TRANSVERSELY PROJECTIVE FOLIATIONS ON SURFACES: EXISTENCE OF MINIMAL FORM AND PRESCRIPTION OF MONODROMY

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ABSTRACT. We introduce a notion of minimal form for transversely projective structures of singular foliations on complex manifolds. Our first main result says that this minimal form exists and is unique when ambient space is two-dimensional. From this result one obtains a natural way to produce invariants for transversely projective foliations on surfaces. Our second main result says that on projective surfaces one can construct singular transversely projective foliations with prescribed monodromy.

1. Introduction and Statement of Results

1.1. **Singular Transversely Projective Foliations.** Classically a smooth holomorphic transversely projective foliation on a complex manifold M is a codimension one smooth holomorphic foliation locally induced by holomorphic submersions on $\mathbb{P}^1_{\mathbb{C}}$ and with transitions functions in $\mathrm{PSL}(2,\mathbb{C})$. Among a number of equivalent definitions that can be found in the literature, we are particularly fund of the following one: $\mathcal{F}$ is a **transversely projective foliation** on a complex manifold M if there exists

1. $\pi : P \to M$ a $\mathbb{P}^1$-bundle over M;
2. $\mathcal{H}$ a codimension one foliation of P transversal to the fibration π;
3. $\sigma : M \to P$ a holomorphic section transverse to $\mathcal{H}$;

such that $\mathcal{F} = \sigma^*\mathcal{H}$. The datum $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \to P)$ is the **transversely projective structure** of $\mathcal{F}$. A nice property of this definition is that the isomorphism class of the $\mathbb{P}^1$-bundle P is an invariant canonically attached to the foliation $\mathcal{F}$, whenever $\mathcal{F}$ has a leaf with non-trivial holonomy, cf. [6, page 177, Ex. 3.24.i].

In the holomorphic category the existence of smooth holomorphic foliations imposes strong restrictions on the complex manifold. For instance there exists a complete classification of smooth holomorphic foliation on compact complex surfaces, cf. [2] and references there within. An interesting corollary of this classification is that a rational surface carries a holomorphic foliation if, and only if, it is a Hirzebruch surface and the foliation is a rational fibration.

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On the other hand the so called Riccati foliations on compact complex surfaces S , i.e., the foliations which are transversal to a generic fiber of a rational fibration, are examples of foliations which are transversely projective when restricted to the open set of S where the transversality of $\mathcal{F}$ with the rational fibration holds.

The problem of defining a *good* notion of singular transversely projective foliation on compact complex manifolds naturally emerges. A first idea would be to consider singular holomorphic foliations which are transversely projective on Zariski open subsets. Albeit natural, the experience shows that such concept is not very manageable: it is too permissive. With an eye on applications one is led to impose some kind of regularity at infinity. A natural regularity condition was proposed by Scárdua in [16]. Loosely speaking, it is imposed that the *transversely projective structure* is induced by a global meromorphic triple of 1-forms. The naturality of such definition has been confirmed by the recent works of Malgrange and Casale on Non-Linear Differential Galois Theory, cf. [11, 3] and references therein.

In this work we will adopt a variant of Scárdua's definition which maintains the geometric flavor of the definition of a smooth transversely projective foliation given at the beginning of the introduction. For us, $\mathcal{F}$ is a **singular transversely projective foliation** if there exists

(1) $\pi : P \to M$ a $\mathbb{P}^1$-bundle over M;
(2) $\mathcal{H}$ a codimension one singular holomorphic foliation of P transverse to the generic fiber of π;
(3) $\sigma : M \dashrightarrow P$ a meromorphic section generically transverse to $\mathcal{H}$;

such that $\mathcal{F} = \sigma^*\mathcal{H}$. Like in the regular case we will call the datum $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \dashrightarrow P)$ a **singular projective transverse structure** of $\mathcal{F}$. Any two such triples $\mathcal{P} = (P, \mathcal{H}, \sigma)$ and $\mathcal{P}' = (P', \mathcal{H}', \sigma')$ are said **bimeromorphically equivalent** whenever they are conjugate by a bimeromorphic bundle transformation $\phi : P \dashrightarrow P'$: we have $\phi^*\mathcal{H}' = \mathcal{H}$ and the diagram

$$\begin{tikzcd}
P \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"] & & P' \arrow[dl, "\pi'"'] \\
& M \arrow[ul, dashed, bend left, "\sigma"] \arrow[ur, dashed, bend right, "\sigma'"'] &
\end{tikzcd}$$

commutes. Actually, this is equivalent to say that $\mathcal{P}$ and $\mathcal{P}'$ induce the same (smooth) transversely projective foliation on a Zariski open subset of M (see Proposition 2.4). Unlike in the regular case, the isomorphism class of P is not determined by the bimeromorphic equivalence class of the transverse structure.

The definition of transverse projective structure used in [16] is actually equivalent to the existence of a triple $\mathcal{P}$ like above (up to bimeromorphic bundle transformation) with the restriction that the section σ is assumed to be holomorphic in [16]. In the case M is projective, both definitions coincide: one can always assume, after a birational bundle transformation, that P is the trivial bundle and $\sigma : M \dashrightarrow P$ is a constant section (see Remark 2.1). The advantage of allowing indeterminacy points for σ in our definition is the possibility to provide a canonical representative $\mathcal{P}$ for the transverse structure unique up to regular bundle isomorphisms.

1.2. **Minimal form for a singular transversely projective structure.** To a singular transversely projective structure $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \dashrightarrow P)$ we associate

- **the polar divisor**, denoted by $(\mathcal{P})_\infty$, is the divisor on M defined by the direct image under π of the tangency divisor between $\mathcal{H}$ and the one-dimensional foliation induced by the fibers of π. Notice that this latter divisor is vertical, i.e., saturated by fibers of π; we will sometimes denote it by $(\mathcal{P})_\infty$ as well, by abuse of notation.
- **the branching divisor**, denoted by $\mathrm{Branch}(\mathcal{P})$, is the divisor on M defined by the direct image under π of the tangency divisor between $\mathcal{H}$ and $\sigma(M)$ (more precisely, the closure of $\sigma(M - \mathrm{indeterminacy}(\sigma))$ in P). This latter divisor is locally defined on $\sigma(M)$ as the divisor of the restriction of any holomorphic 1-form with codimension ≥ 2 zero set defining $\mathcal{H}$.

We will say that a singular transversely projective structure $\mathcal{P}$ is in **minimal form** when the divisor $(\mathcal{P}')_\infty - (\mathcal{P})_\infty$ is effective, i.e. $(\mathcal{P}')_\infty - (\mathcal{P})_\infty \geq 0$, for every projective structure $\mathcal{P}'$ bimeromorphic to $\mathcal{P}$ and $\mathrm{cod}(\mathrm{Branch}(\mathcal{P}) \cap (\mathcal{P})_\infty) \geq 2$.

Theorem 1. *Let $\mathcal{F}$ be a singular transversely projective foliation on a complex surface S. Every transversely projective structure $\mathcal{P}$ of $\mathcal{F}$ is bimeromorphically equivalent to a transversely projective structure in minimal form. Moreover this minimal form is unique up to $\mathbb{P}^1$-bundle isomorphisms.*

From the uniqueness of the minimal form, one can systematically produce invariants for singular transversely projective foliations on complex surfaces. Perhaps the simplest example is the isomorphism class of the bundle P.

We point out that Theorem 1 does not holds on higher dimensional complex manifolds, cf. Example 4.10.

1.3. **The Monodromy Representation.** An important invariant of a projective structure $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \dashrightarrow P)$, is the **monodromy representation**. It is the representation of $\pi_1(M \setminus |(\mathcal{P})_\infty|)$ into $\mathrm{PSL}(2, \mathbb{C})$ obtained by lifting paths on $M \setminus |(\mathcal{P})_\infty|$ to the leaves of $\mathcal{H}$. Given a hypersurface $H \subset M$ and a representation

$$\rho : \pi_1(M \setminus H) \to \mathrm{PSL}(2, \mathbb{C}),$$

one might ask if there exists a foliation $\mathcal{F}$ of M with transversely projective structure $\mathcal{P}$ whose monodromy is ρ. We will show in §5.1 that the answer is in general no: there are local obstructions to solve the *realization problem*. On the other hand if the ambient is two-dimensional and the representation ρ lifts to a representation $\tilde{\rho} : \pi_1(M \setminus H) \to \mathrm{SL}(2, \mathbb{C})$ then we have the

Theorem 2. *Let S be a projective surface and H be a reduced hypersurface on S. If*

$$\rho : \pi_1(S \setminus H) \to \mathrm{PSL}(2, \mathbb{C})$$

is a homomorphism which lifts to a homomorphism $\tilde{\rho} : \pi_1(S \setminus H) \to \mathrm{SL}(2, \mathbb{C})$ then there exists a singular transversely projective foliation $\mathcal{F}$ with a singular transversely projective structure $\mathcal{P}$ (in minimal form) such that $H - (\mathcal{P})_\infty \geq 0$ and ρ is the monodromy representation of $\mathcal{P}$. Moreover, the projective structure $\mathcal{P}$ is the unique one that $\mathcal{F}$ admits (up to birational equivalence) provided that the image of ρ is not virtually abelian.

We point out that the result (and the proof presented below) holds for higher dimensional projective manifolds if one supposes that H is a normal crossing divisor, cf. §5.2 for details. We point out that the $\mathbb{P}^1$-bundle P (and the Riccati foliation $\mathcal{H}$) constructed in Theorem 2 only depend on the choice of a logarithm for the local monodromy around each irreducible component of H; we have a huge degree of freedom for the section σ, and thus for $\mathcal{F}$.

