IRREDUCIBLE COMPONENTS OF THE SPACE OF FOLIATIONS ASSOCIATED TO THE AFFINE LIE ALGEBRA

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Abstract. In this paper, we give the explicit construction of certain components of the space of holomorphic foliations of codimension one, in complex projective spaces. These components are associated to some algebraic representations of the affine Lie algebra $\text{Aff}(\mathbb{C})$. Some of them, the so-called exceptional or Klein-Lie components, are rigid, in the sense that all generic foliations in the component are equivalent (example 1 of §2.2). In particular, we obtain rigid foliations of all degrees. Some generalizations and open problems are given the end of §1.

\section{Introduction}

It is known that the space $F^{(\varrho)}(\mathbb{C}^{n+1})$ of singular holomorphic codimension one foliations of degree $\varrho \geq 0$ on $\mathbb{C}P(n)$, $n \geq 3$, can be considered as an algebraic subset of the space of 1-forms on $\mathbb{C}^{n+1}$ whose coefficients are homogeneous polynomials of degree $\varrho + 1$ (cf. [Ce-LN1], [Ce-LN3] and [CA]). Some of the irreducible components of this algebraic subset have been described; for example, the logarithmic components, which correspond to foliations defined by closed meromorphic 1-forms (cf. [CA]). Other components are the rational (cf. [Ce-LN1]) and the pull-back components (cf. [Ce-LN3]). For $\varrho = 0; 1; 2$ the complete decomposition of $F^{(\varrho)}(\mathbb{C}P(n))$ in irreducible components was obtained in [Ce-LN1].

In this paper, we present new components of $F^{(\varrho)}(\mathbb{C}P(n))$, $n \geq 3$, related with some special representations of the affine Lie algebra $\text{aff}(\mathbb{C}) := \{e_1; e_2; [e_1; e_2] = e_2g\}$ in the algebra of polynomial vector fields of an affine chart $\mathbb{C}^3 \supset \mathbb{C}P(3)$. These new components include as a particular case the "exceptional component" of $F^{(2)}(\mathbb{C}P(3))$, described in [Ce-LN1].

To obtain our result we follow three steps:

1. We construct families of foliations $F_P \supseteq F^{(\varrho)}(\mathbb{C}P(3))$, where $P$ denotes a discrete invariant, arising from representations of the affine algebra.

2. We find sufficient conditions in order to prove stability under deformations of some of these families, i.e. we prove that for certain values of $P$ the deformation of a generic foliation $F_{2}F_{P}$ is still a foliation in $F_{P}$.

3. We get codimension one foliations in $\mathbb{C}P(n)$, $n \geq 4$, by pull-back of the foliations just constructed, and prove that we also have irreducible components in $F^{(\varrho)}(\mathbb{C}P(n))$.

The description of the families in the first step can be geometrically described. To do that, we consider the so-called Klein-Lie curves. They are characterized by the fact of being the rational projective curves fixed by an infinite group of projective automorphism. In $\mathbb{C}P(3)$ such curves, up

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to an automorphism in $\text{PGL}(4; \mathbb{C})$, can be parameterized by $\tilde{i}(t : s) = (t^p : t^q s^p r : t^{p+1} s^p)$; where $1 \cdot r < q < p$ are positive integers with $\gcd(p; q; r) = 1$.

For each $\beta \neq 0$ such that $\gamma + r \notin \mathbb{N}$, we have a representation of the affine Lie algebra $\mathfrak{sl}_2 : \text{aff}(\mathbb{C}) \rightarrow X(\mathbb{C})$, determined by the two vector fields $s := \frac{1}{t} \frac{\partial}{\partial t}$; and $x := t^{\gamma + 1} \frac{\partial}{\partial t}$; Consider the linear semi-simple vector field on $\mathbb{C}^3$ 

$$S = px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y} + rz \frac{\partial}{\partial z};$$ 

