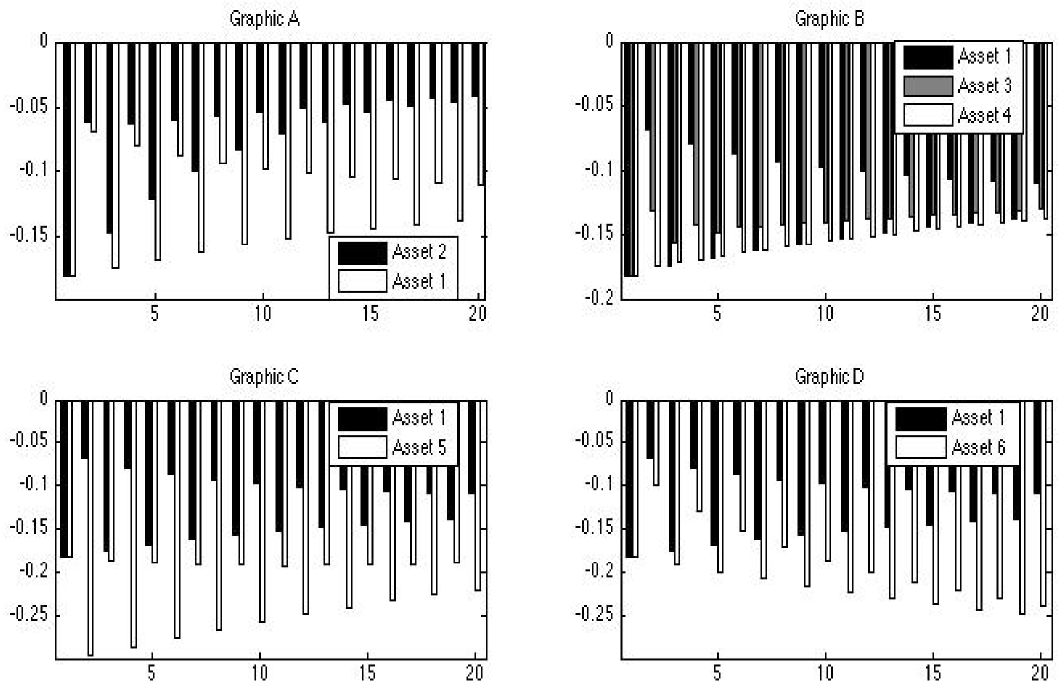

Figure 2.3: Effects of a GARCH(2,2) parameters over the dynamics of the risk measure bound.

- *Graphic A reflects the influence of the factor $\theta_1 + \theta_2$, which is bigger for Asset 1 than for Asset 2, in the dynamics of the risk measure bound. Such factor is directly proportional to the speed convergence of the risk measure bound to its long-term value.*

- *Graphic B reflects the influence of the factors θ_1, θ_2, holding $\theta_1 + \theta_2$ constant, in the dynamics of the risk measure bound. The relevance of θ_1 increases when we move from Asset 1 to Asset 3, and from Asset 3 to Asset 4. The effect is the reduction of he oscillatory behavior of the risk bound.*

- *Graphic C reflects the influence of the factor σ_{-1} in the dynamics of the risk measure bound. Assets 1 and 5 have the same long-term value for the risk bound, but the first has $\sigma_{-1} < \sigma_0$, while the second has $\sigma_{-1} > \sigma_0$. The effect is the reversion of the oscillation direction.*

- *Graphic D reflects the influence of K in the dynamics of the risk measure bound. The effect is the change on the long-term risk bound value.*

2.5 Further Autoregressive Models

The methodology developed in Sections 2.3 and 2.4 can be extended to other autoregressive models with conditional heteroskedasticity, in the cases where the recurrence is given over the variance $\sigma(t)^2$. The following are examples of such models:

ARCH(q) The model is described by

$$\begin{aligned} r(t) &= \mu + \sigma(t)z(t) \\ \sigma(t)^2 &= k + \sum_{j=1}^{q} \alpha(j)\sigma(t-j)^2 z(t-j)^2 \end{aligned}$$

GARCH(p,q) The model is described by

$$\begin{aligned} r(t) &= \mu + \sigma(t)z(t) \\ \sigma(t)^2 &= k + \sum_{j=1}^{q} \gamma(j)\sigma(t-j)^2 + \sum_{j=1}^{q} \alpha(j)\sigma(t-j)^2 z(t-j)^2 \end{aligned}$$

NGARCH(1) The model is described by

$$\begin{aligned} r(t) &= \mu + \sigma(t)z(t) \\ \sigma(t)^2 &= k + \gamma\sigma(t-1)^2 + \alpha\sigma(t-1)^2(z(t-1)-\delta)^2 \end{aligned}$$

GJR-GARCH The model is described by

$$\begin{aligned} r(t) &= \mu + \sigma(t)z(t) \\ \sigma(t)^2 &= k + \sigma(t-1)^2\left[\gamma + \alpha z(t-1)^2 + \phi z(t-1)^2 \chi_{x>0}(z(t-1))\right] \end{aligned}$$

Equations (2.3.3) and (2.3.4) can be used to obtain in each case, via Monte-Carlo methods, the multi-period risk measures. The bound in Equation (2.3.7) also remains valid given certain changes on the coefficients of each model. The precision of such bounds is analyzed in Chapter 4, for a GARCH(1,1) and a GARCH(2,2) model.

In the first two models the values of p and q determine the order of the recurrence that defines the variances, and a direct consequence is the richness and detailing of risk bound behavior. The counterpart is the increase of complexity for the calculation of the measure, which not always is converted on an increased accuracy.

For the last two models the similarity of the recurrence over the volatilities with a GARCH(1,1) model allows us to use an analogous argument to calculate the bound of Equation (2.3.7). The respective values of the θ parameters in each case are

$$\begin{aligned} \theta_{NGARCH} &= \gamma + \alpha(1+\delta^2) \\ \theta_{GJR-GARCH} &= \gamma + \alpha + \frac{\phi}{2} \end{aligned}$$

Examples of autoregressive conditional heteroskedasticity models in which the recurrence is not over $\sigma(t)^2$ include EGARCH and TGARCH models, where the recurrence is over $ln(\sigma(t)^2)$ and $\sigma(t)$, respectively. The bound of Equation (2.3.7) is no longer valid in this case.

Chapter 3

Portfolio Optimization

Using the notation defined on Chapter 2 we can describe the agent's wealth allocation problem as

$$\max_{s \in (S_+^M)^{T-1}} \mathbb{E}(W_T) \ s.t.\ \mathcal{R}(s) > 0, \mathcal{L}(s) > 0. \tag{3.0.1}$$

Where the agent chooses his allocation strategy for each period on the M-dimensional simplex, aiming to maximize his expected final wealth and subject to risk and liquidity constraints, represented by $\mathcal{R}(s) > 0$ e $\mathcal{L}(s) > 0$, respectively.

3.1 Returns modeled by a GARCH(1,1)

Suppose the series logarithmic returns follows a GARCH(1,1) process, given by Equations (2.3.1) and (2.3.2). The term $\mathbb{E}(W_T)$ can be written, assuming equality in Equation (2.1.3), as a function of $s \in (S_+^M)^{T-1}$

$$\begin{aligned}
\mathbb{E}(W_T) &= \mathbb{E}\left[W(0)\prod_{t=0}^{t=T-1}\left(1+\sum_{m=1}^{M} s_m(t) r_m(t)\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left\{W(0)\prod_{t=0}^{t=T-1}\left(1+\sum_{m=1}^{M} s_m(t)(\mu_m + \sigma_m(t) z_m(t))\right) \mid \mathcal{F}_{T-1}\right\}\right] \\
&= \mathbb{E}\left[W(0)\prod_{t=0}^{t=T-2}\left(1+\sum_{m=1}^{M} s_m(t)(\mu_m + \sigma_m(t) z_m(t))\right)\right. \\
&\qquad \left.\mathbb{E}\left\{1+\sum_{m=1}^{M} s_m(T-1)(\mu_m + \sigma_m(T-1) z_m(T-1)) \mid \mathcal{F}_{T-1}\right\}\right] \\
&= \mathbb{E}\left[W(0)\prod_{t=0}^{t=T-2}\left(1+\sum_{m=1}^{M} s_m(t)(\mu_m + \sigma_m(t) z_m(t))\right)\right. \\
&\qquad \left.\left(1+\sum_{m=1}^{M} s_m(T-1)(\mu_m + \sigma_m(T-1)\mathbb{E}\{z_m(T-1) \mid \mathcal{F}_{T-1}\})\right)\right] \\
&= \left(1+\sum_{m=1}^{M} s_m(T-1)\mu_m\right)\mathbb{E}\left[W(0)\prod_{t=0}^{t=T-2}\left(1+\sum_{m=1}^{M} s_m(t)(\mu_m + \sigma_m(t) z_m(t))\right)\right] \\
&= \left(1+\sum_{m=1}^{M} s_m(T-1)\mu_m\right)\mathbb{E}(W_{T-1}). \qquad (3.1.1)
\end{aligned}$$

We used that $\sigma(t,m)$ is $\mathcal{F}_t$-measurable and $z(t,m) \perp \mathcal{F}_t$. Using Equation (3.1.1) we can conclude that

$$\mathbb{E}(W_T) = W(0)\prod_{t=0}^{T-1}\Big(1+\sum_{m=1}^{M} s_m(t)\mu_m\Big). \tag{3.1.2}$$

Using Equation (3.1.2), we can rewrite Equation (3.0.1) as

$$max_{s\in(S_+^M)^{T-1}} W(0)\prod_{t=0}^{T-1}\Big(1+\sum_{m=1}^{M} s_m(t)\mu(m)\Big) \ s.t.\ \mathcal{R}(s) > 0, \mathcal{L}(s) > 0. \tag{3.1.3}$$

The example below illustrates the dynamics of the agent's optimal portfolio over time. To simplify our notation let us define

$$f(t,m) \quad = \quad \Big[k_m\frac{1-(\gamma_m+\alpha_m)^{t-1}}{1-\gamma_m-\alpha_m} + \sigma_m(0)^2(\gamma_m+\alpha_m)^{t-1}\Big].$$

Example 11. *From the analysis in Chapter 2 we know that* $\forall t \in \{1,...,T\}$

$$\mathbb{E}\Bigg[\frac{\mathrm{VaR}_\alpha(X(t)\mid\mathcal{F}_{t-1})}{W(t-1)}\Bigg] \quad \geq \quad \sum_{m=1}^{M} s_m(t-1)\mu_m + \Delta_\alpha\Big(\sum_{m=1}^{M} s_m(t-1)^2 f(t,m)\Big)^{\frac{1}{2}}.$$

Suppose the agent uses such lower bounds as the risk measures in his optimization problem. Suppose also that the agent faces liquidity constraints so that the percentual variation of the wealth invested in a certain asset between t *and* $t+1$ *can't be bigger than* $\rho \in (0,1)$. *So the restrictions* $\mathcal{R}$ *and* $\mathcal{L}$ *in Equation (3.1.3) can be written as*

$$\sum_{m=1}^{M} s_m(t-1)\mu_m + \Delta_\alpha\Big(\sum_{m=1}^{M} s_m(t-1)^2 f(t,m)\Big)^{\frac{1}{2}} \geq -\delta, \forall t \in \{1,..,T\}, \tag{3.1.4}$$

and

$$\mid s_m(t) - s_m(t-1) \mid \leq \rho, \forall m \in \{1,...,M\}, \forall t \in \{1,...,T-1\}. \tag{3.1.5}$$

Parameter δ *represents the biggest negative percentual variation between two periods the agent is willing to face (having as reference the* α*-quantile). So Equations (3.1.3), (3.1.4) and (3.1.5) fully characterize the wealth allocation problem. The table below gives the parameters of the assets present in the agent's portfolio:*

	μ	σ_0	θ	k
Asset 1	*0.05*	*0.0781*	*0.8*	*0.00274*
Asset 2	*0.25*	*0.3906*	*0.7*	*0.03708*
Asset 3	*0.3*	*0.4296*	*0.95*	*0.03052*

The graphic below shows the evolution of the risk present in each asset:

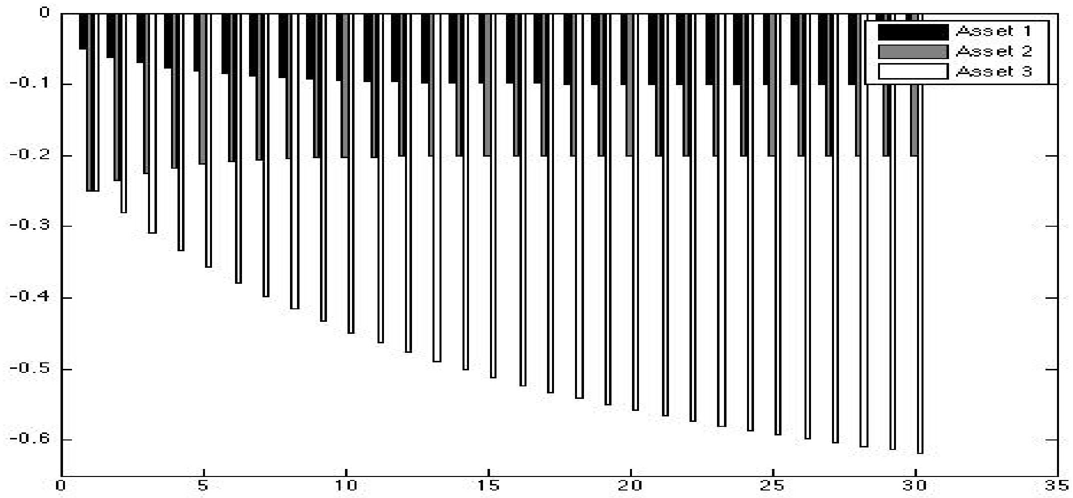

Figure 3.1: Dynamic of the risk measure.