2. Generalities

2.1. **A local description of $\mathcal{H}$.** Let $\Delta^n \subset \mathbb{C}^n$ be a polydisc and $\pi : P \to \Delta^n$ be a $\mathbb{P}^1$-bundle. Since the polydisc is a Stein contractible space we can suppose that P is the projectivization of the trivial rank 2 vector bundle over Δ^n and write $\pi(x, [z_1 : z_2]) = x$. If $\mathcal{H}$ is a codimension one foliation of P generically transversal to the fibers of π then $proj^* \mathcal{H}$ is induced by a 1-form Ω that can be written as

$$\Omega = z_1 dz_2 - z_2 dz_1 + \alpha z_1^2 + \beta z_1 \cdot z_2 + \gamma z_2^2 \,, \tag{1}$$

where α, β, γ are meromorphic 1-forms on Δ^n and $proj : \Delta^n \times \mathbb{C}^2 \dashrightarrow \Delta^n \times \mathbb{P}^1$ is the natural projection. The integrability condition $\Omega \wedge d\Omega = 0$ translates into the relations

$$\begin{cases} d\alpha &= \alpha \wedge \beta \\ d\beta &= 2\alpha \wedge \gamma \\ d\gamma &= \beta \wedge \gamma \end{cases} \tag{2}$$

The divisor of poles of Ω corresponds to the fibers of π that are tangent to $\mathcal{H}$, i.e., if $\mathcal{C}$ denotes the 1-dimension foliation induced by the fibration π then

$$(\Omega)_\infty = \mathrm{tang}(\mathcal{H}, \mathcal{C}) \,.$$

Since $(\Omega)_\infty$ is saturated by fibers of π we will refer to it sometimes as a divisor on P sometimes as a divisor on Δ^n; in fact, $(\Omega)_\infty$ is the restriction of $(\mathcal{P})_\infty$ to Δ^n.

Associated to Ω we have an integrable differential $\mathfrak{sl}(2,\mathbb{C})$-system on the trivial rank 2 vector bundle over Δ^n defined by

$$dZ = A \cdot Z \quad \text{where} \quad A = \begin{pmatrix} -\frac{\beta}{2} & -\gamma \\ \alpha & \frac{\beta}{2} \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{3}$$

The matrix A can be thought as a meromorphic differential 1-form on Δ^n taking values in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and satisfying the integrability condition $dA + A \wedge A = 0$. Darboux's Theorem (see [6], III, 2.8, iv, p.230) asserts that on any simply connected open subset $U \subset \Delta^n \setminus (\Omega)_\infty$ there exists a holomorphic map

$$\Phi : U \to \mathrm{SL}(2,\mathbb{C}) \quad \text{such that} \quad A = \Phi^* M$$

where M is the Maurer-Cartan 1-form on $\mathrm{SL}(2,\mathbb{C})$. Moreover, the map Φ is unique up to a left composition with an element in $\mathrm{SL}(2,\mathbb{C})$. For every $v \in \mathbb{C}^2$ the sections

$$\begin{aligned} \varphi_v : U &\to U \times \mathbb{C}^2 \\ x &\mapsto (x, \Phi(x) \cdot v) \end{aligned}$$

are solutions of the differential system above. It follows that the application

$$\begin{aligned} \phi : U \times \mathbb{P}^1 &\to U \times \mathbb{P}^1 \\ (x, [z_1, z_2]) &\mapsto (x, [\Phi(x)(z_1, z_2)]) \,, \end{aligned}$$

conjugates the foliation $\mathcal{H}|_U$ with the one induced by the submersion $U \times \mathbb{P}^1 \to \mathbb{P}^1$.

Remark 2.1. When M is projective, a singular transversely projective foliation can always be defined by a global triple of meromorphic 1-forms (α, β, γ) satisfying (2). Indeed, a $\mathbb{P}^1$-bundle with a rational section σ is always the projectivization $\mathbb{P}(E)$ of a rank 2 vector bundle (see [7]) and one can moreover assume, up to birational bundle transformation, that P is the trivial bundle and that $\sigma : M \dashrightarrow P$ is a constant section.

Remark 2.2. When the section σ is holomorphic, we can assume, up to regular bundle isomorphism, that it is defined by $z_2 = 0$ over Δ^n and the foliation $\mathcal{F}$ is defined by $\alpha = 0$. After covering M by an atlas of such trivialization charts, the transition between any two local triple (α, β, γ) is given by a regular bundle isomorphism fixing $z_2 = 0$, i.e. having the form

$$\Phi(x, [z_1, z_2]) = (x, [z_1 + g \cdot z_2, f \cdot z_2]) \quad \text{with} \quad f \in \mathcal{O}^* \text{ and } g \in \mathcal{O}. \tag{4}$$

The coefficients of the 1-form $\tilde{\Omega}$ defining $\Phi^*\mathcal{H}_j$ are therefore given by

$$\begin{cases} \tilde{\alpha} & = & \frac{\alpha}{f} \\ \tilde{\beta} & = & \beta + \frac{df}{f} + 2g\alpha \\ \tilde{\gamma} & = & f(\gamma + g\beta + g^2\alpha - dg) \end{cases} \tag{5}$$

We are back to the definition of transversely projective foliations presented in [16].

Remark 2.3. An immediate consequence of Theorem 1 is that a singular transversely projective foliation on a surface S can be defined by a covering equipped with meromorphic 1-forms (α, β, γ) satisfying (2) and compatibility conditions (5) with $f \in \mathcal{M}^*$ and $g \in \mathcal{M}$ meromorphic. Indeed, after convenient bimeromorphic bundle transformations, one can assume the projective triple in minimal form over each chart so that compatibility conditions become biregular.

Before turning our attention to examples of transversely projective foliations we will present some propositions that testify the naturalness of our definition.

Proposition 2.4. *Let $\mathcal{F}$ be a foliation on a complex manifold M. If two transversely projective structures $\mathcal{P}$ and $\mathcal{P}'$ for $\mathcal{F}$ are bimeromorphically equivalent outside a codimension 1 subset $Z \subset M$, then they are bimeromorphically equivalent on the whole of M.*

In particular, if a singular foliation $\mathcal{F}$ is transversely projective in the classical sense in restriction to a Zariski open subset of M, then the meromorphic extension of the projective structure on the whole of M is unique whenever it does exist.

Proof. At the neighborhood U of a point $p \in Z$, one can choose meromorphic trivializations of respective bundles like in Remark 2.2 so that projective structures are defined by respective triples (α, β, γ) and $(\alpha', \beta', \gamma')$ (mind that initially, the sections σ and σ' can be meromorphic). Since α and α' define the same foliation, we have $\alpha = f\alpha'$ for some meromorphic function $f \in \mathcal{M}^*$; by an additional bundle transformation of the form $[z_1, z_2] \mapsto [z_1, f \cdot z_2]$, one can assume $\alpha' = \alpha$. Comparing first line of condition (2) for respective triples now shows that $\beta' = \beta + 2g\alpha$ for some meromorphic function $g \in \mathcal{M}$; after a last transformation of the form $[z_1, z_2] \mapsto [z_1 + g \cdot z_2, z_2]$, one can finally assume $\beta' = \beta$. Formula (5) finally shows that the bundle transformation conjugating the respective structures over $U - Z$ is now the identity and extends on the whole of U. $\square$

Following Levi's Extension Theorem, any singular foliation $\mathcal{F}$ defined outside a codimension 2 subset $Z \subset M$ extends on M. A corollary of Theorem 1 is that the same holds for singular transversely projective structures on surfaces

Proposition 2.5. *Let $\mathcal{F}$ be a foliation on a complex surface S and let $\mathcal{P}$ be a transversely projective structure for $\mathcal{F}|_{M-Z}$ outside a codimension 2 subset $Z \subset M$. Then, there is a (unique) transversely projective structure $\mathcal{P}'$ for $\mathcal{F}$ on M that is bimeromorphically equivalent to $\mathcal{P}$ over $M - Z$.*

Proof. One can assume without loss of generality that Z contains all indeterminacy points of the section σ given by $\mathcal{P}$. At the neighborhood U of a point $p \in Z$, let α be a 1-form defining $\mathcal{F}$ and β a meromorphic 1-form satisfying $d\alpha = \alpha \wedge \beta$. One can cover $U - Z$ by charts U_i on which the projective structure is defined by charts $(\alpha_i, \beta_i, \gamma_i)$ with holomorphic transition condition (5). Proceeding like in the proof of Proposition 2.4, one can assume $\alpha_i = \alpha$ and $\beta_i = \beta$ for all i so that all γ_i coincide to define a meromorphic 1-form γ that extends on the whole of U. On $U - Z$, the projective structure defined by the triple (α, β, γ) is bimeromorphically equivalent to $\mathcal{P}$ by definition. After applying this for an open cover of Z, one obtains a transversely projective structure $\mathcal{P}'$ in the sense of Remark 2.3 and one can conclude with Theorem 1. $\square$

In the previous proof, when M is projective, whatever the dimension is, α and β can be chosen rational (global) and we do not need Theorem 1 to conclude.

Proposition 2.6. *Let $\mathcal{F}$ be a foliation on a complex manifold M and let $\mathcal{P}$ be a transversely projective structure for $\mathcal{F}|_{M-Z}$ outside a codimension 2 subset $Z \subset M$. If the bundle P defining $\mathcal{P}$ extends on the whole of M, then there is a (unique) transversely projective structure $\mathcal{P}'$ for $\mathcal{F}$ on M that is biholomorphically equivalent to $\mathcal{P}$ over $M - Z$.*

Proof. This is a direct application of Levi's Extension Theorem. At the neighborhood of U of a point $p \in Z$, one can trivialize the extension P' of the bundle P. Then $\mathcal{H}$ and σ defined on $U - Z$ by $\mathcal{P}$ extend meromorphically on U. $\square$

3. Examples

3.1. **Riccati foliations.** Recall that a Riccati foliation $\mathcal{F}$ on a $\mathbb{P}^1$-bundle $\Pi : M \to N$ over a complex manifold N is, by definition, a foliation that is transversal to the generic fiber of Π. As we will see $\mathcal{F}$ turns out to be a singular transversely projective foliation.