Suppose that there is another polynomial vector field $X$ on $\mathbb{C}^3$ such that $[S; X] = S$, and so that $o_x s^\xi = \frac{1}{x} S^x o_x (t)^\xi$; $o_x x^\xi = X^x o_x (t)^\xi$; where $o_x (t) = (t^p; t^q; t^r)$ is the affine curve $i \in \mathbb{C}^3$. Then, the algebraic foliation $F = F(S; X)$ on $\mathbb{C}^3$, defined by the 1-form $\omega = i_X (dz_1 \wedge dz_2 \wedge dz_3)$ is associated to a representation of the affine algebra in the algebra of polynomial vector fields in $\mathbb{C}^3$, and it can be extended to a foliation on $\mathbb{CP}(3)$ of certain degree $\gamma$.

We give explicitly several examples in Section 2, all in the case $r = 1$. Note also that both $s$ and $x$ are complete vector fields on $\mathbb{C}$ just in case $\gamma = 1$. This is what happens in Example 1, where $S$ and $X$ are complete and the flow of $S$ is periodic: both necessary conditions for the existence of an action of the affine group on $\mathbb{C}^3$ associated to the foliation.

We define $F^{\gamma}(p; q; r); \gamma; \gamma := fF 2 F(\gamma; 3)jF = F(S; X)$ in some affine chart and we will show that they are irreducible subvarieties of $F(\gamma; 3)$. We also show that if $F 2 F^{\gamma}(p; q; r); \gamma; \gamma$ then the tangent sheaf $T_F$ is isomorphic to $O \otimes O(2 \gamma)$.

In order to carry on the second step, we will need some technical results. Let us first give some definitions.

Definition 1. Let $\omega$ be an integrable 1-form defined in a neighborhood of $p \in \mathbb{C}^3$. We say that $p$ is a generalized Kupka (briefly g.K.) singularity of $\omega$ if $\omega_p = 0$ and, either $\partial \omega_p \neq 0$, or $p$ is an isolated zero of $\partial \omega$.

The local structure of a foliation near a g.K. singularity is well known by now. When $\partial \omega_p \neq 0$ it is of Kupka type and it is locally the product of two foliations: a singular one in dimension two and a nonsingular one of dimension 1, as in Fig. 1 (cf. [K, Me]). When $p$ is an isolated singularity of $\partial \omega$, the singularity is quasi-homogeneous (cf. Theorem A and [LN1]) or logarithmic (cf. Remark 1 and [C{LN2}]).
We also prove that g.K. singularities are stable under deformations, (cf. [C-LN] and Proposition 1).

Definition 2. A codimension one holomorphic foliation $F$ in a complex three manifold $M$ is strongly generalized Kupka (briefly s.g.K.), if all the singularities of $F$ are g.K.

We will show, as a consequence of the stability of g.K. singularities, that s.g.K foliations are stable under deformations. In fact, we first note that the local structure of g.K. singularities implies that the analytic tangent sheaf of a s.g.K foliation is locally free. Using well-known results on holomorphic vector bundle theory (Theorem B), we can prove the following

Theorem 1. Suppose that $F^i(p; q; r); 0$ contains some s.g.K foliation. Then $F^i(p; q; r); 0$ is an irreducible component of $F(0; 3)$.

Theorem 1 and Example 1 in Section 2, give for any $0$, a new irreducible component of the space of foliations of degree $0$. This component is, in fact, the closure of a natural action of $PGL(4; C)$ on $F(0; 3)$. In particular, a foliation corresponding to a generic point in the component, is linearly stable. On the other hand, given $(p; q; r)$ positive integers such that $p > q > r$, the set $f(0; 0)g$ such that $F^i(p; q; r); 0$ contains some s.g.K foliation is finite (Theorem 3). This motivates the following problem:

Problem 1 Given three positive integers $p > q > r$, are there $(0; 0)$ such that $F^i(p; q; r); 0$ contains a s.g.K foliation?