The graphics below show the evolution of the optimal portfolio composition for $\rho = 0.1$ *and* $\delta = 0.05, 0.1, 0.15$ *and* 0.2*, respectively.*

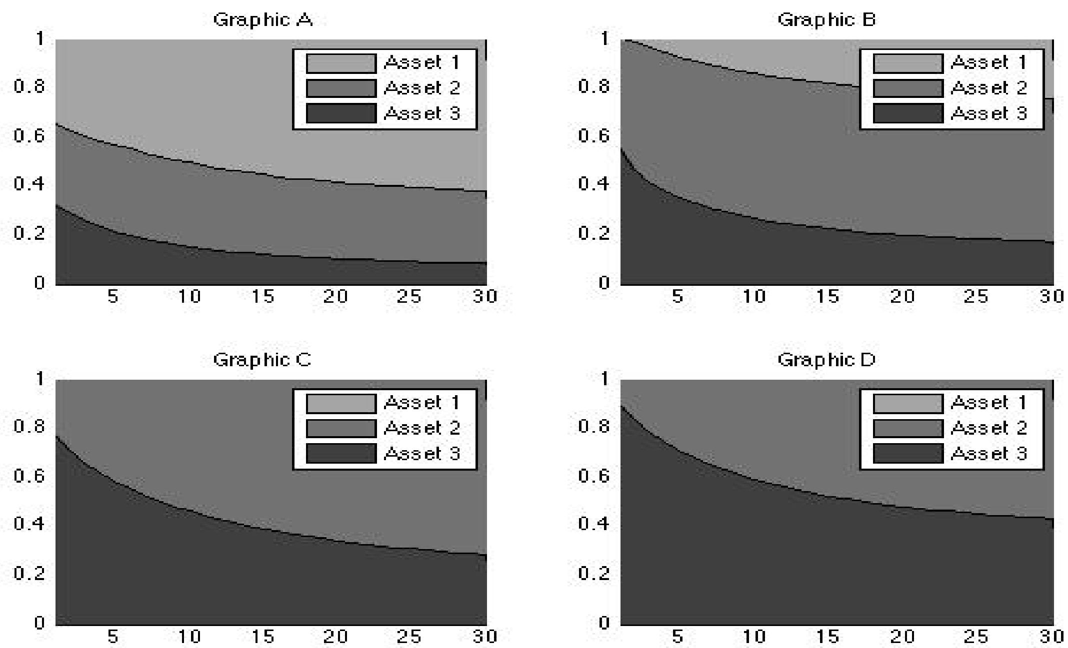

Figure 3.2: Dynamic of the optimal portfolio.

The increase of the risk present in the asset with the largest mean return is reflected on the decreasing share of wealth invested on such asset on the long run. Notice that the composition of the optimal portfolio converges to its long-term configuration together with the risk measures. Also notice that once we relax the risk constraints the share of wealth invested on the asset with the largest return/risk increases.

3.2 Returns modeled by a GARCH(2,2)

Suppose the series logarithmic returns follows a GARCH(2,2) process, given by Equations (2.4.1) and (2.4.2). Equation (3.1.2) remains valid since $\sigma_m(t)$ is still $\mathcal{F}_t$-measurable and $z_m(t) \perp \mathcal{F}_t$. So Equation (3.1.3) continues to describe the agent's wealth allocation problem. The following example illustrates the dynamic of the agent's optimal portfolio. To simplify the notation let's define

$$\begin{aligned} g(t,m) &= \frac{\lambda_{m+}^t - \lambda_{m-}^t}{\lambda_{m+} - \lambda_{m-}}\sigma_m(0)^2 + \frac{\lambda_{m+}\lambda_{m-}^t - \lambda_{m+}^t\lambda_{m-}}{\lambda_{m+} - \lambda_{m-}}\sigma_m(-1)^2 \\ &+ \frac{k_m}{1-(\theta_m(1)+\theta_m(2))}\left(1 - \frac{\lambda_{m+}^t - \lambda_{m-}^t + \lambda_{m+}\lambda_{m-}^t - \lambda_{m+}^t\lambda_{m-}}{\lambda_{m+} - \lambda_{m-}}\right). \end{aligned} \quad (3.2.1)$$

Where $\theta_m(j) = \gamma_m(j) + \alpha_m(j), j \in \{1,2\}$, e $\lambda_{m+} = \frac{\theta_m(1)+\sqrt{\theta_m(1)^2+4\theta_m(2)}}{2}$, $\lambda_{m-} = \frac{\theta_m(1)-\sqrt{\theta_m(1)^2+4\theta_m(2)}}{2}$.

Example 12. *Suppose that the agent from Example 10 decides to use a GARCH(2,2) to model his returns series, and sets his optimization problem as before. In this case the risk restrictions the agent faces are given by*

$$\sum_{m=1}^{M} s(t-1,m)\mu(m) + \Delta_\alpha\left(\sum_{m=1}^{M} s(t-1,m)^2 g(t,m)\right)^{\frac{1}{2}} \geq -\delta, \forall t \in \{1,..,T\}. \quad (3.2.2)$$

The table below gives the parameters of the assets present on the agent's portfolio:

	μ	k	θ_1	θ_2	σ_0	σ_{-1}
Ativo 1	*0.05*	*0.00274*	*0.05*	*0.75*	*0.0781*	*0.01*
Ativo 2	*0.25*	*0.03708*	*0.05*	*0.75*	*0.3906*	*0.01*
Ativo 3	*0.4*	*0.03052*	*0.05*	*0.75*	*0.4296*	*0.6*

The graphic below show the evolution of the risk present in each asset:

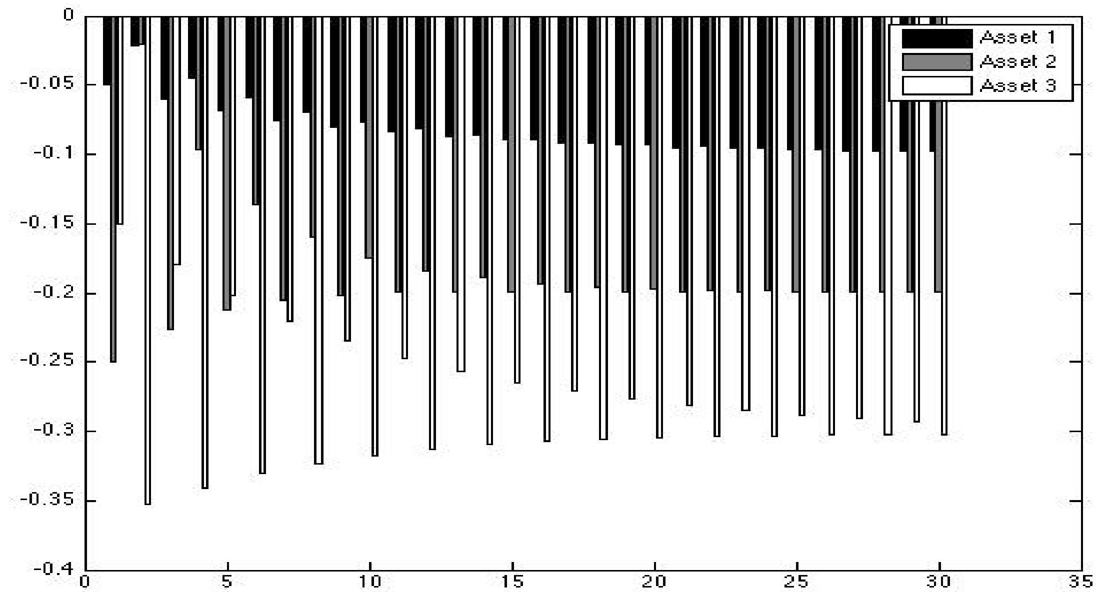

Figure 3.3: Dynamic of the risk measure.

Notice that the risk present on Asset 1 and Asset 3 oscillates, and this is more evident on Asset 3. Also, the risk in Asset 2 converges monotonically to its long-term value. The graphics below illustrate the evolution of the optimal portfolio composition for $\rho = 0.1$ *and* $\delta = 0.05, 0.1, 0.15$ *and* 0.2*, respectively.*

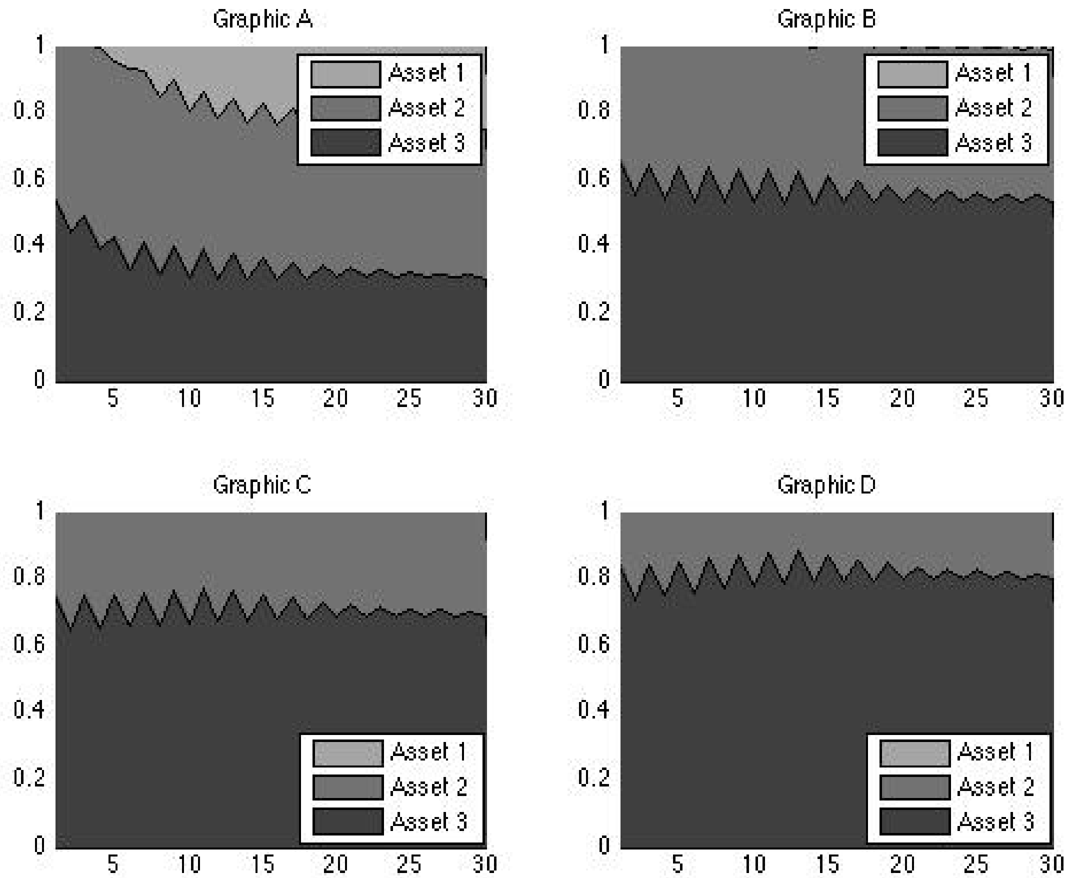

Figure 3.4: Effect of δ over the dynamics of the optimal allocation.

The similarity between the inferior and superior contours in each graphic, and also between different graphics, indicate that the oscillatory behavior of the portfolio composition is due, predominantly, to Asset 3. Notice that the oscillations are mitigated on the long run, since the risk measures converge. Also notice that once the risk threshold is relaxed, the share of wealth invested on the riskier asset increases.