Consider the fiber product

$$\begin{tikzcd} P \arrow[r, "\pi'"] \arrow[d, "\pi"'] & M' \arrow[d, "\Pi'"] \\ M \arrow[r, "\Pi"] & N \end{tikzcd}$$

where $\Pi' : M' \to N$ is a copy of $\Pi : M \to N$ and

$$P = \{(p, p') \in M \times M' \ ; \ \Pi(p) = \Pi'(p')\}$$

is equipped with two structures of $\mathbb{P}^1$-bundle given by projections

$$\pi : (p, p') \mapsto p \quad \text{and} \quad \pi' : (p, p') \mapsto p'.$$

Now, consider on M' the corresponding copy $\mathcal{F}'$ of $\mathcal{F}$ and the pull-back $\mathcal{H} := (\pi')^*\mathcal{F}'$ on P: $\mathcal{H}$ is a Riccati foliation with respect to the bundle structure $\pi : P \to M$. Finally, the projective transverse structure for $\mathcal{F}$ is given by the *diagonal* section

$$\sigma : M \to P \ ; \quad p \mapsto (p,p)$$

with image cutted out by the equation $p' = p$.

When the initial Riccati foliation $\mathcal{F}$ has minimal polar divisor (up to bimeromorphic bundle transformation of $\Pi : M \to N$), then the corresponding triple $(P,\mathcal{H},\sigma)$ above is a minimal form.

For instance, in the case $N = \mathbb{P}^1$, then $M = \mathbb{F}_n$, $n \geq 0$, is a Hirzebruch surface: $M = \mathbb{P}(E)$ where $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$. By construction, $P = \mathbb{P}(\Pi^*E)$ and $\Pi^*E = \mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(n[f])$ where f is any fibre of $\mathbb{F}_n$. In particular, P is decomposable: there are two disjoint holomorphic sections $M \to P$ defined by the two inclusions $\mathcal{O}_{\mathbb{F}_n}, \mathcal{O}_{\mathbb{F}_n}(n[f]) \hookrightarrow \mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(n[f])$.

In the one hand, if σ' is one of these two sections induced by the splitting of E then the transversely projective foliation defined on $M = \mathbb{F}_n$ by $(P,\mathcal{H},\sigma')$ is just the fibration defined by Π endowed with a rich projective transverse structure.

In the special case $n = 1$, one can contract the exceptional section of $M = \mathbb{F}_1$ as well as the $\mathbb{P}^1$-bundle P (see Lemma 4.5); we thus obtain a foliation $\overline{\mathcal{F}}$ on $\mathbb{P}^2$ with a projective transverse structure $(\overline{P},\overline{\mathcal{H}},\overline{\sigma})$ in minimal form (nothing has changed over a Zariski open set). Moreover $\overline{P} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$.

Finally, consider the pull-back $\tilde{\mathcal{F}}$ of $\overline{\mathcal{F}}$ by a regular morphism $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ of degree d, that is a ramified covering. If $\overline{\mathcal{F}}$ has no invariant curve other than the support of $(\overline{\mathcal{P}})_\infty$, then, the pull-back $(\tilde{P},\tilde{\mathcal{H}},\tilde{\sigma})$ of the projective transverse structure for $\overline{\mathcal{F}}$ is still in minimal form; this can be shown by reasonning locally at a smooth point of the branching locus of ϕ. In this case, we have $\tilde{P} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(d))$.

3.2. **Projective structures on curves.** In the special case when $M = C$ is a curve, a (singular) transversely projective foliation (of dimension 0!) on C is just a (singular) projective structure on C. It is thus defined by a ruled surface $\pi : P \to C$, a (singular) Riccati foliation $\mathcal{F}$ on P (with respect to the projection π) and a section $\sigma : C \to P$ generically transversal to $\mathcal{F}$. These foliated surfaces appear in [2], pages 50-57, as typical counter examples of the existence of an unique minimal model for foliated surfaces: there are infinitely many Riccati foliations birational to $\mathcal{F}$ and minimizing the polar divisor $(\mathcal{P})_\infty$ (except regular Riccati foliations).

Indeed, any elementary transformation applied to a singular point of $\mathcal{F}$ can only decrease the degree of the polar divisor. Nevertheless, there is a unique birational model minimizing the polar divisor such that the section σ does not intersect the singular set of the foliation. Our definition of minimal form was inspired by this phenomenon.

In the case $\mathcal{P}$ is a regular transversely projective structure on $\mathcal{C}$, the minimal form is a regular Riccati foliation with a transversal cross-section. According to Gunning (see [8]), the underlying $\mathbb{P}^1$-bundle (ruled surface) is the unique maximally unstable indecomposable $\mathbb{P}^1$-bundle over C provided that the genus of C is $g > 1$.

3.3. **Meromorphic Fibrations.** Consider the foliation $\mathcal{F}$ defined on a complex manifold M by the fibres of a non constant meromorphic function $F : M \dashrightarrow \mathbb{P}^1$. The projective structure of $\mathbb{P}^1$ induces through F a projective transverse structure for $\mathcal{F}$. A triple $\mathcal{P}$ defining this transverse structure is for instance given by $P =$

$M \times \mathbb{P}^1$, the trivial bundle, $\mathcal{H}$ the horizontal foliation on P and $\sigma(x,z) = (x, F(z))$ defined by the graph of F. This triple is in minimal form.

Such foliations admit a huge number of other projective transverse structure: one can put on $\mathbb{P}^1$ any singular projective structure (cf. §3.2) and pull it back to M using F.

An alternate definition for the minimal model was adopted in a previous version of our work (see [10]) having the advantage to provide non trivial bundle for the fibration case. Roughly speaking, we asked for branch locus of codimension ≥ 2 and minimal polar locus.

3.4. **Logarithmic foliations.** Consider the foliation $\mathcal{F}$ defined on a complex manifold M by a closed meromorphic 1-form ω. Then $\mathcal{F}$ admits an one parameter family of transversely projective structures in minimal form given on the trivial bundle by

$$\Omega = dz + (1 + cz)\omega, \quad c \in \mathbb{C}$$

with section $\sigma : \{z = 0\}$. For $c = 0$, the monodromy is additive, while otherwise it is multiplicative.

3.5. **Foliations on the Projective Plane and eccentricity.** Let $\mathcal{P} = (\pi : P \dashrightarrow \mathbb{P}^2, \mathcal{H}, \sigma : \mathbb{P}^2 \dashrightarrow P)$ be a singular transversely projective structure in minimal form of a foliation $\mathcal{F}$ of the projective plane $\mathbb{P}^2$. We define the **eccentricity** of $\mathcal{P}$, denoted by $\mathrm{ecc}(\mathcal{P})$, as follows: if $L \subset \mathbb{P}^2$ is a generic line and $P|_L$ is the restriction of the $\mathbb{P}^1$-bundle P to L then we set $\mathrm{ecc}(\mathcal{P})$ as minus the self-intersection in $P|_L$ of $\overline{\sigma(L)}$. It turns out that the eccentricity of $\mathcal{P}$ can be easily computed once we know the degree of the polar divisor and the branching locus. In order to be more precise, recall that the degree of $\mathcal{F}$ is defined as the number of tangencies of $\mathcal{F}$ with a general line L on $\mathbb{P}^2$. When $\mathcal{F}$ has degree d it is defined through a global holomorphic section of $\mathrm{T}\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(d-1)$, see [2, pages 27–28]. Then we have

Proposition 1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ and $\mathcal{P}$ a singular transversely projective structure for $\mathcal{F}$ in minimal form. Then*

$$\mathrm{ecc}(\mathcal{P}) = \deg(\mathcal{P})_\infty - (\deg(\mathcal{F}) + \deg(\mathrm{Branch}(\mathcal{P})) + 2)\,.$$

Here, $\deg(\mathrm{Branch}(\mathcal{P}))$ denotes the degree of the codimension 1 component of the branching locus (counted with multiplicity). We do not know if it is possible to give upper bounds for $\mathrm{ecc}(\mathcal{P})$ just in function of the degree of $\mathcal{F}$. A positive result on this direction would be relevant for what is nowadays called the Poincaré Problem.

Proof. Recall first the following general fact. If $\pi : S \to B$ is a $\mathbb{P}^1$-bundle over a projective curve B, $\mathcal{C}$ is the foliation tangent to the fibers of π and $\mathcal{R}$ is a Riccati foliation on S then $\mathcal{R}$ is defined by a global holomorphic section of $\mathrm{T}S \otimes \pi^*(\mathrm{T}^*B) \otimes \mathcal{O}_S(\mathrm{tang}(\mathcal{R}, \mathcal{C}))$ [2, page 57]. If $C \subset S$ is a reduced curve not $\mathcal{R}$-invariant then [2, proposition 2,page 23]

$$\deg\left(\pi^*(\mathrm{T}B) \otimes \mathcal{O}_S(-\mathrm{tang}(\mathcal{R}, \mathcal{C}))\,|_C = C^2 - \mathrm{tang}(\mathcal{R}, C)\,.$$

Now, let $L \subset \mathbb{P}^2$ be a generic line and let P_L be the restriction of the $\mathbb{P}^1$-bundle $\pi : P \to \mathbb{P}^2$ to L. On P_L we have $\mathcal{G}$, a Riccati foliation induced by the restriction of $\mathcal{H}$, and a curve C corresponding to $\sigma(L)$. Notice that

$$T\mathcal{G} = (\pi|_L)^*\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(-(\mathcal{P})_\infty)\,.$$

We also point out that the tangencies between $\mathcal{G}$ and C arise from tangencies between $\mathcal{F}$ and L and from intersections between $\mathrm{Branch}(\mathcal{P})$ and L. Thus

$$\begin{aligned} T\mathcal{G}\cdot C &= C\cdot C - \mathrm{tang}(\mathcal{G},C) \\ &= -\mathrm{ecc}(\mathcal{P}) - \deg(\mathcal{F}) - \deg(\mathrm{Branch}(\mathcal{P}))\,. \end{aligned}$$

Combining this with the expression for $T\mathcal{G}$ above we obtain that

$$2-\deg((\mathcal{P})_\infty) = -\mathrm{ecc}(\mathcal{P}) - \deg(\mathcal{F}) - \deg(\mathrm{Branch}(\mathcal{P}))\,,$$

and the proposition follows. □

Fibrations and foliations defined by meromorphic closed 1-forms provide examples of transversely projective foliations with vanishing excentricity (minimal form with trivial $\mathbb{P}^1$-bundle and *constant* sections, see §3.3 and 3.4) while Riccati foliations constructed in section 3.1 have negative excentricity since the section defining $\mathcal{F}$ has positive self-intersection. We will now present an example with positive excentricity.