The examples in §2.2 are s.g.K foliations in $CP(3)$, all of them belonging to some $F^i(p; q; r); 0$. Consequently, the tangent sheaf for these examples splits. This motivates the following questions:

Problem 2 Is it true that $T_F$ splits for any s.g.K foliation $F$ on $CP(3)$? More generally, let $F$ be a codimension one foliation on $CP(3)$ such that for any $p > 2 CP(3)$ the sheaf of germs of vector fields at $p$ tangent to $F$ is free with two generators. Does $T_F$ split?

We observe that all examples that we have of s.g.K. foliations on $CP(3)$ have at most two quasi-homogeneous singularities. A natural question is the following:

Problem 3. Are there s.g.K foliations on $CP(3)$ with more than two quasi-homogeneous singularities?

Finally, concerning the third step, in §3.2 we will consider foliations on $CP(n), n > 4$, which are pull-backs of s.g.K foliations on $CP(3)$ by a generic linear rational map $f: CP(n) \rightarrow CP(3)$. Denote by $F^i(p; q; r); 0, n^{\geq 2} F(0; n)$ the set of foliations so obtained from $F((p; q; r); 0), F^i(p; q; r); 0, n^{\geq 2} F(0; n) := F j F = f \circ G; G 2 F((p; q; r); 0, n^{\geq 2} F(0; n)$

We prove the following:

Theorem 2. Let $F$ be a foliation on $CP(n); n > 4$ and $i: CP(3)! CP(n)$ be a linear embedding of a 3-plane in general position with respect to $F$. Suppose that $G = i^*(F)$ is a s.g.K foliation in $F(0; 3)$ and that it is generated by two one-dimensional foliations on $CP(3)$. Then there exists a linear rational map $f: CP(n) \rightarrow CP(3)$ such that $F = f^*(G)$. In particular $F^i(p; q; r); 0, n^{\geq 2} F(0; n)$ is an irreducible component of $F(0; n)$.

§2 Preliminary results and examples

Notation. Throughout the paper, we will consider $(z_1 : z_2 : z_3 : z_4)$ as homogeneous coordinates in $CP(3)$. The basic affine open subsets, will be $E_1 = f(1: w: v: u)(u; v; w) 2 C^3 g; E_2 = f(r : 1: s : t)(r; s; t) 2 C^3 g; E_3 = f(r : s : 1: t)(r; s; t) 2 C^3 g$ and $E_0 = f(x: y : z : 1)(x; y; z) 2 C^3 g$. 
2.1 Generalized Kupka and quasi-homogeneous singularities. Let $p$, $q$, $r > 0$ be relatively prime integers and $S$ be the semi-simple vector field on $\mathbb{C}^3$ defined as in (1) by $S = px_x + qy_y + rz_z$. We say that a vector field $X$, holomorphic in a neighborhood of the origin, is $S$-quasi-homogeneous of weight $\lambda$, if we have the following Lie bracket identity: $[S; X] = \lambda X$. Remark that necessarily $\lambda + r$ is a non-negative integer and $X$ is a polynomial vector field. In fact, if $X = P_1 + P_2 + P_3$, then $X$ is $S$-quasi-homogeneous of weight $\lambda$ is equivalent to the fact that, after giving weights $p$, $q$ and $r$ to the variables $x$, $y$ and $z$, respectively, the polynomials $P_1$, $P_2$ and $P_3$ are weighted homogeneous of degrees $\lambda + p$, $\lambda + q$ and $\lambda + r$, respectively.