The graphics below illustrate the evolution of the optimal portfolio composition for $\delta = 0.05$ *and* $\rho = 0.1, 0.05, 0.025$ *and* 0.01*, respectively.*

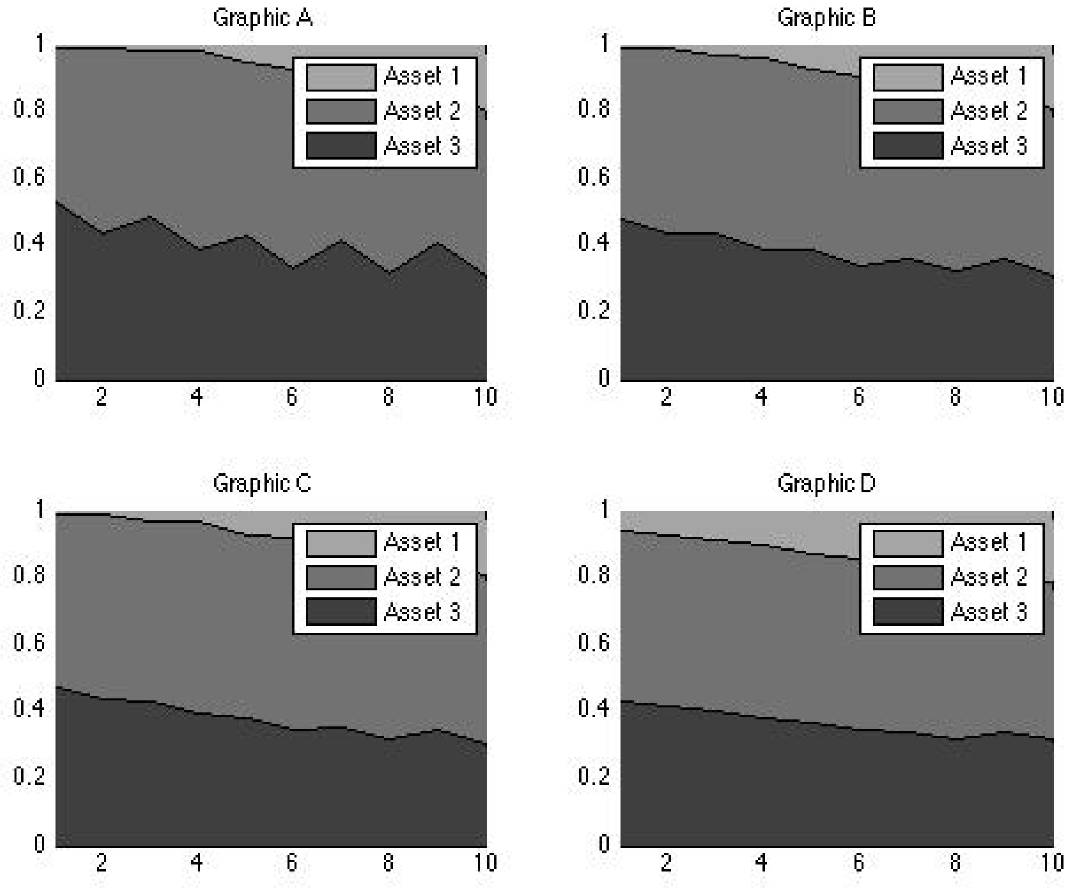

Figure 3.5: Effect of ρ over the dynamic of the optimal allocation.

There are two effects that must be highlighted on the figure above:

1. *The stronger liquidity constraints generate a reduction on the oscillatory behavior of the optimal portoflio*

2. *Once the liquidity constraints becomes stronger the agent is forced to shift the short-term portfolio composition so that the risk constraint can be satisfied on the long-term.*

Chapter 4

Numerical Analysis

The analytic expressions for the bounds found in Chapter 2 and the wealth allocation problems in Chapter 3 show how simpler the risk management and computation of optimal strategies become when one uses such bounds as measures of risk, since Monte-Carlo methods are no longer needed for scenario simulations. On the other hand, when one uses such bound as risk measures it may be incurring in great rentability losses, since the risk of the portfolios can be significantly overestimated. The present chapter uses Monte-Carlo methods to calculate the error between the risk bounds and the risk measures, in order to determine if the approach proposed in the examples of Chapter 3 is valid.

4.1 Error Analysis for the GARCH(1,1) Model

To calculate the error incurred when one uses the terms of Equation (2.3.7) as a risk measure for each subsequent period, we will study the the case with a single asset on the portfolio, as in Example 9. The next figure shows the evolution of the MRVaR and the analytic lower bound for the reference parameters described below.

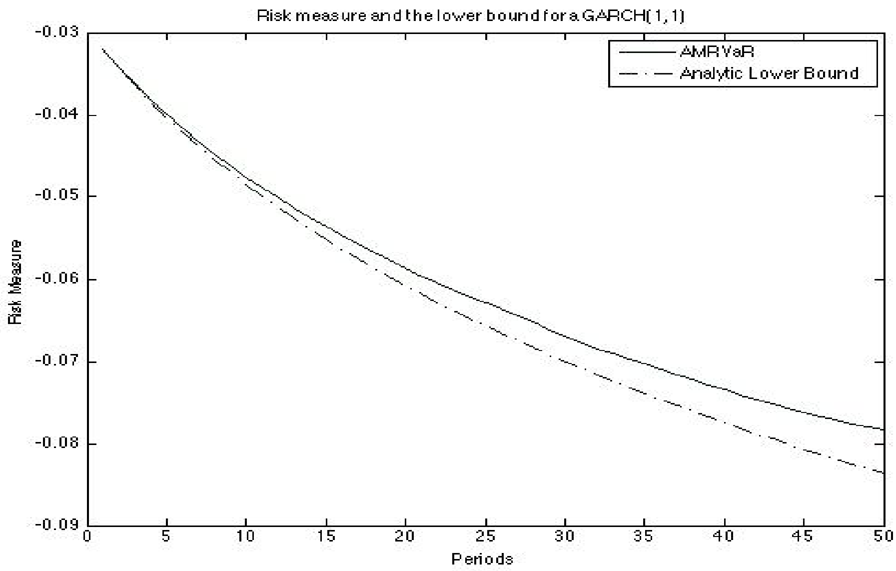

Figure 4.1: Risk measure and analytic lower bound for a GARCH(1,1)

The graphics below show the evolution of the error between the risk measure, calculated by Monte-Carlo methods with 500.000 scenarios, and the analytic bound.

The reference values where chosen so that they would better represent the values regularly found on the case studies of Chapter 5. In each case one of the reference parameters, given on the table below, was changed, except on the case of α and γ, where such parameters must also satisfy $\alpha + \gamma = 0.99$.

	μ	σ_0	α	γ	k
Asset 1	0	0.025	0.14	0.85	0.0001

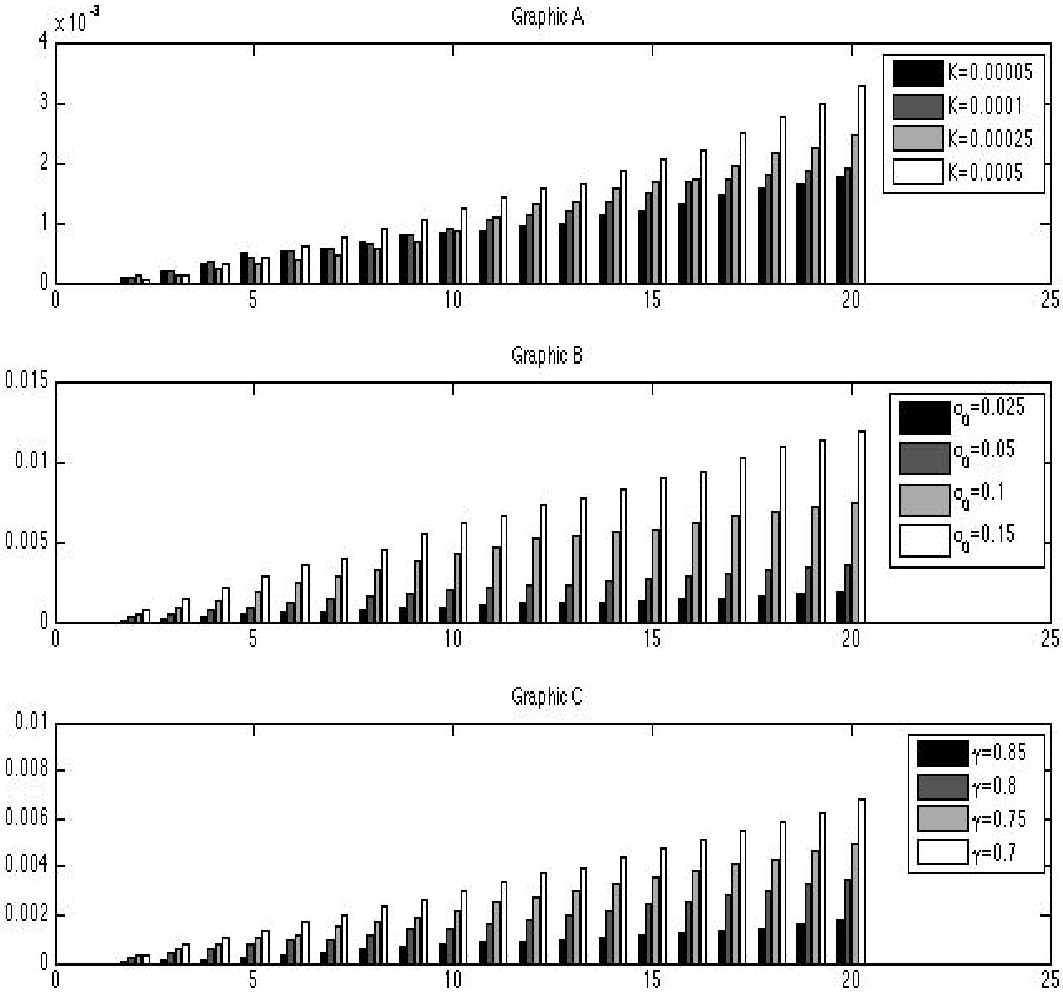

Figure 4.2: Error evolution with a GARCH(1,1)

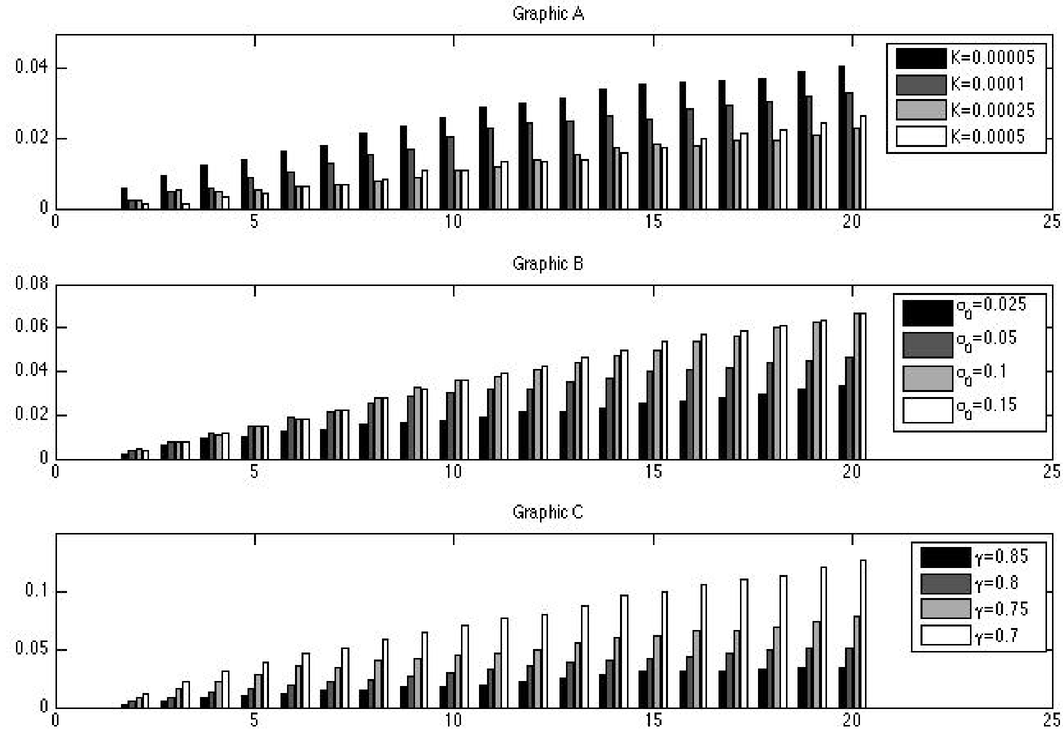

Figure 4.3: Relative error evolution with a GARCH(1,1)

For comparison purposes, the $VaR_{10\%}$ in the first period is -0.032 for the reference parameters. Considering the 20 days windows, the following points must be highlighted:

1. The absolute and relative errors increase with the number of periods

2. Raising K and σ_0 makes the absolute error increase on the long-term. The reduction of γ, which corresponds to an increase in α, makes the absolute error bigger.

3. Raising parameters σ_0 and γ increases the relative error on the long-term. Raising K decreases the relative error.

4. In Figure 4.2, Graphic A shows that even with a substantial increase on the value of K the order of the error is maintained at 10^{-2}.

5. In Figure 4.2, Graphic B shows that the error evolves roughly linearly with time and is proportional to the value of σ_0.