3.6. **Hilbert Modular Foliations on the Projective Plane.** In [12] some Hilbert Modular Foliations on the Projective Plane are described. For instance in Theorem 4 of loc. cit. a pair of foliations $\mathcal{H}_2$ and $\mathcal{H}_3$ of degrees 2 and 3 is presented. Both foliations admit transversely projective structures with reduced polar divisor whose support consists of a rational quintic and a line, cf. [4, 12]; by construction, the branching locus is entirely contained in the polar divisor. For $\mathcal{H}_2$ the eccentricity is equal to $2 = 6-(2+0+2)$ while for $\mathcal{H}_3$ it is equal to $1 = 6-(3+0+2)$. Similarly if one consider the pair of foliations $\mathcal{H}_5$ and $\mathcal{H}_9$ presented in Theorem 2 of loc. cit. then $\mathcal{H}_5$ has eccentricity $8 = 15-(5+0+2)$ and $\mathcal{H}_9$ has eccentricity $4 = 15-(9+0+2)$. Since $\mathcal{H}_5$ is birationally equivalent to $\mathcal{H}_9$ and $\mathcal{H}_2$ is birationally equivalent to $\mathcal{H}_3$ these examples show that the eccentricity is not a birational invariant of transversely projective foliations.

Actually, working directly with explicit expressions given in [4], one can check that the $\mathbb{P}^1$-bundle associated with the minimal form of $\mathcal{H}_2$ (resp. $\mathcal{H}_3$) is $\mathbb{P}(\mathcal{O}(2)\oplus\mathcal{O})$ (resp. $\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O})$) with section σ given by the inclusion of $\mathcal{O}$ in each case.

3.7. **Brunella's Very Special Foliation.** The very special foliation admits a birational model on $\mathbb{P}^2$; it is induced in an affine chart (x,y) by the 1-form (cf. [13])

$$\omega = (-y^2 - x + 2xy)dx + (3xy - 3x^2)dy\,. \tag{6}$$

It has three invariant curves. The line $\{x=0\}$, the line at infinity and the rational cubic $\{x^2+x-3xy+y^3=0\}$. Notice that the rational cubic has a node at $[1:1:1]$. It admits a unique projective, actually affine structure whose minimal form is given on the trivial $\mathbb{P}^1$-bundle by

$$\Omega := z_1dz_2 - z_2dz_1 + \frac{\omega}{x(x^2+x-3xy+y^3)}z_1^2 + \frac{1}{3}\frac{dx}{x}\cdot z_1z_2 + 0\cdot z_2^2,$$

the section σ given by $z_2=0$ and the infinity of the affine structure, by the invariant section $z_1=0$. The polar divisor is reduced, with support equal to the three $\mathcal{F}$-invariant curves. Over the line at infinity, the section σ identifies with a branch of singularities and the minimal form is obtained on the bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))$; the eccentricity is 1.

3.8. **Quasi-minimal foliations.** A foliation $\mathcal{F}$ is said *quasi-minimal* when all leaves but a finite number are dense with regards to the transcendental topology.

Proposition 2. *Let $\mathcal{F}$ be a quasi-minimal singular transversely projective foliation of $\mathbb{P}^2$ and $\mathcal{P}$ be a transversely projective structure for $\mathcal{F}$ in minimal form. If the monodromy representation of $\mathcal{P}$ is not minimal then*

$$\mathrm{ecc}(\mathcal{P}) > 0\,.$$

For instance, Hilbert modular foliations introduced in section 3.6 are quasi-minimal with monodromy in $\mathrm{PSL}(2,\mathbb{R})$, cf. [12, Theorem 1]. We have already seen that $\mathcal{H}_2$ and $\mathcal{H}_3$ have indeed positive eccentricity.

Proof. Let $\mathcal{F}$ be a quasi-minimal singular transversely projective foliation of $\mathbb{P}^2$ with transverse structure $\mathcal{P} = (\pi : P \to M, \mathcal{H}, \sigma : M \dashrightarrow P)$ in minimal form. If the monodromy of $\mathcal{H}$ is non-solvable and not minimal then there exists a non-algebraic proper closed set $\mathcal{M}$ of P formed by a union of leaves and singularities of $\mathcal{H}$.

If $L \subset \mathbb{P}^2$ is a generic line then $\mathrm{ecc}(\mathcal{P}) = -C^2$ where $C = \sigma(L)$. If $\mathrm{ecc}(\mathcal{P}) \leq 0$, i.e., $C^2 \geq 0$ then every leaf of $\mathcal{G}$, the restriction of $\mathcal{H}$ to $\pi^{-1}(L)$ must intersects $\mathcal{M} \cap \pi^{-1}(L)$. In the case $C^2 > 0$ this follows from [14, Corollary 8.2]. When $C^2 = 0$ we have that $\pi^{-1}(L) = \mathbb{P}^1 \times \mathbb{P}^1$ and every non algebraic leave must intersect every fiber of the *horizontal* fibration (otherwise the restriction of the second projection to it would be constant).

Therefore for L generic enough $\sigma^*\mathcal{M}$ is a non-algebraic proper closed subset of $\mathbb{P}^2$ invariant under $\mathcal{F}$. Thus $\mathcal{F}$ is not quasi-minimal. This contradiction implies the result. □

4. Existence and Uniqueness of Minimal Forms

We return to the local setup with notations of section 2.1.

4.1. **The behaviour of $\mathcal{H}$ over a generic point of $(\Omega)_\infty$.** Let W be an analytic subset of the support of $(\Omega)_\infty$. We will set $S(W)$ as

$$S(W) = \pi^{-1}(W) \cap \mathrm{sing}(\mathcal{H})\,.$$

We will start by analyzing $\mathcal{H}$ over the irreducible components H of $(\Omega)_\infty$ for which $\pi^{-1}(H)$ is $\mathcal{H}$-invariant.

Lemma 4.1. *Let H be an irreducible component of the support of $(\Omega)_\infty$ and*

$$V = \{p \in H \text{ such that } \pi^{-1}(p) \subsetneq \mathrm{sing}(\mathcal{H}) \text{ and } H \text{ is smooth at } p\}.$$

If $\pi^{-1}(H)$ is $\mathcal{H}$-invariant then the foliation $\mathcal{H}$ has a local product structure along V. In particular $\pi|_{S(V)} : S(V) \to V$ is an étale covering of V of degree 1 or 2.

Since $\mathrm{cod}\,\mathrm{sing}(\mathcal{H}) \geq 2$, we note that V is a dense open subset of H. By a **local product structure** we mean that there exists a Riccati foliation $\mathcal{R}$ on $\Delta^1 \times \mathbb{P}^1$ (Δ^1 the unit disc) with a single pole at $0 \in \Delta^1$ such that for every $p \in V$, the restriction of $\mathcal{H}$ over some small neighborhood $U \subset M$ of p is conjugate by a fibre bundle isomorphism $\phi : \pi^{-1}(U) \to (\Delta^{n-1} \times \Delta^1) \times \mathbb{P}^1$ to the product Riccati foliation $\Pi^*\mathcal{R}$ where $\Pi : (\Delta^{n-1} \times \Delta^1) \times \mathbb{P}^1 \to \Delta^1 \times \mathbb{P}^1$ is the projection to the last coordinates. The Riccati foliation $\mathcal{R}$ may be thougth as the restriction of $\mathcal{H}$ to a small disc transversal to V; we will refer to $\mathcal{R}$ as the **transversal type** of $\mathcal{H}$ along V (or H).

Proof. Let $p \in V$ and $F \in \mathcal{O}_{\Delta^n,p}$ be a local equation around for the poles of Ω. Since $p \in V$ at least one of the holomorphic 1-forms $F\alpha, F\beta, F\gamma$ is non-zero at p. After applying a change of coordinates of the form

$$(x, [z_1 : z_2]) \mapsto (x, [a_{11}z_1 + a_{12}z_2 : a_{21}z_1 + a_{22}z_2])$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}),$$

we can assume that $F\alpha, F\beta$ and $F\gamma$ are non-zero at p.

From the relation $d\alpha = \alpha \wedge \beta$ we promptly see that the holomorphic 1-form $F\alpha$ besides being non-singular is also integrable. It follows from Frobenius integrability Theorem and the $\mathcal{H}$-invariance of H that there exist a local system of coordinates $(x, y_2, \ldots, y_n) : U \to \mathbb{C}^n$ where p is the origin of $\mathbb{C}^n$, $F = x^n$ for a suitable $n \in \mathbb{N}$ and $F\alpha = h_0 dx$ for some $h_0 \in \mathcal{O}^*_{\Delta^n,p}$. Again from the relation $d\alpha = \alpha \wedge \beta$ and the fact that $F\beta(p) = (x^n\beta)(p) \neq 0$ it follows that there exists $h_1 \in \mathcal{O}^*_{\Delta^n,p}$ such that

$$\beta = -\frac{dh_0}{h_0} + h_1 \frac{dx}{x^n}.$$

After performing the holomorphic change of variables

$$(x, [z_1 : z_2]) \mapsto (x, [h_0 z_1 : z_2 + (1/2)h_1 z_1])$$

we can suppose that $(\alpha, \beta) = (\frac{dx}{x^n}, 0)$.