Moreover, $S$ and $X$ give a representation of the affine Lie algebra in the algebra of polynomial vector fields. If we suppose that $S$ and $X$ are linearly independent at generic points, then these vector fields generate an algebraic foliation on $\mathbb{C}^3$, which is given by the integrable 1-form $\omega = i_s i_x (dx \wedge dy \wedge dz)$. Since $\omega$ is a polynomial 1-form, this foliation can be extended to a singular foliation of $\mathbb{C}P(3)$, which will be denoted by $F(\cdot)$ or by $F(S; X)$. Observe that $S$ extends to a holomorphic vector field $X$ on $\mathbb{C}P(3)$ and that its trajectories are contained in the leaves of $F(\cdot)$. On the other hand, in general, the vector field $X$ is meromorphic in $\mathbb{C}P(3)$, but the foliation defined by it on $\mathbb{C}^3$ extends to a foliation on $\mathbb{C}P(3)$, which will be denoted by $G(X)$, whose leaves are also contained in the leaves of $F(\cdot)$. Remark that the singular set of $F(\cdot)$, denoted by $\text{sing}(F(\cdot))$, is invariant by the flow of $S$, $\exp(tS) := S_t$. This follows from the relation

$$\left(2\right) L_S(\cdot) = m_{\omega}; m = \lambda + \text{tr}(S) = \lambda + p + q + r;$$

as the reader can check. Relation (2) implies also that, if $p_0 \not\in \text{sing}(S)$, then $F(\cdot)$ is, in a neighborhood of $p_0$, equivalent to the product of a foliation in dimension two by a one-dimensional disk, like in g. 1. In fact, let $(U; (u; v; w))$ be a holomorphic coordinate system such that $S_U = 0$. Then, it is not difficult to see that, the integrability condition and (2) imply that

$$\omega - (u; v; w) = e^{\mu u} - (0; v; w) = e^{\mu u}(A(v; w)dv + B(v; w)dw);$$

which proves the assertion.

In the affine chart $\mathbb{C}^3 \cong \mathbb{C}P(3)$, where $S$ is like in (1), the leaves of $F(\cdot)$ are $S$-cones with vertex at $0 \in \mathbb{C}^3$, that is, immersed surfaces invariant by the flow of $S$. If $\text{sing}(F(\cdot))$ has codimension two, then each one of its components is the closure of an orbit of $S$. Now, we impose a condition which imposes the local stability of this kind of singularity by small perturbations of the form defining the foliation.

Let $\omega$ be an integrable 1-form in a neighborhood of $p_0 \in \mathbb{C}^3$ and $\omega$ be a holomorphic 3-form such that $i_p^1 \neq 0$. Then $\omega = i_Z(\omega)$, where $Z$ is a holomorphic vector field. It is not difficult to see that $p_0$ is a g.K. singularity of $\omega$ if, and only if, $p_0$ is an isolated singularity of $Z$.

Definition 3. We say that $p_0$ is a quasi-homogeneous (briefly q.h.) singularity of $\omega$ if $p_0$ is an isolated singularity of $Z$ and the germ of $Z$ at $p_0$ is nilpotent (as a derivation in the local ring of formal power series at $p_0$).

This definition is justified by the following result (cf. [LN]):

Theorem A. Let $p_0 \in \mathbb{C}^3$ be a quasi-homogeneous singularity of an integrable 1-form $\omega$. Then, there exist two holomorphic vector fields $S$ and $X$ and a local chart $(U; (x; y; z))$ around $p_0$ such that $x(p_0) = y(p_0) = z(p_0) = 0$ and:

(a) $\omega = i_S i_x (dx \wedge dy \wedge dz)$.
(b) $S = px_x + qy_y + rz_z$, where $p$, $q$ and $r$ are positive integers with $\gcd(p; q; r) = 1$.
(c) $p_0$ is an isolated singularity for $X$, $X$ is a polynomial in the chart $(U; (x; y; z))$ and $[S; X] = \omega$. Where $\omega$, $1$. 
Definition 4. Let $p_0 \in \mathbb{C}^3$ be a q.h. singularity of $!$. We say that it is of type $(p; q; r; \epsilon)$, if for some local chart and vector fields $S$ and $X$, then properties (a), (b) and (c) of Theorem A are satisfied.