6. In Figure 4.2, Graphic C shows that parameter γ is the only capable of a substantial increase on the order of magnitude of the error term, reaching 10^{-1}. Such increase is inversely proportional to the value of γ

4.2 Error Analysis for the GARCH(2,2) Model

Using the same methodology as in the previous section, consider the following reference parameters

	μ	k	α_1	γ_1	α_2	γ_2	σ_0	σ_{-1}
Asset 1	0	0.0001	0.1	0.39	0.1	0.39	0.025	0.025

The figure describing the evolution of the risk measure and the analytic lower bound for such reference parameters is presented below

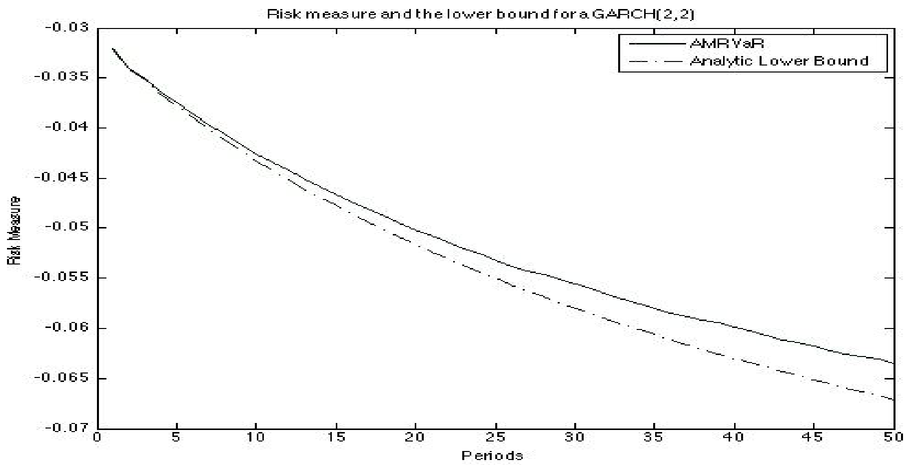

Figure 4.4: Risk measure and analytic lower bound for a GARCH(2,2)

Notice that in this case the error tends to a non-zero stationary value. In the case of a GARCH(2,2) we have several other possible behaviors for the risk measure, although most are unlikely from happening, and we present here one of them, where the error is not monotonically increasing in the number of periods:

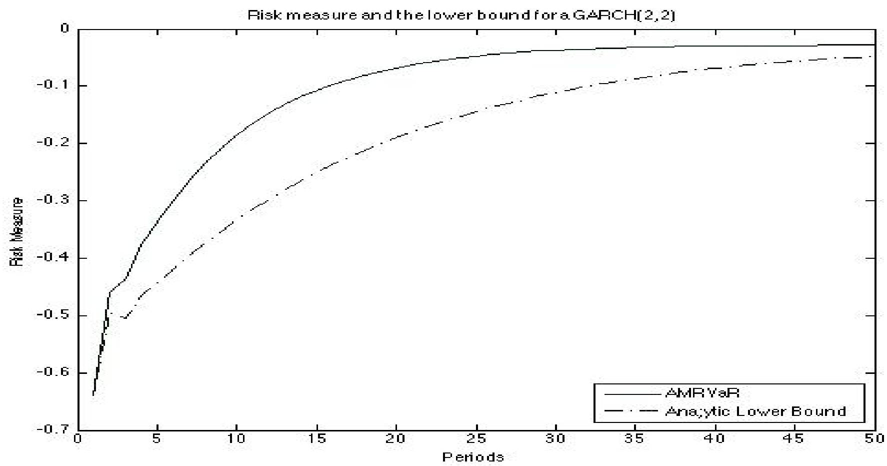

Figure 4.5: Risk measure and analytic lower bound for an unconventional GARCH(2,2)

The values of α, γ must satisfy $\alpha_1 + \gamma_1 + \alpha_2 + \gamma_2 < 1$. In the last two graphics the values 0.03 and 0.25 were assigned to the pair α, γ that was not being analyzed. The results are given in the graphics below:

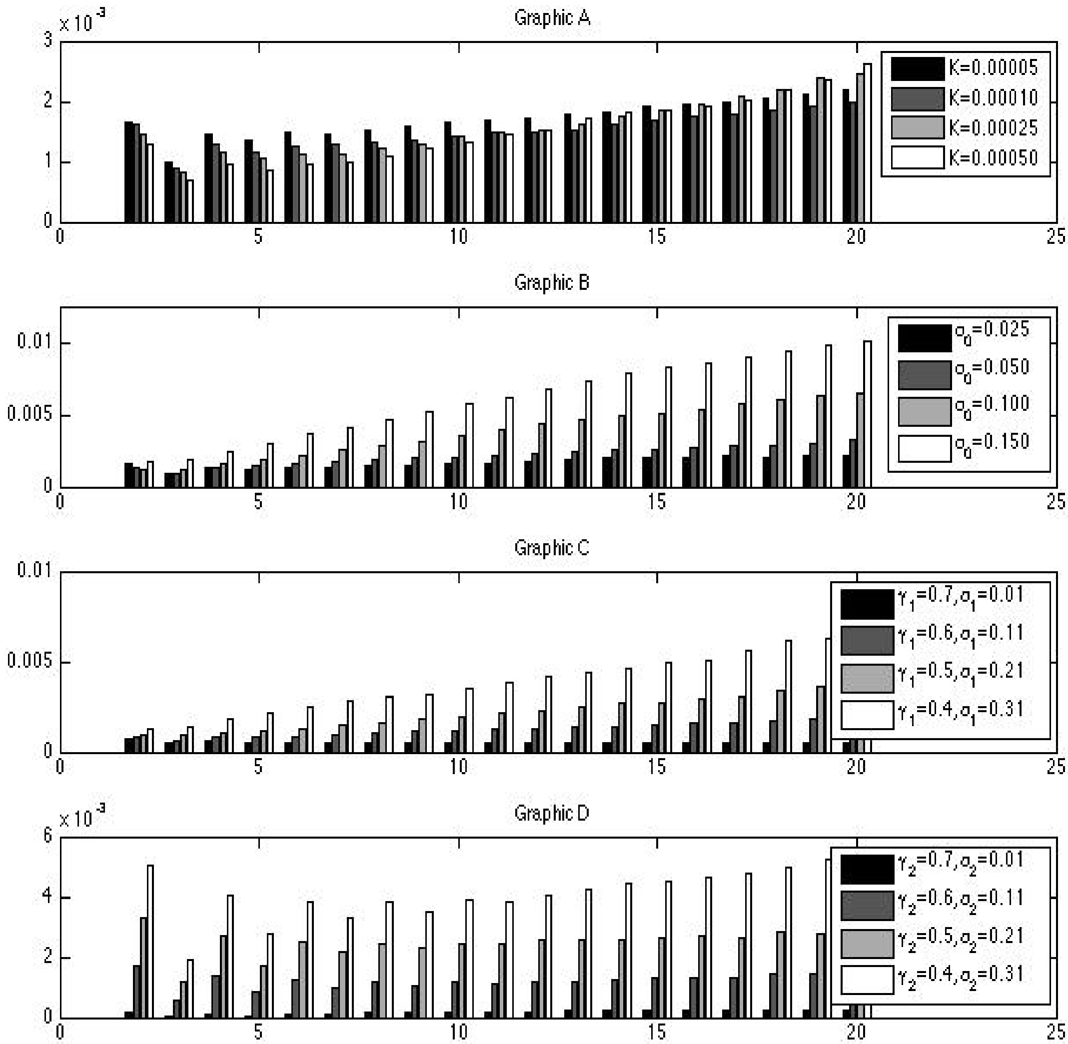

Figure 4.6: Error evolution with a GARCH(2,2)

The following points must be highlighted on Figure 4.3:

1. There is a fall on the absolute error when going from the second to the third period, but after this the error increases with the number of periods.

2. The absolute error is inversely proportional to the value of K. So we may also state that the absolute error is inversely proportional to the long-term risk value.

3. The absolute error is directly proportional to the value of σ_0.

4. The absolute error is inversely proportional to the value of γ_1.

5. The absolute error is inversely proportional to the value of γ_2. Such factor has a smaller influence over the error than γ_1. This can be noticed if one compares the error magnitudes in each case.

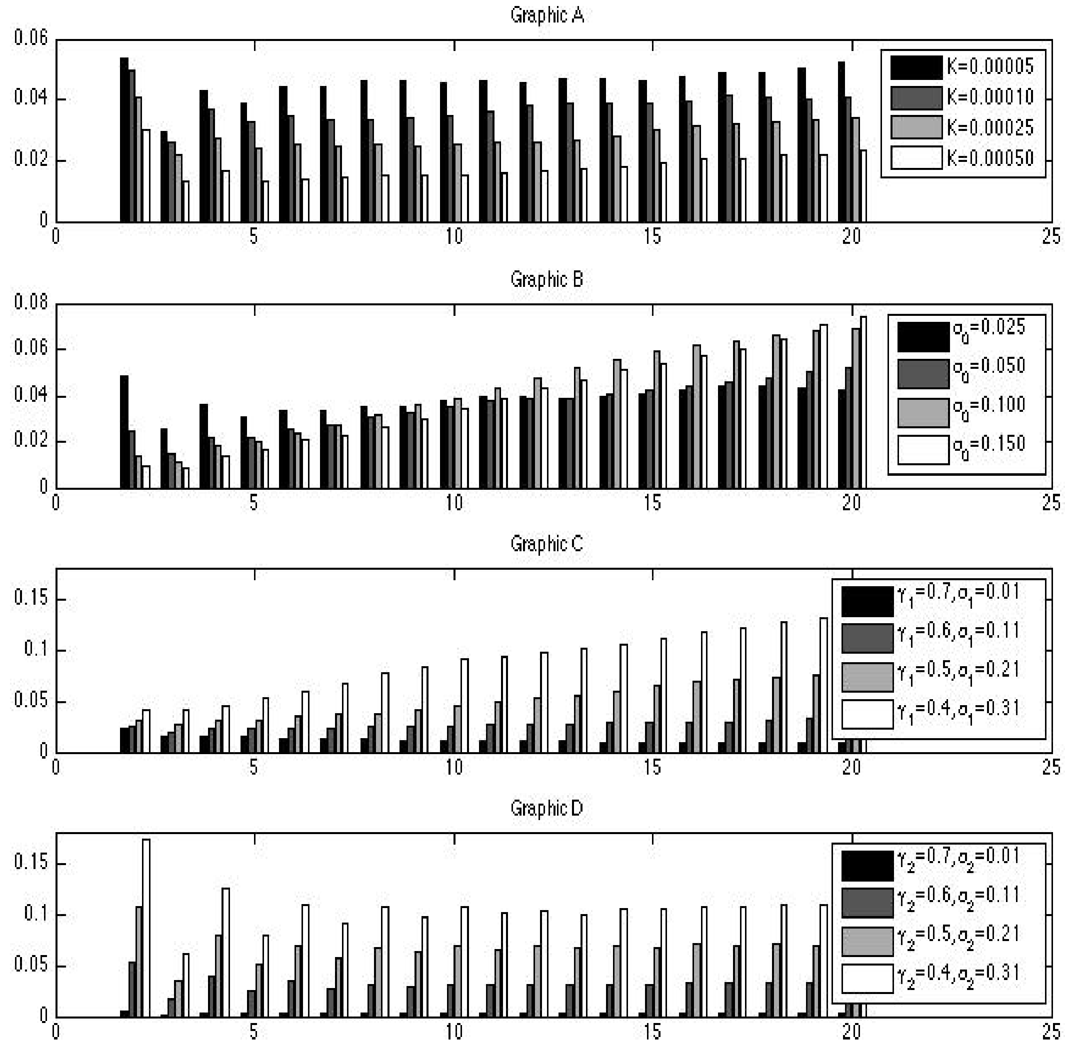

Figure 4.7: Relative error evolution with a GARCH(2,2)

The following points must be highlighted:

1. The relative error decreases from the second to the third period, but then it becomes monotonically increasing with the number of periods.

2. The relative error is inversely proportional to the value of K,γ_1 e γ_2.

3. The relative error is directly proportional to the value of σ_0.

4. There is an oscillatory behavior on the risk measure, that increases when γ_2 decreases.

5. The variation of the parameters γ generates the largest relative errors.

Chapter 5

Case Study

Following the same lines from Chapter 3 and 4, it is fair to question how adequate, in practical terms, Equation (2.3.7) is to be used as a risk measure. Using as standard risk measure the $\mathrm{VaR}_{10\%}$ we know that, by definition, in 10% of all days the portfolio returns will be lower than the value of the risk measure. Rigorously, when considering the terms of Equation (2.3.7) with $t \geq 2$ what we are actually estimating is the mean of such $VaR_{10\%}$, and we can expect some distortion on this 10% level, when comparing the portfolio returns with the risk measure over a longer horizon. To perform the analysis we selected 10 stocks from BM&FBOVESPA. They are:

PETR3, VALE3, EMBR3, ELET3, SUZB5, CSNA3, CRUZ3, ABEV3, BBDC3, BBAS3

Such group of assets was chosen taking into account diversity over the volume of daily transactions and branch of the companies. The historical series analyzed refer to three periods of the Brazilian capital markets:

1. From 02/01/1999 to 12/30/2002: Internet crisis period, in 1999-2000, and 09/11, in 2001.
2. From 01/02/2007 to 12/30/2009: 2008 financial crisis.
3. From 01/02/2013 to 11/28/2014: Recent period.

The models used for the logarithmic returns series were GARCH(1,1) and GARCH(2,2). The standard risk measure is the $\mathrm{VaR}_{10\%}$. The models were calibrated using the previous 1000 trading days, and, according to the framework proposed in Chapter 2, opening prices were used. The following figures show the evolution of the estimated coefficients over time, and the analysis in Chapter 4 can be used as a reference to estimate errors magnitudes. The only exception is EMBR3, that was omitted from such graphics, since its coefficients values have a completely different behavior and scale.The tables in Sections 5.1 and 5.2 show the number of times the risk measure was greater then the actual return of the asset. The graphics in Sections 5.3 and 5.4 also show such frequencies at each subsequent period, and we introduce a third approach in order to compare the efficiency of our proposed method. The red line in each graphic represents the frequency of times the risk measure was above the actual value of the asset return, using as risk measure for all subsequent periods the $\mathrm{VaR}_{10\%}$ of the current period. All historical data were treated according to the methodology proposed by Meucci[7].