The conditions $d\beta = 2\alpha \wedge \gamma$ and $d\gamma = \beta \wedge \gamma$ imply that γ depends only on x: $\gamma = b(x)\frac{dx}{x^n}$, with b holomorphic. Note that on this new coordinate system we can no longer suppose that $F\gamma(p) = x^n\gamma(p) \neq 0$. Thus on this new coordinate system

$$\Omega = z_1 dz_2 - z_2 dz_1 + z_1^2 \frac{dx}{x^n} + z_2^2 b(x) \frac{dx}{x^n}\,.$$

It follows that on $\pi^{-1}(q)$, $q \in V$, we have one or two singularities of $\mathcal{H}$: one when $b(0) = 0$ and two otherwise. □

Let us now analyze $\mathcal{H}$ over the irreducible components H of $(\Omega)_\infty$ for which $\pi^{-1}(H)$ is not $\mathcal{H}$-invariant. In the notation of lemma 4.1 we have the

Lemma 4.2. *If $\pi^{-1}(H)$ is not $\mathcal{H}$-invariant then $\pi|_{S(V)} : S(V) \to V$ admits an unique holomorphic section.*

Proof. Let $p \in V$ be an arbitrary point. Without loss of generality we can assume that $F\alpha, F\beta$ and $F\gamma$ are non-zero at p and that H is not invariant by the foliation induced by α, cf. proof of lemma 4.1.

Assume also that $\ker \alpha(p)$ is transverse to H. Thus there exists a suitable local coordinate system $(x, y, y_3, \ldots, y_n) : U \to \mathbb{C}^n$ where p is the origin, $F = x^n$ for some $n \in \mathbb{N}$ and $F\alpha = h_0 dy$ for some $h_0 \in \mathcal{O}^*_{\Delta^n,p}$.

The condition $d\alpha = \alpha \wedge \beta$ implies that $\beta = n\frac{dx}{x} + h_1 \cdot \alpha$ with h_1 meromorphic at p. Since $F\beta = x^n\beta$ is holomorphic and does not vanish at p the same holds for h_1, i.e., $h_1 \in \mathcal{O}^*_{\Delta^n,p}$. Thus if we apply the holomorphic change of coordinates

$$((x, y, y_3, \ldots, y_n), [z_1 : z_2] \mapsto ((x, y, y_3, \ldots, y_n), [z_1 : h_0 \cdot z_2 + h_1 \cdot z_1])$$

we have $d\beta = 0$.

Combining $0 = d\beta = 2\alpha \wedge \gamma$ with $d\gamma = \beta \wedge \gamma$ we deduce that $\gamma = x^n h_3(y)\alpha$ for some meromorphic function h_3. Since γ has poles contained in $H = \{x = 0\}$, h_3 is in fact holomorphic and consequently $\mathcal{H}$ is induced by the 1-form

$$x^n(z_1 dz_2 - z_2 dz_1) + (dy)z_1^2 + (x^n h_3(y)dy)z_2^2 \,. \tag{7}$$

It is now clear that the singular set of $\mathcal{H}$ is given by $\{x = 0\} \cap \{z_1 = 0\}$. Thus there exists an open subset $V_0 \subset V$ for which $S(V_0)$ is isomorphic to V_0. Since $S(V)$ does not contain fibers of $\pi_{|S(V)}$ this is sufficient to prove the lemma. □

Remark 4.3. These irreducible components of $(\Omega)_\infty$ are a kind of *fake* or *apparent* singular set for the transversely projective structures. Over them the transverse type is a Riccati foliation with a dicritical singularity and a saddle singularity. The dicritical singularity corresponds to a true singularity of $\mathcal{H}$ whereas the saddle singularity is an apparent one arising from the tangency between $\mathcal{H}$ and the surface where the Riccati foliation lives. After the fibred birational change of coordinates

$$((x, y, y_3, \dots, y_n), [z_1 : z_2] \mapsto ((x, y, y_3, \dots, y_n), [z_1 : x^n z_2])$$

the foliation induced by (7) is completely transversal to the fibres of the $\mathbb{P}^1$-fibration and has a product structure as in the case H is $\mathcal{H}$-invariant.

4.2. **Elementary Transformations.** Let $\pi : P \to M$ be a $\mathbb{P}^1$-bundle over M, $H \subset M$ a smooth hypersurface and $s : H \to P$ a holomorphic section. An **elementary transformation** $\mathrm{elm}_S : \Delta^n \times \mathbb{P}^1 \dashrightarrow \Delta^n \times \mathbb{P}^1$ with center in $S = s(H)$ can be described as follows: we first blow-up S on M and then we contract the strict transform of $\pi^{-1}(H)$. In local coordinates, if $F = 0$ is a reduced equation of H and S is the intersection of $H \times \mathbb{P}^1$ with the hypersurface $z_2 = 0$ then elm_S can be explicitly written as

$$\begin{array}{rcl} \mathrm{elm}_S : \Delta^n \times \mathbb{P}^1 & \dashrightarrow & \Delta^n \times \mathbb{P}^1 \\ (x, [z_1 : z_2]) & \mapsto & (x, [F(x)z_1 : z_2]) \end{array}$$

modulo $\mathbb{P}^1$-bundle isomorphisms on the source and the target.

In the case P is the projectivization of a rank 2 holomorphic vector bundle over a complex manifold M then the elementary transformations just described are projectivizations of the so called elementary modifications, see [5, pages 41–42].

More generally, let $s : \Delta^n \dashrightarrow \Delta^n \times \mathbb{P}^1$ be a meromorphic section of the trivial $\mathbb{P}^1$-bundle given in local coordinates by $s(x) = (x, [f(x) : g(x)])$ and $H \subset \Delta^n$ a possibly singular hypersurface defined by a reduced equation $F(x) = 0$. Assume that the indeterminacy locus $Z = \{f = g = 0\}$ of s is contained in H in the analytic sense: there are $a, b \in \mathcal{O}(\Delta^n)$ such that $F = a \cdot f + b \cdot g$. Then we define the elementary transformation with center $S := s(H)$ by

$$\begin{array}{rcl} \mathrm{elm}_S : \Delta^n \times \mathbb{P}^1 & \dashrightarrow & \Delta^n \times \mathbb{P}^1 \\ (x, [z_1 : z_2]) & \mapsto & (x, [a(x)z_1 + b(x)z_2 : g(x)z_1 - f(x)z_2]) \end{array}$$

modulo $\mathbb{P}^1$-bundle isomorphisms at the target. At a smooth point $p \in H$ where s is holomorphic, one can easily check that ϕ is an elementary transformation in the previous sense.

Conversely, any birational bundle transformation over Δ^n defined by a matrix $A(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in gl(2, \mathcal{O}(\Delta^n))$ with reduced determinant $ad - bc = F$ is an elementary transformation in the previous sense; the center is given over $H = \{F = 0\}$ by either $[z_1 : z_2] = [b(x) : -a(x)]$, or $[d(x) : -c(x)]$.

Remark 4.4. When M has dimension 2, one can still define the elementary transformation along any meromorphic section S of H whenever H is singular. In order to see this, one can first desingularize the curve H by blowing-ups of M, apply the elementary transformation along the strict transform of S and then contract the exceptional divisor by means of the following

Lemma 4.5. *Let $\widetilde{\pi} : \widetilde{P} \to \widetilde{M}$ be a $\mathbb{P}^1$-bundle over a compact complex surface $\widetilde{M}$ and let $r : \widetilde{M} \to M$ be a bimeromorphic morphism with exceptional divisor D. Then there exists a $\mathbb{P}^1$-bundle $\pi : P \to M$ and a map $\phi : \widetilde{P} \to P$ such that $\phi|_{\widetilde{\pi}^{-1}(\widetilde{M}\setminus D)}$ is a $\mathbb{P}^1$-bundle isomorphism.*

Proof. It is sufficient to consider the case where $r : \widetilde{M} \to M$ is a single blow-up at a point $p \in M$. Therefore, D is a projective line of self-intersection -1. The restriction $\widetilde{P}|_D$ is then a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, namely a Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-n))$ for some integer $n \in \mathbb{N}$. If $n = 0$, then $\widetilde{P}|_D \simeq D \times \mathbb{P}^1$ has a second "horizontal" fibration by rational curves, over which the normal bundle of $\widetilde{P}|_D$ inside $\widetilde{P}$ has degree -1; the bundle P is obtained after contracting these horizontal fibres.

When $n > 0$, then $\widetilde{P}|_D$ has a section C having self-intersection $-n$: the normal bundle of C in P is $\mathcal{O}(-1) \oplus \mathcal{O}(-n)$; after doing an elementary transformation of P with center C, the new bundle restricts to D as $\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-n)) = \mathbb{F}_{n-1}$. After n such elementary transformations we are back to the case $\widetilde{P}|_D \simeq D \times \mathbb{P}^1$. ☐

Example 4.6. *The elementary transformation of the trivial bundle over $\mathbb{C}^2$ with center $S(H)$ defined as the restriction of the meromorphic section $S(x, y) = [x^2 : y^3]$ over $H = \{y^2 - x^3\}$ is given by $\phi(x, y, [z_1 : z_2]) = (x, y, [(y^2 - x^3)z_1 : xyz_1 + z_2])$.*

Proposition 4.7. *Let M be a complex surface and $\phi : P \to P'$ be a bimeromorphic bundle transformation between two $\mathbb{P}^1$-bundles over M. Then ϕ is obtained by applying successively finitely many elementary transformations.*

Proof. Through local trivializing coordinates for P and P', the map ϕ is defined as

$$(x, [z_1 : z_2]) \mapsto (x, [z_1' : z_2']) = (x, [a(x)z_1 + b(x)z_2 : c(x)z_1 + d(x)z_2])$$

where $A(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is holomorphic in x with codimension ≥ 2 zero set. The divisor $(\phi)_\infty := (\det(A(x)))_0$ does no depend on the choice of the trivialization charts and defines a global divisor on M. Also, the kernel of $A(x)$ defines an holomorphic section S over each irreducible component H of the support $|(\phi)_\infty|$. It suffices to show that $\phi = \tilde{\phi} \circ \mathrm{elm}_S$ with $(\tilde{\phi})_\infty < (\phi)_\infty$ (the order of $(\tilde{\phi})_\infty$ along H is 1 less that the one of $(\phi)_\infty$). To see this, just note that at the neighborhood of a smooth point p of H, one can choose trivialization charts through which the matrix $A(x)$ defining ϕ has kernel generated by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which just means that