Remark 1. If the singularity $p_0$ is g.K. but the germ of $Z$ at $p_0$ is semi-simple, then the foliation $F (!)$ can be de ned locally by an action of $C^2$. More precisely, there exists a germ of vector field $X$ at $p_0$ such that $[Z; X] = 0$ and

$$i_x i_Z (dx \wedge dy \wedge dz) = f :! ;$$

where $f (p_0) \neq 0$. This fact is a consequence of the results of [Ce-LN-2]. We call this type of singularity a logarithmic type singularity.

Remark 2. Let $p_0$ be a q.h. singularity of type $(p; q; r; \epsilon)$ of an integrable 1-form $!$. If $S$ and $X$ are as in Theorem A, then the multiplicity of $X$ at the singularity $p_0$ (the Milnor number) is given by

$$\mu = \frac{(\epsilon + p)(\epsilon + q)(\epsilon + r)}{pqr}.$$

In particular, $pqr$ must divide $(\epsilon + p)(\epsilon + q)(\epsilon + r)$. The proof of this fact can be found in [LN].

We can now state the stability result:

Proposition 1. Let $(-s)_{z \in B}$ be a holomorphic family of integrable 1-forms defined in a neighborhood of a compact ball $B = f z 2 \mathbb{C}^3$: $jz \cdot \mathbb{G}$, where $\mathbb{G}$ is a neighborhood of $0 \in \mathbb{C}^k$. Suppose that $B$ is a q.h. singularity of $-s$ of type $(p; q; r; \epsilon)$. There exists $s > 0$ such that if $0 < s < 2$, then $-s$ has a q.h. singularity $z(s)$ in $B$, of type $(p; q; r; \epsilon)$. Moreover, the function $s \mathbb{G} z(s)$ is holomorphic and $z(0) = 0$.

The arguments of the proof of Proposition 1 are contained in the proof of Lemma 6 of x4.3 of [Ce-LN-1]. We leave the details for the reader.

As a consequence of Proposition 1 and of Theorem 5 of [C-LN], we get the following:

Corollary. Let $F_0$ be a codimension one s.g.K. foliation on a compact complex threefold $M$. Then there exists a neighborhood $U$ of $F_0$ in the space of codimension one foliations, such that any $F \in U$ is s.g.K.

We use Theorem 5 of [C-LN] to guarantee the stability of the singularities of Kupka and logarithmic types.

Remark 3. If $p_0$ is a g.K. singularity of a foliation $F$, then the sheaf of germs of vector fields at $p_0$ tangent to $F$, is locally free and has two generators.

In fact, if $F$ is defined by $!$ in a neighborhood of $p_0$ and $d! = i_z^1$, where $i_z^1 \neq 0$, then the germ of $Z$ at $p_0$ has an isolated singularity at $p_0$. The integrability of $!$ implies that $i_z^1 = 0$, so that, by De Rham's division Theorem (cf. [DR] and [C-LN]), we can write $! = i_y^1$, where $y$ is a 2-form. Since we are in dimension three, we have $\mu = i_Y^1$, where $Y$ is a vector field. This implies that $! = i_y^1 = i_z^1$. Now, if $X$ is a germ of vector field such that $i_x^1 = 0$, we have $X = aY + bZ$ where $a$ and $b$ are holomorphic outside $\text{sing}(!)$. Since $\text{sing}(!)$ has codimension two, it follows from Hartog's Theorem that $a$ and $b$ can be extended to a neighborhood of $p_0$, which proves the assertion.

Remark 4. Let $p_0$ be an isolated singularity of a codimension one foliation $F$ on a threefold (for instance a Morse singularity). Then the sheaf of germs of vector fields at $p_0$ tangent to $F$ is not locally free. In fact, it follows from Malgrange's Theorem (cf. [M]), that $F$ has a local holomorphic first integral. This implies the assertion, as the reader can check (see also [LN-1]).
Remark 5. If $F$ is a s.g.K foliation on $M$, we can associate to $F$ a rank two vector bundle over $M$, the tangent bundle of $F$, which will be denoted by $T_F$, as follows. Take a covering $(U_\circ)_{\circ\in A}$ of $M$ by open sets such that for any $\circ\in A$ there are two holomorphic vector fields on $U$, say $X_\circ$ and $Y_\circ$, such that the sheaf of vector fields tangent to $F|_{U_\circ}$ is generated by these vector fields. If $U_\circ:=U_\circ\setminus U^\circ\setminus U^\circ;\setminus U$, then in $U_\circ$ we can write

