A new proof for maximal monotonicity of subdifferential operators

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Abstract

In this paper we present a new proof for maximal monotonicity of subdifferential operators. This result was proved by Rockafellar in [6] where other fundamental results were also proved. The proof presented here is simpler and makes use of classical results from subdifferential calculus as Brønsted-Rockafellar’s theorem and Fenchel duality formula.

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1 Introduction

Let $X$ be a real Banach space with dual $X^*$. A proper convex function on $X$ is a function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, not identically $+\infty$, such that

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

whenever $x \in X$, $y \in X$ and $0 < \lambda < 1$. The subdifferential of $f$ is the point-to-set operator $\partial f : X \rightrightarrows X^*$ defined at $x \in X$ by

$$\partial f(x) = \{u \in X^* \mid f(y) \geq f(x) + \langle y - x, u \rangle, \text{ for all } y \in X\},$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical duality product between $X$ and $X^*$. For each $x \in X$, the elements $u \in \partial f(x)$ are called subgradients of $f$ at $x$.

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A point-to-set operator $A : X \rightrightarrows X^*$ is said to be monotone if

$$\langle x - y, u - v \rangle \geq 0, \text{ whenever } u \in A(x), v \in A(y).$$

It is easy to check that $\partial f$ is monotone. The monotone operator $A$ is called maximal monotone if, in addition, its graph

$$G(A) = \{(x, u) \mid u \in A(x)\} \subset X \times X^*$$

is not properly contained in the graph of any other monotone operator $A' : X \rightrightarrows X^*$. This is equivalent to say that

$$\langle x - x_0, u - v_0 \rangle \geq 0, \text{ for all } (x, v) \in G(A) \Rightarrow (x_0, v_0) \in G(A).$$

Rockafellar proved in a fundamental work [6] that the subdifferential of a proper convex lower semicontinuous (l.s.c. from now on) function is maximal monotone. Beside this result, that paper contain other useful and interesting results (see Theorem 6.1 of [4] for an application). Simpler proofs of Rockafellar’s result were given in [5], [7], [1] and [8]. Our aim is to give (another) new and simple proof of the maximal monotonicity of the subdifferential.

For a proper convex function $f$, the Fenchel-Legendre conjugate of $f$ is the function $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(u) = \sup\{(x, u) - f(x) \mid x \in X\}.$$

If $f$ is also l.s.c., then $f^*$ is proper and from its definition, follows directly the Fenchel-Young inequality: for all $x \in X$, $u \in X^*$,

$$f(x) + f^*(u) \geq \langle x, u \rangle, \text{ with equality if and only if } u \in \partial f(x). \quad (1)$$

For instance, if we consider $f(x) = \frac{1}{2}\|x\|^2$, it is not difficult to see that $f^*(u) = \frac{1}{2}\|u\|^2$, where $\|\cdot\|$ denotes both norms of vectors spaces $X$ and $X^*$.

The concept of $\varepsilon$-subdifferential of a convex function $f$ was introduced by Brønsted and Rockafellar [3]. It is a point-to-set operator $\partial \varepsilon f : X \rightrightarrows X^*$ defined at each $x \in X$ as

$$\partial \varepsilon f(x) = \{u \in X^* \mid f(y) \geq f(x) + \langle y - x, u \rangle - \varepsilon, \text{ for all } y \in X\},$$

where $\varepsilon \geq 0$. Note that $\partial f = \partial_0 f$ and $\partial f(x) \subset \partial \varepsilon f(x)$, for all $\varepsilon \geq 0$. Using the conjugate function $f^*$ of $f$ it is easy to see that

$$u \in \partial \varepsilon f(x) \iff f^*(u) + f(x) \leq \langle x, u \rangle + \varepsilon. \quad (2)$$

The following fundamental theorem of Brønsted and Rockafellar [3], estimates how well $\partial \varepsilon f$ approximates $\partial f$.\[2\]
Theorem 1.1 If \( f \) is a l.s.c. proper convex function on \( X \) and \( u \in \partial f(x) \), for any \( \eta > 0 \), there exist vectors \( z \in X \) and \( w \in X^* \) such that \( \|z - x\| \leq \eta \), \( \|w - u\| \leq \varepsilon/\eta \) and \( w \in \partial f(z) \).