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} = \begin{pmatrix} \tilde{a}(x) & b(x) \\ \tilde{c}(x) & d(x) \end{pmatrix} \begin{pmatrix} F(x) & 0 \\ 0 & 1 \end{pmatrix}$$

where $F(x)$ is a reduced equation of H. ☐

Remark 4.8. When the dimension of M is > 2, the elementary transformation of a $\mathbb{P}^1$-bundle P with center defined by restriction $S = s(H)$ of a meromorphic section $s : M \dashrightarrow P$ over an hypersurface H does not exist in general. For instance, consider in coordinates $(x, y, z) \in \mathbb{C}^3$ the section of the trivial bundle $P = \mathbb{C}^3 \times \mathbb{P}^1$ defined by $s(x, y, z) = [x : y]$ and its restriction $S = s(H)$ to the hyperplane $H = \{z = 0\}$. The elementary transformation with center S is well defined over $\mathbb{C}^3 - \{0\}$, giving rise to a new bundle $P' = \mathrm{elm}_S P$. Nevertheless, this transformation, or equivalently the bundle P' does not extend at $0 \in \mathbb{C}^3$. Indeed, if P' were trivial at 0, then elm_S would be defined by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{gl}(2, \mathcal{O})$ with determinant $ad - bc = z$ and z should divide $ax + by$ and $cx + dy$; the matrix above thus takes the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} fy & -fx \\ gy & -gx \end{pmatrix} + z \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$$

and its determinant yields (after simplification and division by z)

$$(f\tilde{c} - g\tilde{a})x + (f\tilde{d} - g\tilde{b})y + (\tilde{a}\tilde{d} - \tilde{b}\tilde{c})z = 1$$

which is impossible. Nevertheless, after blowing-up the origin of $\mathbb{C}^3$, the section s becomes holomorphic and the elementary transformation with center the strict transform of S is well-defined. This provides in turn an example showing that lemma 4.5 is no longer true in dimension greater than two.

We are now interested in describing the foliation $(\mathrm{elm}_S)_*\mathcal{H} = (\mathrm{elm}_S^{-1})^*\mathcal{H}$ when $\mathcal{H}$ is a Riccati foliation on the bundle P. More specifically we want to understand how the divisors $\mathrm{tang}(\mathcal{H}, \mathcal{C})$ and $\mathrm{tang}(\mathrm{elm}_*\mathcal{H}, \mathcal{C})$ are related, where $\mathcal{C}$ denotes the one dimension foliation induced by the fibers of $\Delta^n \times \mathbb{P}^1 \to \Delta^n$. We point out that the analysis we will now carry on can be found in the case $n = 1$ in [2, pages 53–56]. The arguments that we will use are essentially the same. We decided to include them here thinking on readers' convenience.

Let k be the order of $(\Omega)_\infty$ along H. Since $\mathrm{elm}_S^{-1}(x, [z_1 : z_2]) = (x, [z_1 : F(x) z_2])$ it follows that

$$(\mathrm{elm}_S^{-1})^*\Omega = F(z_1 dz_2 - z_2 dz_1) + \alpha z_1^2 + F\left(\beta + \frac{dF}{F}\right) z_1 \cdot z_2 + F^2 \gamma z_2^2 \, .$$

Thus the foliation $(\mathrm{elm}_S)_*\mathcal{H}$ is induced by the meromorphic 1-form

$$\widetilde{\Omega} = (z_1 dz_2 - z_2 dz_1) + \frac{\alpha}{F} z_1^2 + \left(\beta + \frac{dF}{F}\right) z_1 \cdot z_2 + F \gamma z_2^2$$

In order to describe $(\widetilde{\Omega})_\infty$ we will consider three mutually exclusive cases:

(1) <u>S is not contained in $\pi^{-1}(H) \cap \mathrm{sing}(\mathcal{H})$:</u> $(F^k \alpha)_{|H} \equiv 0$.
Therefore we have
$$(\widetilde{\Omega})_\infty = (\Omega)_\infty + H \, .$$

(2) <u>$S \subsetneq \pi^{-1}(H) \cap \mathrm{sing}(\mathcal{H})$:</u> $(F^k \alpha)_{|H} \equiv 0$ while $(F^k \beta)_{|H} \not\equiv 0$.
If $k \geq 2$ then
$$(\widetilde{\Omega})_\infty = (\Omega)_\infty \, .$$

If $k = 1$ we have two possible behaviors
$$(\widetilde{\Omega})_\infty = \begin{cases} (\Omega)_\infty - H & \text{when } \beta + \frac{dF}{F} \text{ is holomorphic.} \\ (\Omega)_\infty & \text{otherwise .} \end{cases}$$

(3) $\underline{S = \pi^{-1}(H) \cap \mathrm{sing}(\mathcal{H})}$: $(F^k\alpha)|_H \equiv (F^k\beta)|_H \equiv 0$ while $(F^k\gamma)_{|H} \not\equiv 0$.
If $k = 1$ then

$$(\widetilde{\Omega})_\infty = (\Omega)_\infty .$$

If $k \geq 2$, then we have

$$(\widetilde{\Omega})_\infty = (\Omega)_\infty - (k - k')H$$

with k' the smallest positive integer for which $(F^{k'}\alpha)|_H \equiv (F^{k'+1}\beta)|_H \equiv 0$.

Remark 4.9. The unique way to make k decrease is to perform an elementary transformation with center contained in $\mathrm{sing}(\mathcal{H})$. When $k = k'$, then a case-by-case analysis shows that the transversal type of H is given up to bundle isomorphism by

- Linear case: $x(z_1dz_2 - z_2dz_1) + (\lambda z_1z_2)dx$ with $\lambda \in \mathbb{C}$.
- Poincaré-Dulac case: $x(z_1dz_2 - z_2dz_1) + (x^nz_1 + nz_1z_2)dx$ with $n \in \mathbb{N}$.
- Saddle-Node case: $x^k(z_1dz_2 - z_2dz_1) + (o(x)z_1^2 + (1 + o(x))z_1z_2 + o(x)z_2^2)dx$.
- Nilpotent case: $x^k(z_1dz_2 - z_2dz_1) + (x(1 + o(x))z_1^2 + o(x)z_1z_2 + o(x)z_2^2)dx$.

Moreover, each of these cases is stable under additional elementary transformation with center contained in $\mathrm{sing}(\mathcal{H})$. For instance, in the linear case, the invariant λ is shifted by ± 1 after each such transformation. In particular, when $\lambda \in \mathbb{Z}$, one arrives at a regular Riccati foliation ($\lambda = 0$ and $k = 0$) after finitely many elementary transformations: we say that the singular branch H is **apparent**. Apart from this very special case, k is actually minimal, i.e. cannot decrease by finitely many additional elementary transformations.

4.3. **Existence of a Minimal Form.** Let $\mathcal{P} = (\pi : P \to S, \mathcal{H}, \sigma : M \dashrightarrow P)$ be a transversely projective structure for a foliation $\mathcal{F}$ on a complex surface M. Let H be an irreducible component of $(\mathcal{P})_\infty$ of multiplicity $k(H)$ and, as in §4.1, let $S(H)$ be given by $S(H) = \pi^{-1}(H) \cap \mathrm{sing}(\mathcal{H})$. Thus (see lemmata 4.1 and 4.2) $S(H)$ is an analytic subset of $\pi^{-1}(H)$ formed by a finite union of fibers together with a one or two-valued holomorphic section s of $P|_H$. Note that to assure that s is in fact holomorphic, and not just meromorphic, we have used that H is a curve, i.e., we have used that S is a surface.

If H is not $\mathcal{H}$-invariant, then we are in the situation of lemma 4.2: s is actually single valued and after an elementary transformation with center s, $k(H)$ decreases by one; after $k(H)$ successive elementary transformations, the resulting foliation is smooth over a generic point of H (compare with remark 4.3).

Assume now that H is $\mathcal{H}$-invariant. If the transversal type of H is already in the list of remark 4.9 (and not apparent), then $k(H)$ cannot decrease by additional elementary, or bimeromorphic transformation. In order to minimize $k(H)$, we have just to consider two cases: either s is single valued with degenerate transversal type (not in the list of remark 4.9), or s is two-valued with apparent transversal type (linear with $\lambda \in \mathbb{N}^*$). In the first case, $k(H)$ decreases by one after an elementary transformation with center s. In the second case, s actually splits into two global single valued sections s_+ and s_- corresponding to the singular points with eigenvalues λ and $-\lambda$ respectively. By an elementary transformation with center s_+, the transversal type is still apparent with new eigenvalue $\lambda - 1$; iterating λ times this process will lead to $k(H) = 0$.

In resume after applying a finite number of elementary transformations we arrive at a projective structure, still denoted by $\mathcal{P}$, for which $(\mathcal{P})_\infty$ has minimal multiplicity in the same bimeromorphic equivalence class. Moreover, Proposition 4.7 and discussion ending the previous section show that any bimeromorphically equivalent projective structure $\mathcal{P}'$ will have the same polar divisor if and only if it is derived from $\mathcal{P}$ by applying finitely many elementary transformations with center contained in the singular set. We now use this lack of uniqueness to put the section σ in general position with respect to the singular locus of $\mathcal{H}$.

Let H be an irreducible codimension one component of $\mathrm{Branch}(\mathcal{P}) \cap (\mathcal{P})_\infty$. The restriction s of σ to $\pi^{-1}(H)$ is thus contained in the singular locus of $\mathcal{H}$. After an elementary transformation centered in s, $k(H)$ remains unchanged and $(\mathcal{P})_\infty$ is still minimal. But it is easy to check that the branching order of H is shifted by -1 (apply formulae at the end of section 4.2 with $\sigma(x) = (x, [1:0])$). After finitely many elementary transformations, H is no more in the support of $\mathrm{Branch}(\mathcal{P})$.