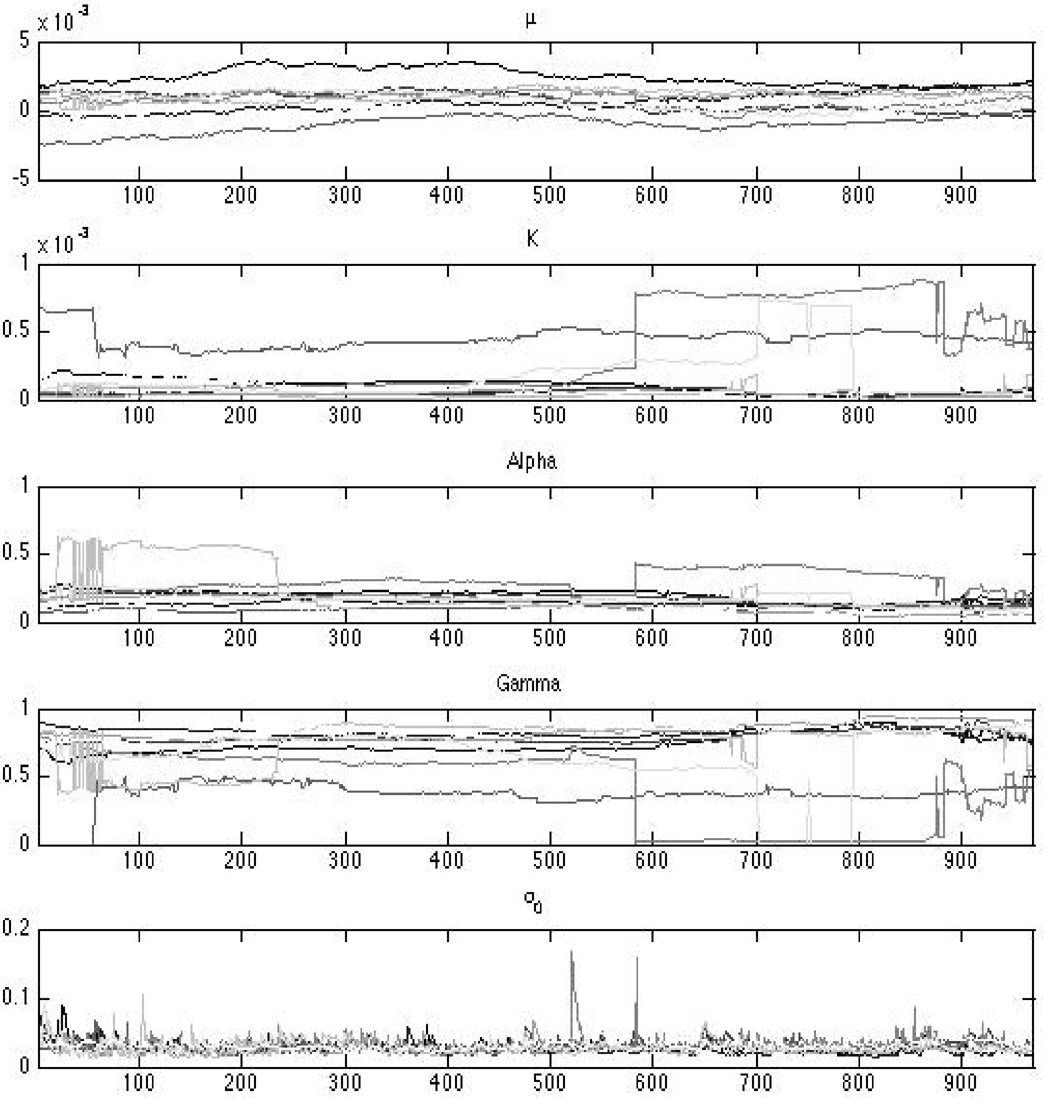

Figure 5.1: Evolution of the GARCH(1,1) parameters during the first period

Notice that the average returns assumed both positive and negative values during this period and volatility remained close to 0.02. Also, coefficients γ and α remained close to 0.8 and 0.2, respectively.

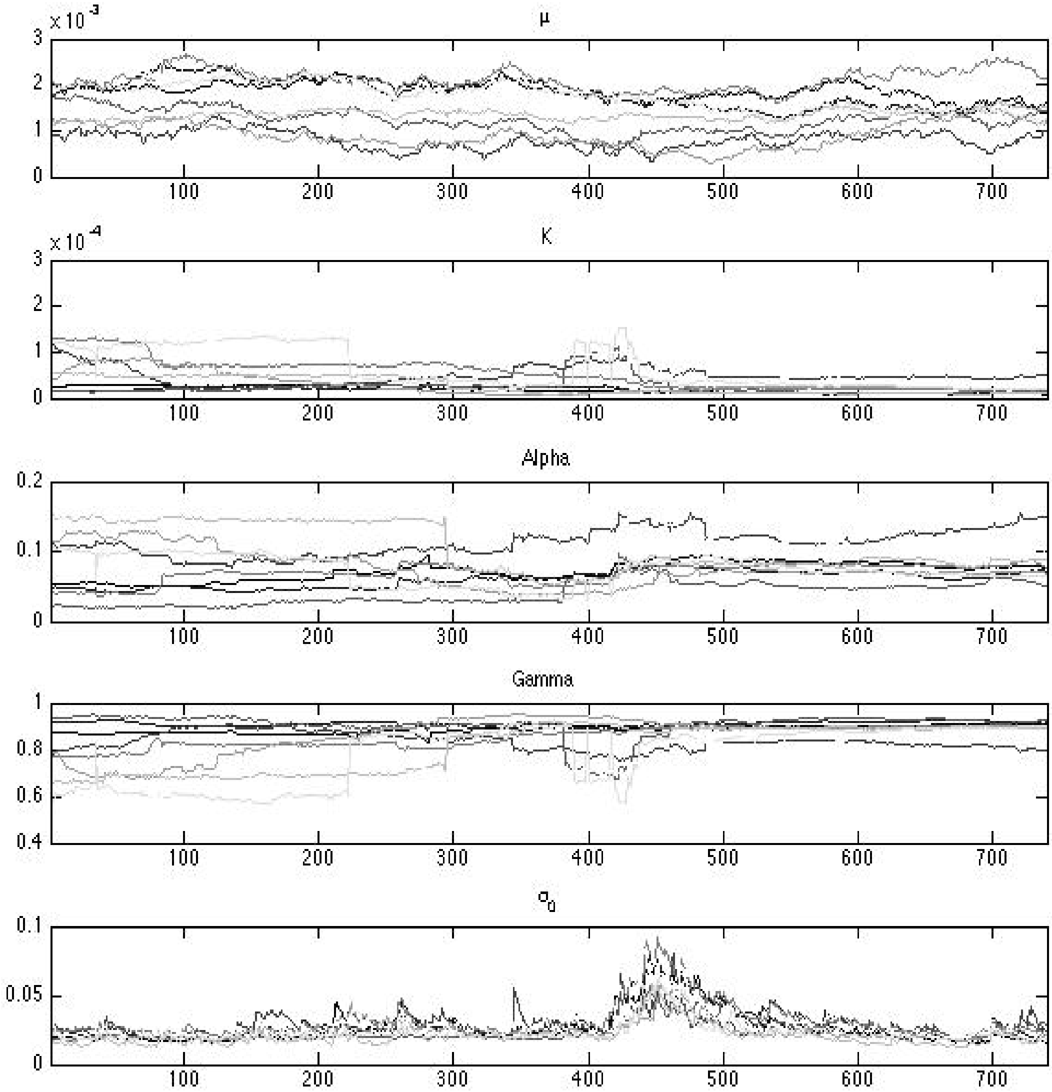

Figure 5.2: Evolution of the GARCH(1,1) parameters during the second period

Notice that the average returns remained positive during this period, although assuming smaller values than the ones reached on the previous period, and volatility remained close to 0.02, reaching its peak during the 2008 year. Also, coefficients γ and α remained close to 0.8 and 0.2, respectively.

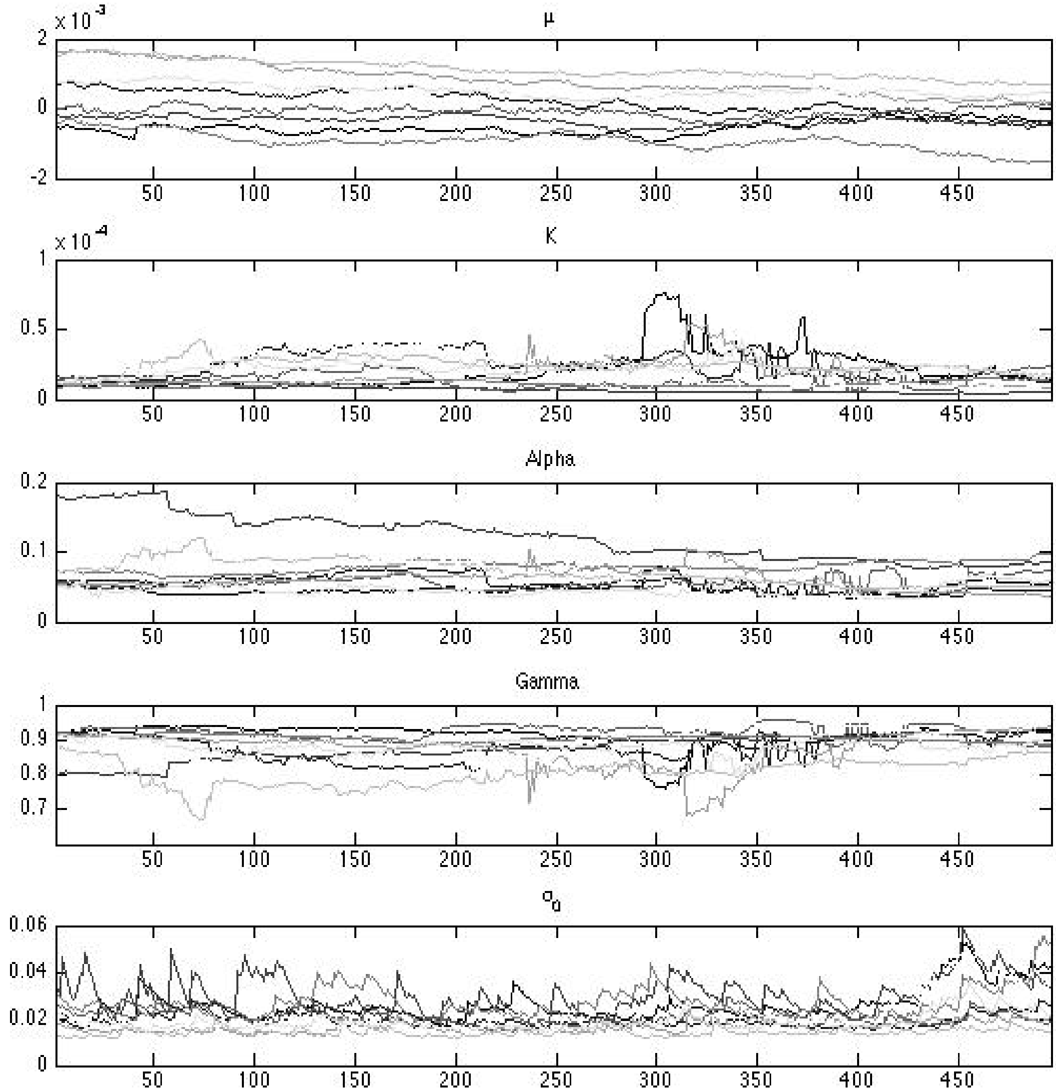

Figure 5.3: Evolution of the GARCH(1,1) parameters during the third period

Notice that the average returns assumed both positive and negative values during this period, with a negative trend over time, and volatility remained close to 0.02, although it presented several peaks during this period. Also, coefficients γ and α remained close to 0.9 and 0.1, respectively.