Next we present the classical Fenchel duality formula, which proof can be found in [2, page 11]

Theorem 1.2 Let us consider two proper and convex functions \( f \) and \( g \) such that \( f \) (or \( g \)) is continuous at a point \( \hat{x} \in X \) for which \( f(\hat{x}) < \infty \) and \( g(\hat{x}) < \infty \). Then, there exists \( u \in X^* \) such that

\[
\inf_{x \in X} \{f(x) + g(x)\} = \max_{u \in X^*} \{-f^*(u) - g^*(u)\}.
\]  (3)

These theorems above will be of fundamental importance in the proof of Theorem 2.1, which is presented in the next section.

2 Main result

In this section a new proof for maximal monotonicity of subdifferential of a l.s.c proper convex function is presented as a direct application of Theorems 1.1 and 1.2.

Theorem 2.1 If \( f \) is a l.s.c. proper convex function on \( X \), then \( \partial f \) is a maximal monotone operator from \( X \) to \( X^* \).

Proof. Let us suppose \((x_0, v_0) \in X \times X^* \) is such that

\[
\langle x - x_0, v - v_0 \rangle \geq 0
\]

holds true whenever \( v \in \partial f(x) \). We aim to prove that \( v_0 \in \partial f(x_0) \).

Define \( f_0 : X \rightarrow \mathbb{R} \cup \{+\infty\} \),

\[
f_0(x) = f(x + x_0) - \langle x, v_0 \rangle.
\]  (4)

Applying Theorem 1.2 to \( f_0 \) and \( g(x) = \frac{1}{2}\|x\|^2 \) we conclude that there exists \( u \in X^* \) such that

\[
\inf_{x \in X} \left\{ f_0(x) + \frac{1}{2}\|x\|^2 \right\} = -f_0^*(u) - \frac{1}{2}\|u\|^2.
\]

As \( f_0 \) is l.s.c., proper and convex, both sides on the above equation are finite. Therefore, reordering this equation we obtain

\[
\inf_{x \in X} \left\{ f_0(x) + \frac{1}{2}\|x\|^2 \right\} + f_0^*(u) + \frac{1}{2}\|u\|^2 = 0.
\]  (5)
In particular, there exists a (minimizing) sequence \( \{y_n\} \) such that
\[
\frac{1}{n^2} \geq f_0(y_n) + \frac{1}{2} \|y_n\|^2 + f_0^*(u) + \frac{1}{2} \|u\|^2 \\
\geq \langle u, y_n \rangle + \frac{1}{2} \|y_n\|^2 + \frac{1}{2} \|u\|^2 \\
\geq \frac{1}{2}(\|y_n\| - \|u\|)^2 \geq 0,
\]
where the second inequality follows from Fenchel-Young inequality. Using the above equation we obtain
\[
f_0(y_n) + f_0^*(u) - \langle u, y_n \rangle \leq 1/n^2.
\]
Hence, \( u \in \partial_{1/n^2} f_0(y_n) \) and by Theorem 1.1 it follows that there exist sequences \( \{z_n\} \) in \( X \) and \( \{w_n\} \) in \( X^* \) such that
\[
w_n \in \partial f_0(z_n), \quad \|w_n - u\| \leq 1/n \quad \text{and} \quad \|z_n - y_n\| \leq 1/n.
\]
Using the initial assumption, we also obtain
\[
\langle z_n, w_n \rangle \geq 0.
\]
Using (6) we obtain
\[
\|y_n\| \to \|u\|, \quad \langle y_n, u \rangle \to -\|u\|^2, \quad \text{as} \ n \to \infty,
\]
which, combined with (7) and (8) yields \( u = 0 \). Therefore, \( y_n \to 0 \). As \( f_0 \) is l.s.c., \( x = 0 \) minimizes \( f_0(x) + \frac{1}{2} \|x\|^2 \) and, using (5) we have
\[
f_0(0) + f_0^*(0) = 0.
\]
Therefore \( 0 \in \partial f_0(0) \), which is equivalent to \( v_0 \in \partial f(x_0) \). □

References