To finish the proof of Theorem 1 we have to establish the uniqueness of the minimal form up to biregular bundle transformations. Although this follows quite directly from the above construction together with previous results of the section, we provide below a proof which remains valid whatever the dimension of the ambient manifold M is, provided that a minimal form does exist.

4.4. **Uniqueness of the Minimal Form.** Let $\mathcal{P} = (\pi : P \to S, \mathcal{H}, \sigma : S \dashrightarrow P)$ and $\mathcal{P}' = (\pi : P' \to S, \mathcal{H}', \sigma : S \dashrightarrow P')$ be two transversely projective structures in minimal form for the same foliation $\mathcal{F}$ and in the same bimeromorphic equivalence class. Let $\phi : P \dashrightarrow P'$ be a fibred bimeromorphism. We want to show that ϕ is in fact biholomorphic.

Since both $\mathcal{P}$ and $\mathcal{P}'$ are in minimal form we have that $(\mathcal{P})_\infty = (\mathcal{P}')_\infty$. Thus for every $p \in S \setminus |(\mathcal{P})_\infty|$ there exists a neighboorhood U of p such that $\mathcal{H}|_{\pi^{-1}(U)}$ and $\mathcal{H}'|_{\pi'^{-1}(U)}$ are smooth foliations transverse to the fibers of π and π', respectively. If ϕ is not holomorphic when restricted to $\pi^{-1}(U)$ then it most contract some fibers of π. This would imply the existence of singular points for $\mathcal{H}'|_{\pi'^{-1}(U)}$ and consequently contradict our assumptions. Thus ϕ is holomorphic over every $p \in S \setminus |(\mathcal{P})_\infty|$.

Suppose now that $p \in |(\mathcal{P})_\infty|$ is a generic point and that Σ_p is germ of curve at p transverse to $|(\mathcal{P})_\infty|$. The restriction of ϕ to $\pi^{-1}(\Sigma)$ (denoted by ϕ_Σ) induces a bimeromorphism of $\mathbb{P}^1$-bundles over Σ. Since Σ has dimension one this bimeromorphism can be written as a composition of elementary transformations. Since p is generic on the fiber $\pi^{-1}(p)$ we have two of three distinguished points: one or two singularities of $\mathcal{H}$ and one point from the section σ. But ϕ_Σ must send these points to the corresponding ones over the fiber $\pi'^{-1}(p)$. This clearly implies that ϕ_Σ is holomorphic. From the product structure of $\mathcal{H}$ in a neighborhood of p, cf. lemma 4.1 and remark 4.3 after lemma 4.2, it follows that ϕ is holomorphic in a neighborhood of $\pi^{-1}(p)$.

At this point we have already shown that there exists Z, a codimension two subset of S, such that $\phi|_{\pi^{-1}(S \setminus Z)}$ is holomorphic.

Let now $p \in Z$ and U be a neighborhood of p where both P and P' are trivial $\mathbb{P}^1$-bundles. Thus after restricting and taking trivializations of both P and P' we have that $\phi|_{\pi^{-1}(U)}$ can be written as

$$\phi|_{\pi^{-1}(U)}(x, [y_1 : y_2]) = (x, [a(x)y_1 + b(x)y_2 : c(x)y_1 + d(x)y_2]) \, ,$$

where a, b, c, d are germs of holomorphic functions. But then the points $x \in U$ where ϕ is not biholomorphic are determined by the equation $(ad - bd)(x) = 0$. Since $(ad - bd)(x)$ is distinct from zero outside the codimension two set Z it is distinct from zero everywhere. Therefore we conclude that ϕ is fact biholomorphic and in this way conclude the prove of the uniqueness of the minimal form. This also concludes the proof of Theorem 1. $\square$

Example 4.10. Consider the example of transversely projective foliation given in section 3.7: in homogeneous coordinates, the foliation $\mathcal{F}$ is defined in $\mathbb{C}^3$ by

$$\omega = (-y^2z - xz^2 + 2xyz)dx + (3xyz - 3x^2z)dy + (x^2z - 2xy^2 + x^2y)dz,$$

the Riccati foliation $\mathcal{H}$ is defined on the trivial bundle $P = \mathbb{C}^3 \times \mathbb{P}^1$ by

$$\Omega := z_1dz_2 - z_2dz_1 + \frac{\omega}{xz(x^2z + xz^2 - 3xyz + y^3)}z_1^2 + \left(\frac{1}{3}\frac{dx}{x} + \frac{2}{3}\frac{dz}{z}\right)z_1z_2 + 0 \cdot z_2^2$$

and the section σ defining $\mathcal{F}$ is the horizontal one $z_2 = 0$. This triple is in minimal form. Now, if we replace the horizontal section σ by $\sigma'(x, y, z) = [-2y(y - z) : 3(x - 2y - z)]$, we obtain a new transversely projective foliation which is not in minimal form: the polar divisor $xz(x^2z + xz^2 - 3xyz + y^3)$ is minimal but the branching divisor is z. To obtain a minimal form, we do not have other choice than to apply the elementary transformation with center $\sigma(H)$ over $H = \{z = 0\}$; nevertheless, the map $(x, y) \mapsto \sigma(x, y, 0) = [2y^2 : 3(2y - x)]$ has an indeterminacy point at the origin and one can check by arguments similar to those used in remark 4.8 that such elementary transformation does not exist. The projective structure $\mathcal{P}' = (P, \mathcal{H}, \sigma')$ does not admit a minimal form.

5. The Monodromy Representation

5.1. **A Local Obstruction.** Let H be a (singular) hypersurface at the neighborhood U of $0 \in \mathbb{C}^n$ and $\rho : \pi_1(U \setminus H) \to \mathrm{PSL}(2, \mathbb{C})$ a representation.

Proposition 5.1. *If ρ is the monodromy representation of a transversely projective structure $\mathcal{P}$ defined on U with polar divisor supported in H, then ρ lifts to* $\mathrm{SL}(2, \mathbb{C})$.

Proof. We can suppose without loss of generality that U is a polydisc and that $\mathcal{P}$ is minimal form. Over U every $\mathbb{P}^1$-bundle is trivial therefore $\mathcal{H}$ induces an integrable differential $\mathfrak{sl}(2, \mathbb{C})$-system on the trivial rank 2 vector bundle over U, cf. §2, formula (3). Clearly ρ *lifts* to the monodromy of the $\mathrm{sl}(2, \mathbb{C})$-system and the proposition follows. $\square$

Example 5.2. Let $H = \{x_1 \cdot x_2 = 0\}$ be the union of the coordinate axis in $\mathbb{C}^2$ and

$$\rho : \pi_1(\mathbb{C}^2 \setminus H) \simeq \mathbb{Z}^2 \to \mathrm{PSL}(2, \mathbb{C})$$

the representation sending the 2 generators respectively to

$$(z_1 : z_2) \mapsto (z_2 : z_1) \quad \text{and} \quad (z_1 : z_2) \mapsto (-z_1 : z_2).$$

Clearly, this representation cannot lift to $\mathrm{SL}(2, \mathbb{C})$. In fact, one can construct a Riccati foliation $\mathcal{H}$ on a $\mathbb{P}^1$-bundle $\pi : P \to \mathbb{C}^2 - \{0\}$ having polar divisor $(\mathcal{P})_\infty = 2 \cdot H$ and monodromy representation ρ (see below). The $\mathbb{P}^1$-bundle so constructed does not admit meromorphic section on any punctured neighborhood of $0 \in \mathbb{C}^2$, otherwise the projective structure should extend by Proposition 2.5 and contradict the Proposition above.

In order to construct $\mathcal{H}$, first consider the saddle-node

$$z_1dz_2 - z_2dz_1 + \frac{1}{2}\frac{dx}{x}z_1^2 + \frac{dx}{x^2}z_1z_2 - \frac{1}{2}\frac{dx}{x}z_2^2.$$

Associated Stokes matrices are $S^+ = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $S^- = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ with $t^2+2=0$ so that the local monodromy $A = S^+S^- = \begin{pmatrix} -1 & t \\ t & 1 \end{pmatrix}$ has trace 0; setting $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have $B^{-1}AB = -A$ so that A and B commute in $\mathrm{PGL}(2,\mathbb{C})$ and are actually conjugated to the two generators of the dihedral group above. Moreover, the pair $(\tilde{S}^+, \tilde{S}^-) := (B^{-1}S^-B, B^{-1}S^+B)$ is conjugated to the initial Stokes data (S^+, S^-) by a diagonal matrix, which means that B actually extends as a symmetry of the differential equation.

Now, one can first construct $\mathcal{H}$ over $\mathbb{C}^2 - H$ by suspension of the representation, and then complete it along H by glueing singularities whose transversal type is the above saddle-node: along a given branch of H, say $x_1 = 0$, the local monodromy around is given by A; the condition that the monodromy B around $x_2 = 0$ extends as a symmetry of the foliation is necessary and sufficient for the gluing to be well defined along each branch of $H - \{0\}$ ($\simeq \mathbb{C}^*$).

A word of warning: it is not true that the monodromy of a transversely projective structure $\mathcal{P}$ lifts globally to $\mathrm{SL}(2,\mathbb{C})$. For instance we have smooth Riccati foliations on $\mathbb{P}^1$-bundle over elliptic curves with monodromy group conjugated to the abelian group

$$G = < (z_1 : z_2) \mapsto (z_2 : z_1); (z_1 : z_2) \mapsto (-z_1 : z_2) > .$$

5.2. **Prescribing the monodromy: Proof of Theorem 2.** First we will assume that H is an hypersurface with smooth irreducible components and with at most normal crossings singularities. Instead of working with the projective surface S we will work with a projective manifold M of arbitrary dimension n.