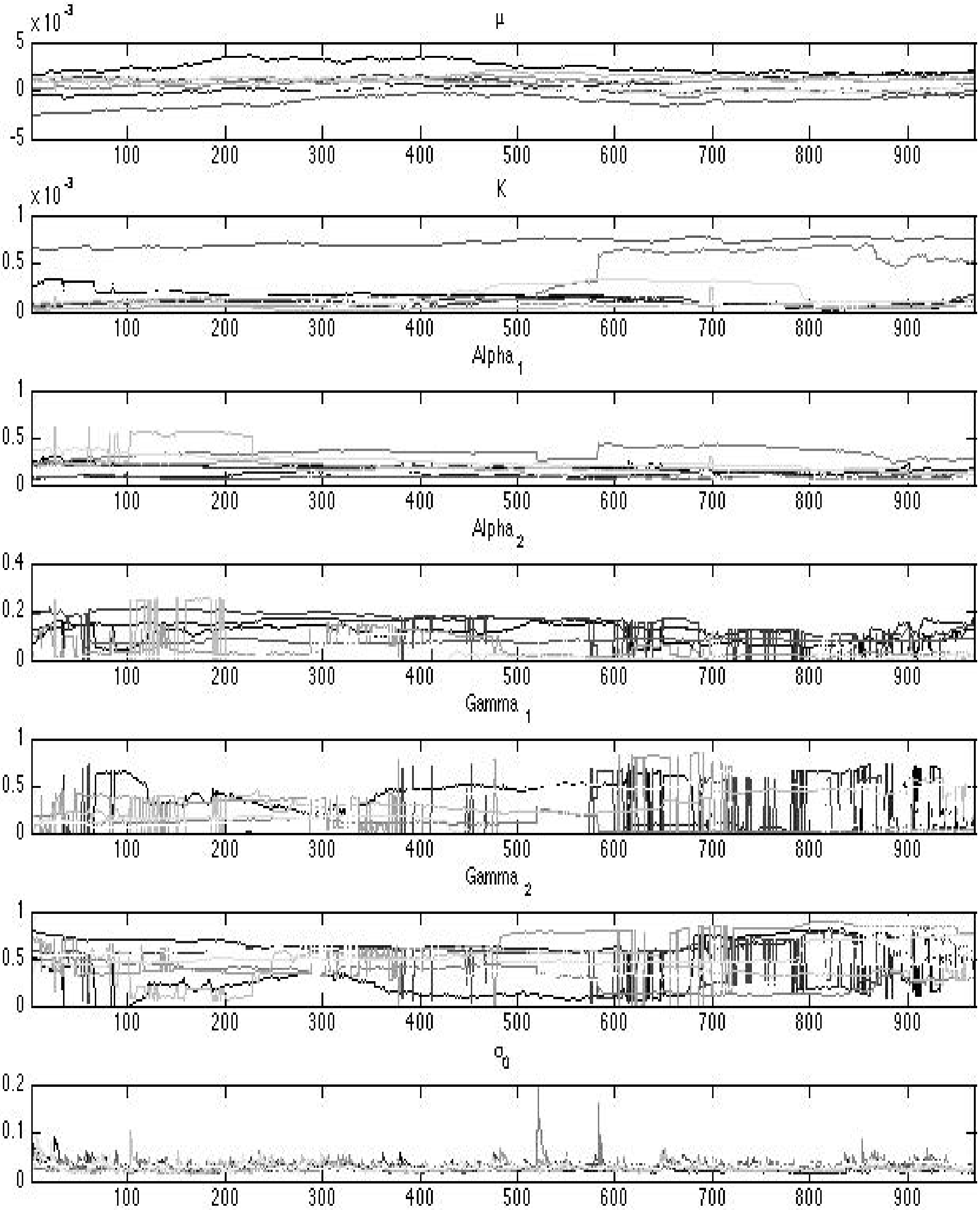

Figure 5.4: Evolution of the GARCH(2,2) parameters during the first period

Notice that the average returns assumed both positive and negative values during this period and volatility remained close to 0.02. Also, coefficients γ and α did not have a clear trend during this period.

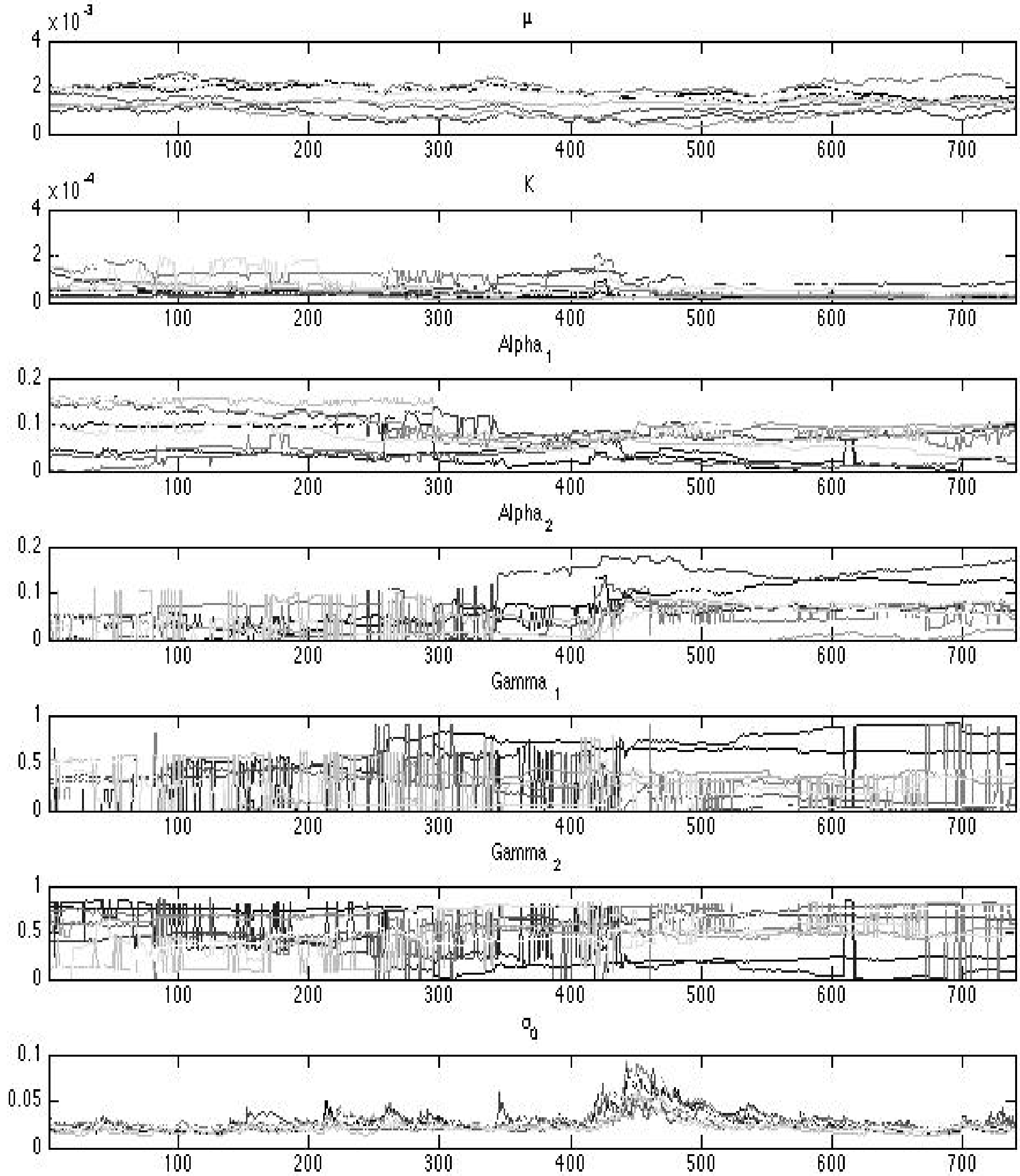

Figure 5.5: Evolution of the GARCH(2,2) parameters during the second period

Notice that the average returns remained positive during this period and volatility remained close to 0.02, reaching its peak during the 2008 year. Also, coefficients α_1 and α_2 remained close to 0.1 and 0.05, respectively, and γ did not have a clear trend during this period.

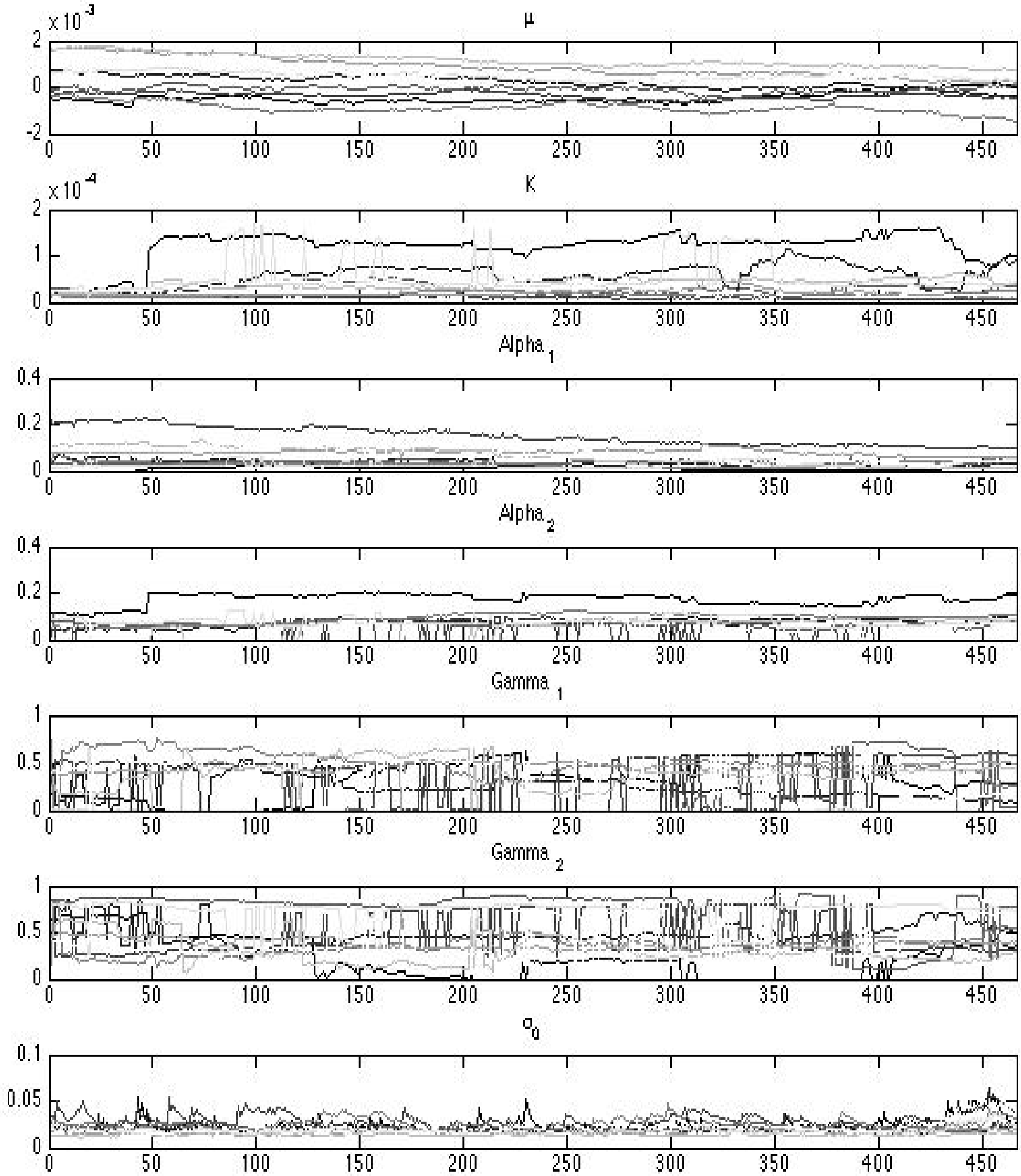

Figure 5.6: Evolution of the GARCH(2,2) parameters during the third period

Notice that the average returns assumed both positive and negative values during this period, with a negative trend over time, and volatility remained close to 0.02, although it presented several peaks during this period. Also, coefficients α_1 and α_2 remained close to 0.1 and 0.1, respectively, and γ did not have a clear trend during this period..

5.1 Results tables for a GARCH(1,1) model

The tables below show the percentage of times the risk metric proposed by Equation (2.3.7) (with a 10%-VaR) remained above the actual return value from the price series.

	t=1	t=2	t=3	t=4	t=5	t=10	t=25	t=50
PETR3	07.32%	07.01%	06.70%	05.98%	05.88%	04.12%	03.20%	02.68%
VALE3	06.08%	05.77%	05.88%	05.46%	05.15%	04.33%	03.61%	02.58%
EMBR3	01.24%	00.82%	00.62%	00.41%	00.41%	00.21%	00.21%	00.21%
ELET3	10.62%	10.21%	09.79%	09.48%	08.87%	08.04%	06.91%	06.19%
SUZB5	06.39%	06.29%	06.08%	06.08%	06.19%	06.08%	06.08%	05.98%
CSNA3	06.80%	06.08%	05.98%	06.91%	06.29%	05.57%	05.26%	05.36%
CRUZ3	07.94%	08.25%	08.04%	07.84%	07.22%	07.11%	05.05%	04.64%
ABEV3	08.14%	07.94%	07.11%	07.01%	06.39%	05.57%	03.61%	03.09%
BBDC3	07.53%	07.42%	06.91%	05.98%	05.67%	05.46%	05.15%	04.64%
BBAS3	07.42%	07.22%	06.91%	06.80%	06.39%	07.22%	06.60%	06.19%

Table 5.1: First period results

	t=1	t=2	t=3	t=4	t=5	t=10	t=25	t=50
PETR3	12.03%	11.62%	11.76%	11.22%	11.62%	11.49%	12.43%	12.57%
VALE3	11.62%	11.76%	11.76%	11.35%	11.35%	12.16%	12.70%	12.97%
EMBR3	09.19%	08.92%	09.05%	09.32%	09.46%	09.86%	08.92%	10.54%
ELET3	07.97%	07.84%	07.97%	07.43%	07.30%	06.62%	06.89%	06.49%
SUZB5	11.89%	11.76%	11.49%	11.22%	11.89%	12.43%	13.11%	13.78%
CSNA3	10.00%	10.14%	10.00%	10.68%	10.54%	11.08%	11.35%	10.95%
CRUZ3	09.59%	09.73%	09.86%	10.14%	10.27%	10.14%	11.08%	11.08%
ABEV3	11.22%	11.08%	11.49%	11.22%	11.62%	12.03%	12.97%	13.65%
BBDC3	11.08%	10.95%	11.22%	11.08%	10.95%	11.08%	11.49%	11.35%
BBAS3	10.27%	10.14%	09.86%	09.73%	09.73%	10.41%	11.49%	11.76%

Table 5.2: Second period results

	t=1	t=2	t=3	t=4	t=5	t=10	t=25	t=50
PETR3	10.30%	10.52%	10.52%	10.09%	10.09%	10.73%	11.16%	13.30%
VALE3	13.09%	13.09%	13.09%	12.45%	12.02%	12.66%	13.09%	13.95%
EMBR3	06.44%	06.44%	07.30%	07.08%	06.44%	06.44%	06.65%	07.08%
ELET3	08.37%	09.01%	08.58%	08.58%	08.80%	09.44%	08.37%	07.51%
SUZB5	07.94%	07.73%	08.15%	08.15%	08.15%	08.15%	06.87%	06.01%
CSNA3	09.66%	10.30%	10.30%	09.44%	09.44%	10.09%	11.59%	13.73%
CRUZ3	10.94%	11.16%	11.16%	10.73%	10.73%	11.59%	11.16%	11.37%
ABEV3	11.59%	11.59%	12.02%	12.02%	11.73%	12.02%	12.02%	11.16%
BBDC3	10.73%	10.52%	10.30%	10.30%	10.09%	10.73%	12.23%	12.88%
BBAS3	10.73%	10.09%	09.87%	10.73%	10.73%	10.94%	11.59%	13.30%

Table 5.3: Third period results

5.2 Results tables for a GARCH(2,2) model

The tables below show the percentage of times the risk metric proposed by Equation (2.3.7) (with a 10%-VaR) as remained above the actual return value from the price series.

	t=1	t=2	t=3	t=4	t=5	t=10	t=25	t=50
PETR3	7.32%	06.70%	06.08%	05.77%	05.77%	04.12%	03.09%	02.68%
VALE3	06.08%	05.88%	05.88%	05.15%	05.15%	04.12%	03.51%	02.78%
EMBR3	01.13%	01.03%	00.82%	00.62%	00.62%	00.31%	00.21%	00.31%
ELET3	10.62%	09.79%	09.48%	09.18%	08.45%	07.53%	06.60%	05.98%
SUZB5	06.60%	06.08%	06.08%	06.08%	06.08%	06.08%	06.19%	06.08%
CSNA3	07.01%	05.77%	06.19%	06.70%	06.08%	05.77%	05.57%	05.46%
CRUZ3	08.25%	08.04%	08.14%	07.94%	07.63%	07.11%	05.05%	04.95%
ABEV3	07.84%	07.84%	07.11%	07.11%	06.49%	05.98%	04.33%	03.81%
BBDC3	08.04%	07.01%	06.91%	06.19%	05.77%	05.26%	05.05%	04.54%
BBAS3	07.42%	07.11%	07.01%	06.80%	06.39%	07.22%	06.39%	05.88%

Table 5.4: First period results

	t=1	t=2	t=3	t=4	t=5	t=10	t=25	t=50
PETR3	11.62%	11.89%	11.76%	11.62%	11.62%	11.62%	12.70%	12.30%
VALE3	11.62%	11.89%	11.49%	11.49%	11.62%	12.30%	12.84%	13.11%
EMBR3	09.46%	09.05%	09.05%	09.19%	09.46%	09.86%	08.92%	10.68%
ELET3	07.84%	07.84%	07.84%	07.57%	07.57%	06.62%	07.03%	06.62%
SUZB5	12.03%	11.62%	11.35%	11.62%	12.16%	12.70%	13.51%	14.05%
CSNA3	10.27%	10.00%	10.14%	10.54%	10.54%	10.95%	11.35%	10.95%
CRUZ3	9.32%	09.73%	09.46%	10.00%	10.00%	10.14%	11.08%	11.08%
ABEV3	11.08%	11.49%	11.35%	11.49%	11.89%	12.16%	12.97%	13.78%
BBDC3	11.08%	11.22%	11.22%	10.95%	10.68%	11.22%	11.35%	11.22%
BBAS3	10.81%	10.14%	09.86%	10.00%	09.73%	10.95%	11.89%	11.76%

Table 5.5: Second period results

	t=1	t=2	t=3	t=4	t=5	t=10	t=25	t=50
PETR3	11.16%	12.02%	12.02%	11.37%	11.37%	12.23%	12.45%	14.16%
VALE3	13.30%	12.45%	12.45%	12.02%	12.23%	12.88%	13.09%	13.95%
EMBR3	07.08%	06.87%	06.87%	06.44%	06.65%	06.44%	06.65%	07.08%
ELET3	08.37%	09.01%	08.58%	08.80%	09.23%	09.44%	08.58%	07.51%
SUZB5	08.15%	08.37%	08.15%	07.51%	07.73%	07.94%	06.87%	05.58%
CSNA3	09.87%	10.30%	10.30%	09.23%	09.44%	10.94%	11.59%	13.73%
CRUZ3	11.16%	11.16%	11.16%	10.73%	10.73%	11.80%	11.16%	11.37%
ABEV3	11.37%	11.80%	12.02%	12.02%	11.16%	11.80%	12.02%	11.16%
BBDC3	10.52%	10.52%	10.94%	10.94%	10.94%	10.94%	11.80%	12.88%
BBAS3	10.30%	10.30%	10.30%	10.94%	10.73%	10.94%	11.37%	13.30%

Table 5.6: Third period results

5.3 Comparison between the proposed approach and a standard one

In this section we perform the same analysis as the one presented in the previous two sections, using a time window of 50 periods and presenting the values graphically. The graphics show the percentage of times that the risk metric proposed by Equation (2.3.7) remained above the actual return experienced by the portfolio in the subsequent periods. Such approach follows the work done by C. Azevedo[5].

To be able to compare the accuracy of such risk metric we proposed as benchmark the following risk calculation approach. In the current date, always identified by $t = 0$, one calculates a 10%-VaR using the previous 1000 trading days, which is the same information set as the one used for calibrating the GARCH models, through a fitted Gaussian distribution, and in order to forecast the risk of the portfolio in the fifty subsequent periods one simply repeats the present value, considering that such distribution is stationary.

This benchmark is given by the red line presented in the graphics below. One would expect the risk metric proposed by Equation (2.3.7) to remain closer to the 10% value then the benchmark in most of the subsequent periods and assets studied. Since such metric is actually a lower bound for the true one, one would also expect that, if far from the 10% value, it should be smaller than this threshold.