Construction of the $\mathbb{P}^1$-bundle and of the foliation. If $\rho : \pi_1(M \setminus H) \to \mathrm{SL}(2;\mathbb{C})$ is a representation then it follows from Deligne's work on Riemann-Hilbert problem [9] that there exists E, a rank 2 vector bundle over M, and a meromorphic flat connection

$$\nabla : E \to E \otimes \Omega^1_M(\log H)$$

with monodromy representation given by ρ. From the $\mathbb{C}$-linearity of ∇ we see that its solutions induce $\mathcal{H}$, a codimension one foliation of $\mathbb{P}(E)$. If $\pi_{\mathbb{P}(E)} : \mathbb{P}(E) \to M$ denotes the natural projection then over $\pi^{-1}_{\mathbb{P}(E)}(M \setminus H)$ the restriction of $\mathcal{H}$ is nothing more than suspension of $[\rho] : \pi_1(M \setminus H) \to \mathrm{PSL}(2,\mathbb{C})$ as defined in [4, Example 2.8].

Let U be a sufficiently small open set of M and choose a trivialization of $E_{|U} = U \times \mathbb{C}^2$ with coordinates $(x, z_1, z_2) \in U \times \mathbb{C} \times \mathbb{C}$. Then for every section $\sigma = (\sigma_1, \sigma_2)$ of $E_{|U}$ we have that

$$\nabla_{|U}(\sigma) = \begin{pmatrix} d\sigma_1 \\ d\sigma_2 \end{pmatrix} + A \cdot \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is two by two matrix with $\alpha, \beta, \gamma, \delta \in \Omega^1_M(\log H)$ satisfying the integrability condition $dA + A \wedge A = 0$. Thus $\nabla = 0$ induces the system

$$\begin{aligned} dz_1 &= z_1\alpha + z_2\beta \\ dz_2 &= z_1\gamma + z_2\delta\,. \end{aligned}$$

Thus the solution of the above differential system are contained in the leaves of the foliation defined over $\pi^{-1}_{\mathbb{P}(E)}(U)$ by

$$\Omega_U = z_1 dz_2 - z_2 dz_1 - z_2^2\beta + z_1 z_2(\gamma - \alpha) + z_1^2\delta\,.$$

Clearly the foliations defined in this way patch together to give $\mathcal{H}$, a codimension one foliation on $\mathbb{P}(E)$ transverse to fibers of π which are not over H.

Construction of the meromorphic section. The next step in the proof of Theorem 2 is to assure the existence of a *generic* meromorphic section of $\mathbb{P}(E)$. This is done in the following

Lemma 5.3. *There exists a meromorphic section*

$$\sigma : M \dashrightarrow \mathbb{P}(E)\,,$$

with the following properties:

(i) $\underline{\sigma \textit{ is generically transversal to } \mathcal{G};}$
(ii) $\overline{\mathrm{sing}(\sigma^*\Omega) \setminus (\sigma^*\mathrm{sing}(\mathcal{G}) \cup \mathrm{Ind}(\sigma))}$ *has dimension zero.*

Proof. Let $\mathcal{L}$ be an ample line bundle over M. By Serre's Vanishing Theorem we have that for $k \gg 0$ the following properties holds:

(a) $E \otimes \mathcal{L}^k$ is generated by global sections;
(b) for every $x \in M$, $E \otimes \mathcal{L}^k \otimes m_x$ and $E \otimes \mathcal{L}^k \otimes m_x^2$ are also generated by global sections.

Using a variant of the arguments presented in [15, proposition 5.1] it is possible to settle that there exists a Zariski open $V \subset \mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k})$ such that for every $s \in V$ the zeros locus of s is non-degenerated, of codimension two, with no irreducible component contained in the support of H and whose image does not contains any irreducible component of $\mathrm{sing}(\mathcal{F})$. We leave the details to the reader.

Let now $\mathcal{U} = \{U_i\}_{i \in I}$ be a finite covering of M by Zariski open subsets such that the restrictions of E and of the cotangent bundle of M to each U_i are both trivial bundles. For each $i \in I$ consider

$$\begin{aligned} \Psi_i : U_i \setminus (U_i \cap H) \times \mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k}) &\to \mathbb{C}^n \\ (x, s) &\mapsto s^*\Omega_i(x) \end{aligned}$$

where Ω_i is the 1-form over $\pi^{-1}_{\mathbb{P}(E)}(U_i)$ defining $\mathcal{G}|_{U_i}$ and $\Omega^1_{U_i}$ is implicitly identified with the trivial rank n vector bundle over U_i. It follows from (a) and (b) that for every $x \in M$ there exists sections in $\mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k})$ with prescribed linear part at p. Thus if $Z_i = \Psi_i^{-1}(0)$ then

$$\dim Z_i = \mathrm{h}^0(M, E \otimes \mathcal{L}^{\otimes k}).$$

If $\rho_i : Z_i \to \mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k})$ is the natural projection then there exists a Zariski open set $W_i \subset \mathrm{H}^0(M, E \otimes \mathcal{L}^{\otimes k})$ such that

$$\dim Z_i \le \dim \rho^{-1}(s) + \mathrm{h}^0(M, E \otimes \mathcal{L}^{\otimes k}).$$

Thus $\dim \rho^{-1}(s) = 0$ for every $s \in W_i$.

A section $s \in \left(\bigcap_{i\in I} W_i\right) \cap V$ will induce a meromorphic section σ of $\mathbb{P}(E)$ with the required properties. □

Uniqueness. It remains to prove the uniqueness in the case that ρ is non-solvable. We will need the following

Lemma 5.4. *Suppose that $\pi : \mathbb{P}(E) \to M$ has a meromorphic section σ such that the foliation $\mathcal{F} = \sigma^*\mathcal{H}$ have non unique transversely projective structure. Then the monodromy representation of $\mathcal{H}$ is meta-abelian or there exists an algebraic curve C, a rational map $\phi : \mathbb{P}(E) \dashrightarrow C \times \mathbb{P}^1$ and Riccati foliation on $C \times \mathbb{P}^1$ such that $\mathcal{H} = \phi^*\mathcal{R}$.*

Proof. After applying a fibred birational map we can assume that $\mathbb{P}(E) = M \times \mathbb{P}^1$ and that σ is the $[1 : 0]$-section, i.e., if

$$\Omega = z_1 dz_2 - z_2 dz_1 + \alpha z_1^2 + \beta z_1 \cdot z_2 + \gamma z_2^2\,,$$

is the one form defining $\mathcal{H}$ then $\mathcal{F}$ is induced by α.

Since $\mathcal{F}$ has at least two non bimeromorphically equivalents projective structures then it follows from [16, proposition 2.1] (see also [4, lemma 2.20]) that there exists a rational function ℓ on M such that

$$d\alpha = -\frac{d\ell}{2\ell} \wedge \alpha\,.$$

Thus, after a suitable change of coordinates we can assume that $\beta = \frac{d\ell}{\ell}$. From the relation $d\beta = 2\alpha \wedge \gamma$ we deduce the existence of a rational function $f \in k(M)$ such that $\gamma = f\alpha$. Therefore $d\gamma = \beta \wedge \gamma$ implies that

$$\left(\frac{df}{f} - \frac{dl}{l}\right) \wedge \alpha = 0.$$

If $\mathcal{F}$ does not admit a rational first integral then $f = \ell$. Consequently, on the new coordinate system,

$$\Omega = z_1 dz_2 - z_2 dz_1 + \alpha z_1^2 + \frac{d\ell}{2\ell} z_1 \cdot z_2 + \ell\alpha z_2^2\,.$$

If $\Phi(x, [z_1 : z_2]) = (x, [z_1 : \sqrt{\ell} z_2])$ then we get

$$\frac{\Phi^*\Omega}{\sqrt{\ell}} = z_1 dz_2 - z_2 dz_1 + (z_1^2 + z_2^2)\frac{\alpha}{\sqrt{\ell}} \implies d\left(\frac{\Phi^*\Omega}{\sqrt{\ell}(z_1^2 + z_2^2)}\right) = 0,$$

meaning that after a ramified covering the foliation $\mathcal{H}$ is induced by a closed 1-form. Thus $\mathcal{H}$ has meta-abelian monodromy.

When $\mathcal{F}$ admits a rational first integral then it follows from [16, Theorem 4.1.(i)](see also [4, proposition 2.19]) that there exists an algebraic curve C, a rational map $\phi : \mathbb{P}(E) \dashrightarrow C \times \mathbb{P}^1$ and Riccati foliation on $C \times \mathbb{P}^1$ such that $\mathcal{H} = \phi^*\mathcal{R}$. □

Back to the proof of Theorem 2 we apply lemma 5.3 to produce a section $\sigma : M \dashrightarrow \mathbb{P}(E)$ generically transversal to $\mathcal{H}$. If the transversely projective structure of $\mathcal{F} = \sigma^*\mathcal{H}$ is non unique then lemma 5.4 implies that there exists an algebraic curve C, a rational map $\phi : \mathbb{P}(E) \dashrightarrow C \times \mathbb{P}^1$ and Riccati foliation on $C \times \mathbb{P}^1$ such that $\mathcal{H} = \phi^*\mathcal{R}$. Recall that we are assuming here that ρ is non-solvable.

As we saw in the proof of lemma 5.3 we have a lot of freedom when choosing σ. In particular we can suppose that $\phi \circ \sigma : M \dashrightarrow C \times \mathbb{P}^1$ is a dominant rational map. Thus $\mathcal{F}$ is the pull-back of Riccati foliation with non-solvable monodromy by a dominant rational map. The uniqueness of the transversely projective structure of $\mathcal{F}$ follows from [16, proposition 2.1].

This is sufficient to conclude the proof of Theorem 2 under the additional assumption on H: normal crossing with smooth ireducible componentes. Notice that up to this point everything works for projective manifolds of arbitrary dimension.

To conclude we have just to consider the case where H is an arbitrary curve on a projective surface S. We can proceed as in the proof of Theorem 1, i.e., if we denote by $p : (\tilde{S}, \tilde{H} = p^*H) \to (S, H)$ the desingularization of H then there exists $\tilde{\rho} : \pi_1(\tilde{S}, \tilde{H}) \to \mathrm{SL}(2, \mathbb{C})$ such that $\rho = p_*\tilde{\rho}$. Thus we apply the previous arguments over $\tilde{S}$ and go back to S using lemma 4.5. □

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