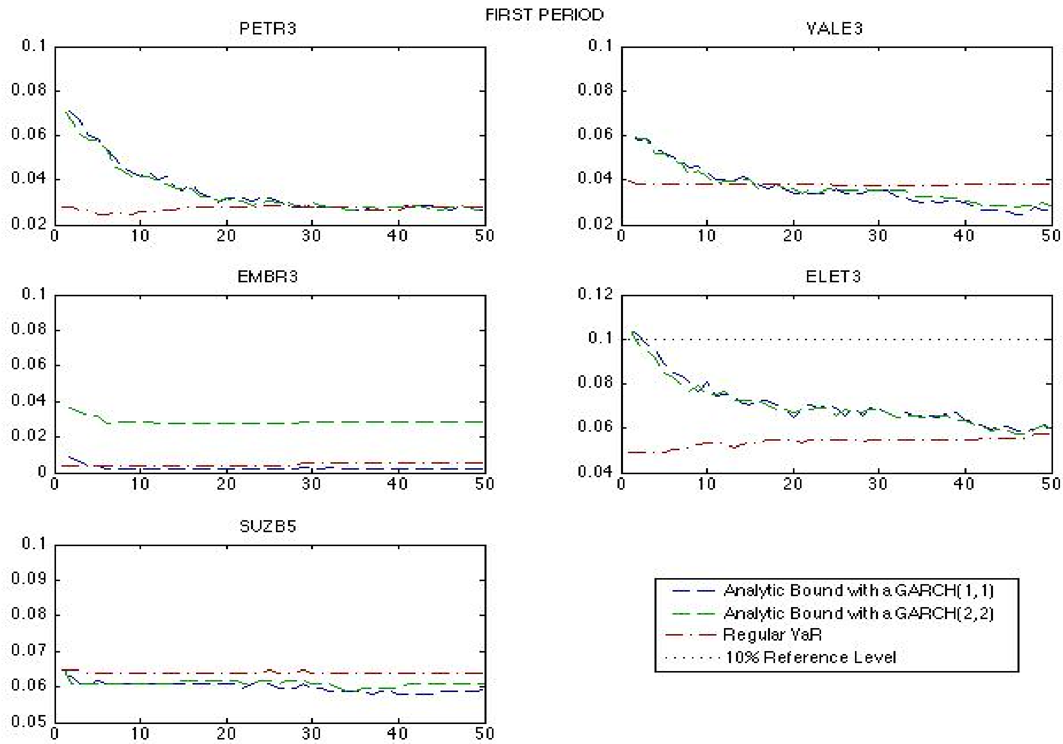

Figure 5.7: Accuracy comparison for the first five assets in the first period

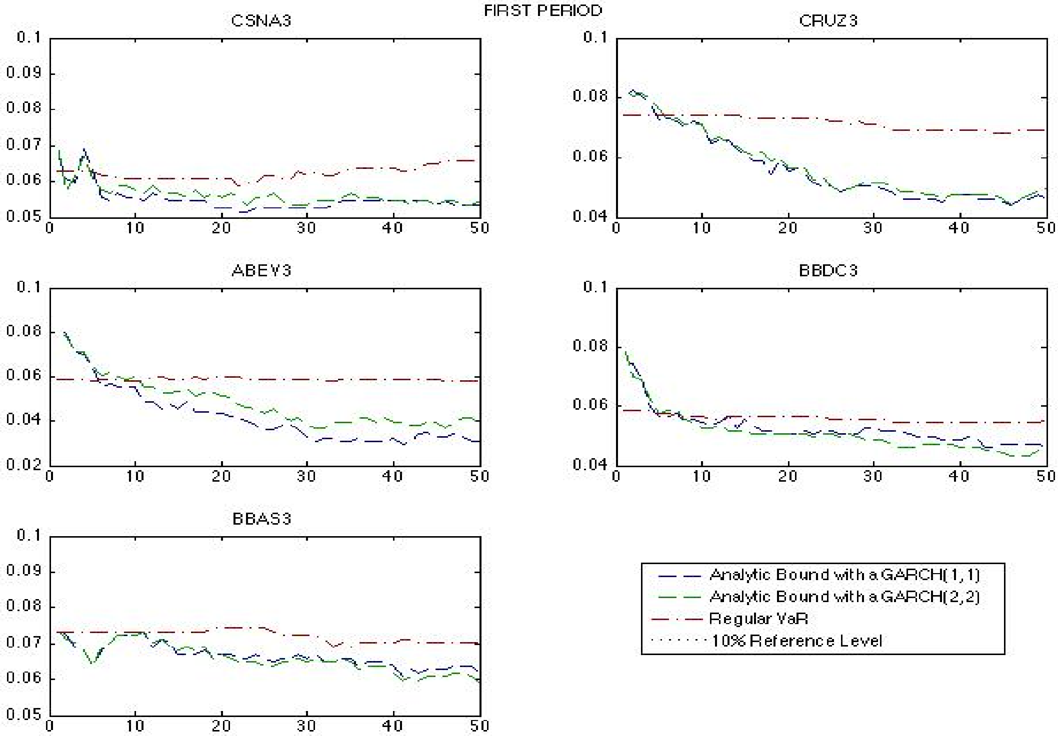

Figure 5.8: Accuracy comparison for the last five assets in the first period

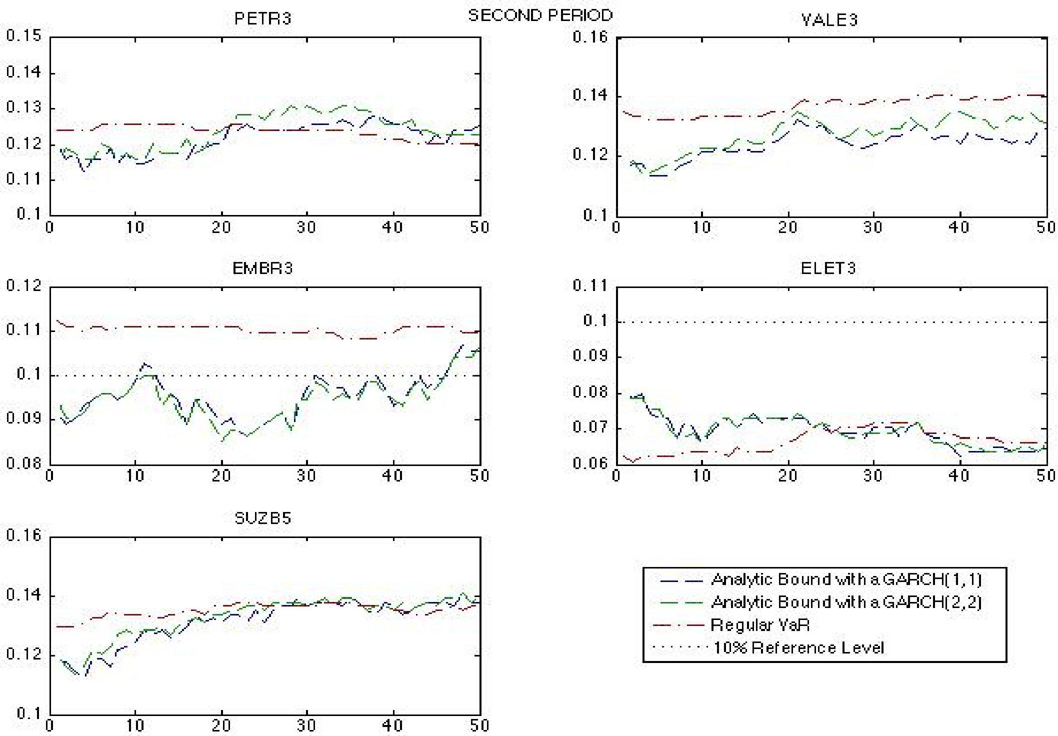

Figure 5.9: Accuracy comparison for the first five assets in the second period

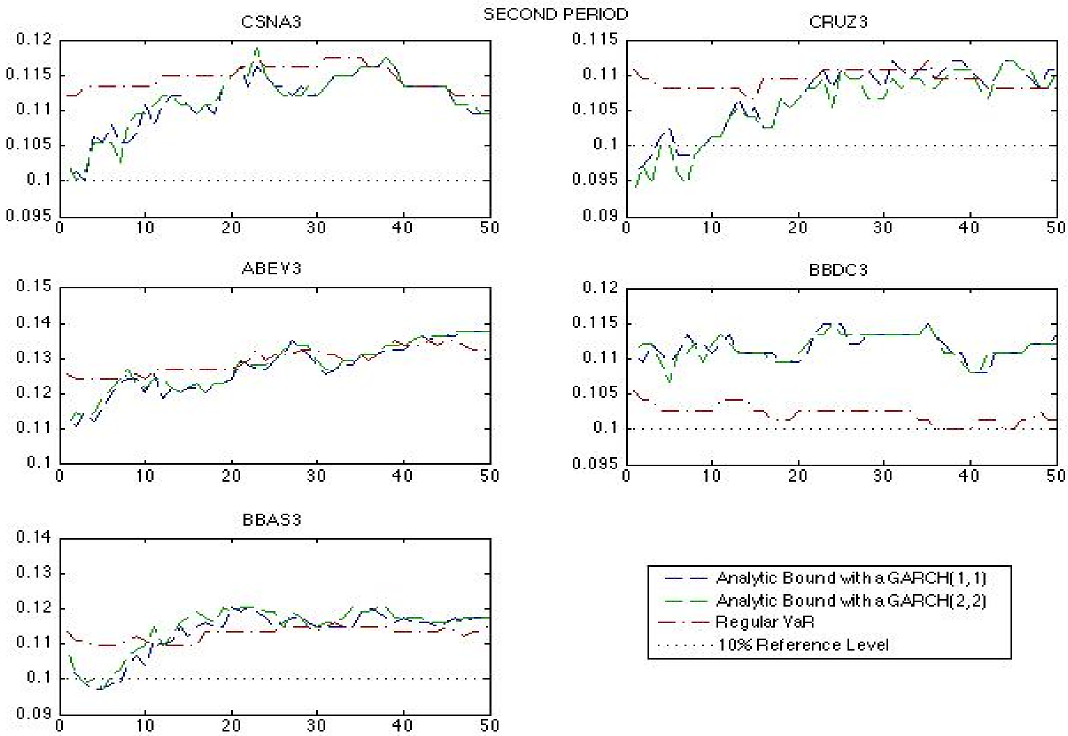

Figure 5.10: Accuracy comparison for the last five assets in the second period

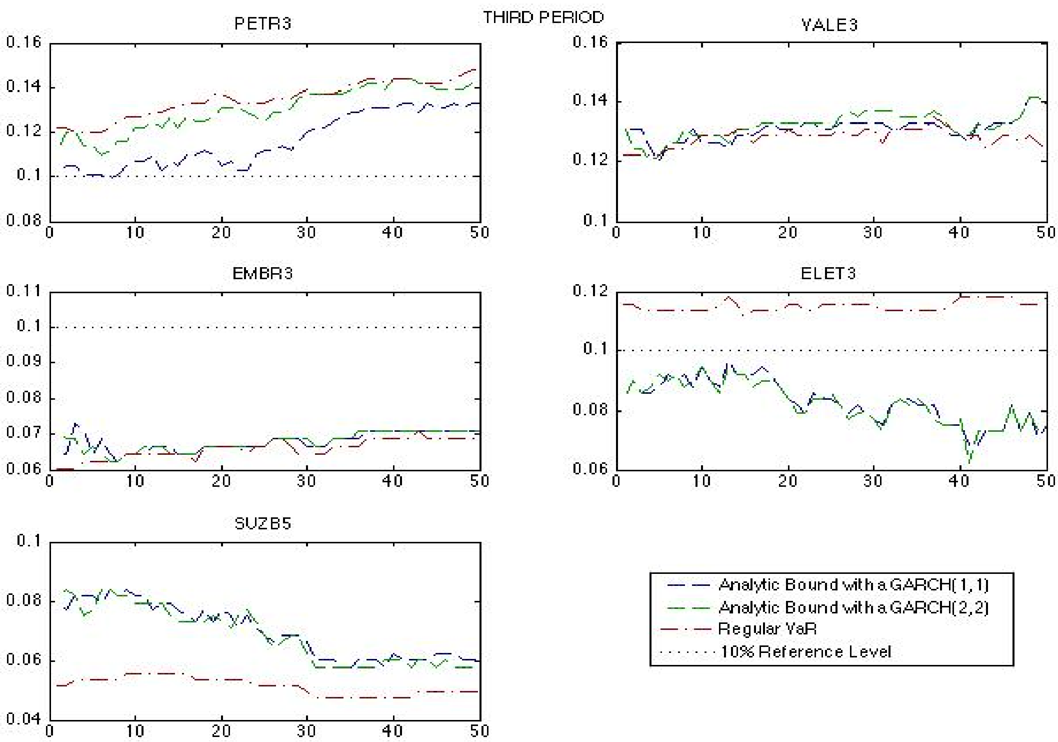

Figure 5.11: Accuracy comparison for the first five assets in the third period

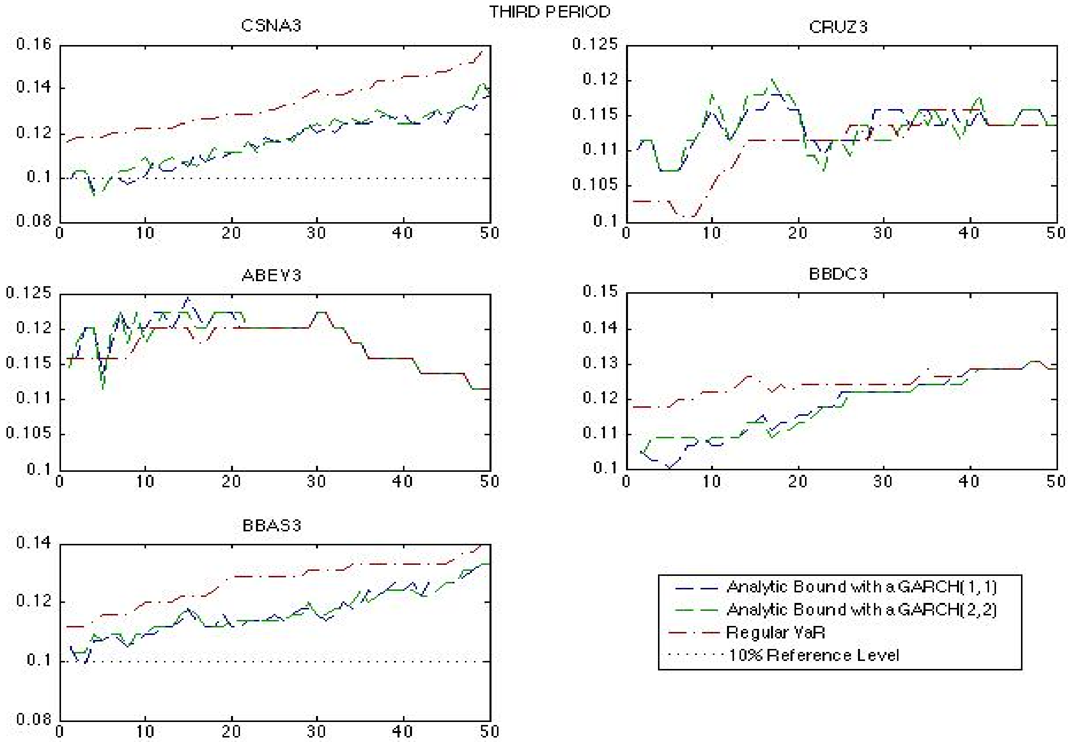

Figure 5.12: Accuracy comparison for the last five assets in the third period

We may state the following conclusions from the tables and graphics presented:

1. The risk metric proposed by Equation (2.3.7) using both GARCH models performs better than the benchmark in small horizons, $t \leq 10$, for the majority of assets and periods studied.

2. For longer horizons, $t > 10$, the first period presented most of the cases unfavorable for the proposed risk metric. Such low accuracy when compared to the benchmark may be due to the fact that the time window used for calibrating the model comprises the Russian crisis, and also to the high volatility exhibited in such period.

3. Still analyzing longer horizons, $t > 10$, the second and third period showed that the proposed risk metric was more accurate than the benchmark for the majority of the assets studied.

4. In the first period risk was recurrently overestimated, as expected. In the second and third periods, risk was recurrently underestimated, although most of the values remained close to the 10% level.

5. The GARCH (2,2) and GARCH(1,1) models lead to a similar result, and in a few cases the GARCH(2,2) furnishes a more accurate risk measure.

Chapter 6

Conclusions

When dealing with multi-period risk one may conclude it is important to first analyze if the application requires the use of absolute or relative wealth variations. The second case furnishes a more suitable measure, although it may still rely on Monte Carlo methods for its calculation depending on the model used for the asset's price series.

The analytic lower bound proposed for the Multi-period Relative Value-at-Risk, which we introduced in order to address specifically relative wealth fluctuations, can itself be used as a risk metric, incurring in a relative error with order of magnitude of 1% for regular values of the parameters of a GARCH model and a time windows of 30 periods.

Regarding the model used for the asset prices, we noticed that higher order GARCH models allow higher degrees of freedom for the risk metric, increasing the complexity of the risk evolution. This is reflected, for example, in the composition of the optimal investment portfolio and in the shape of the efficient frontier curve.

Our case studies showed that when using the lower bound itself as a risk metric one obtain a more accurate risk measure when compared to a classic Value-at-Risk approach. This introduces an alternative to Monte-Carlo simulations which is specially valuable for applications where the processing time is a limiting variable, so that an analytic metric, although still relying on the calibration of a GARCH model, can be used to improve the risk calculation algorithm.

Bibliography

[1] Patrick Billingsley, *Convergence of probability measures*, John Wiley & Sons, 2013.

[2] Tim Bollerslev, *Generalized autoregressive conditional heteroskedasticity*, J. Econometrics **31** (1986), no. 3, 307–327. MR 853051 (87j:62169)

[3] Sérgio Alvares R. de S. Maffra, *Risco e seleção de portfólios com a medida cvar e o modelo go-garch*, Master's thesis, Instituto de Matemática Pura e Aplicada, 2013.

[4] Robert F. Engle, *Arch/garch models in applied financial econometrics*, Journal (2000).

[5] Cristiane A. Ferreira, *Avaliação de modelos de risco através de backtesting*, Master's thesis, Instituto de Matemática Pura e Aplicada, 2013.

[6] Ralf Korn and Elke Korn, *Option pricing and portfolio optimization*, Graduate Studies in Mathematics, vol. 31, American Mathematical Society, Providence, RI, 2001, Modern methods of financial mathematics, Translated from the 1999 German original by the authors. MR 1802499 (2001h:91003)

[7] Attilio Meucci, *Risk and asset allocation*, Springer Finance, Springer-Verlag, Berlin, 2005. MR 2155219 (2006b:91003)

[8] Georg Ch. Pflug, *Some remarks on the value-at-risk and the conditional value-at-risk*, Probabilistic constrained optimization, Nonconvex Optim. Appl., vol. 49, Kluwer Acad. Publ., Dordrecht, 2000, pp. 272–281. MR 1819417

[9] Georg Ch. Pflug and Werner Römisch, *Modeling, measuring and managing risk*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. MR 2424523 (2009g:91008)

[10] Steven E. Shreve, *Stochastic calculus for finance. I*, Springer Finance, Springer-Verlag, New York, 2004, The binomial asset pricing model. MR 2049045 (2004m:91003)

[11] ______, *Stochastic calculus for finance. II*, Springer Finance, Springer-Verlag, New York, 2004, Continuous-time models. MR 2057928 (2005c:91001)

[12] Stanislav Uryasev and R. Tyrrell Rockafellar, *Conditional value-at-risk: optimization approach*, Stochastic optimization: algorithms and applications (Gainesville, FL, 2000), Appl. Optim., vol. 54, Kluwer Acad. Publ., Dordrecht, 2001, pp. 411–435. MR 1835507 (2002f:90073)