$$
\frac{1}{2}X^\circ = a_\circ^\circ X_\circ + b_\circ^\circ Y_\circ,
$$
where the matrix $A_\circ$ is in $\text{SL}(2; \text{O}(U_\circ))$.

Clearly, $(A_\circ)_{\circ\in A}$ is a cocycle of matrices, that is, if $U_\circ:=(U_\circ\setminus U^\circ\setminus U;\setminus U^\circ),\setminus U$, then $A_\circ:A_\circ\circ\circ=A_\circ\circ\circ\circ=1$ on $U^\circ\circ\circ$.

Let $W$ be the disjoint union $\bigcup(U_\circ\setminus C^2)$ and $\equiv$ be the equivalence relation on $W$ denoted by

$$
U_\circ\setminus C^2\setminus 3(x_\circ;v_\circ)\equiv(x^-;v^-)2U^-\setminus C^2,\quad x_\circ\equiv x^- \equiv x 22U^\circ\setminus v_\circ\equiv v^-A_\circ(x);
$$

where in the above relation, we consider $v_\circ$ and $v^-$ as line vectors. We define $T_F=W=\equiv$ and $1/4T_F!$ $M$ by $1/4x;v_\circ=x$, where $\{x;v_\circ\}$ is the quotient class of $\{x;v_\circ\}$ in $W$. It is not difficult to prove that $T_F$ is a complex manifold and $T_F!\equiv M$ is a vector bundle.

We observe that to any holomorphic (resp. meromorphic) section of $T_F$ on some open set $U\setminus M$ corresponds an unique holomorphic (resp. meromorphic) vector field tangent to $F$. In fact, given a section $Z_\circ\circ\in U\setminus M$, we can write on $U\setminus U^\circ\setminus U;\setminus U^\circ$ $A\equiv C^2$.

De\nition 5. We say that a codimension one foliation $F$ on a complex threefold $M$ is generated by two foliations of dimension one, say $G_1$ and $G_2$, if for any $p\in M$ there exists a neighborhood $U$ of $p$ and holomorphic vector fields $X_1$ and $X_2$ on $U$ such that:

(a) $G_1$ is defined in $U$ by $X_1$, $x_1=1,2$.

(b) $F|_U$ is defined by the 1-form $i=ix_1ix_2$, where $i$ is a nonvanishing 3-form on $U$. In particular, we have that $G_1$ and $G_2$ are tangent to $F$.

(b.1) If $p\in M$ $\text{nsing}(G_1)\setminus \text{sing}(G_1)$ and $T_pG_1 6 T_pG_2 \setminus T_pM$, then $T_pF = T_pG_1 \odot T_pG_2$.

(b.2) $\text{sing}(F) = \text{sing}(G_1) \setminus \text{sing}(G_2)$ \setminus $D$, where

$$
D = fp\setminus M\setminus \text{sing}(G_1)\setminus \text{sing}(G_2)\setminus T_pG_1 = T_pG_2\setminus G_2;
$$

Proposition 2. Let $F$ be a s.g.K foliation on $M$ and $T_F$ be its tangent bundle. Then:

(a) To any line sub-bundle $L$ of $T_F$, corresponds a foliation by curves $G_1$ on $M$ with the following properties:

(a.1) $G_1$ is tangent to $F$.

(a.2) $\text{sing}(G_1) \setminus 1/2\text{sing}(F)$.

(b) $T_F$ splits as a sum of two line bundles if, and only if, $F$ is generated by two foliations of dimension one.

The proof of the proposition is straightforward and is left for the reader.

In the next section we will see some examples of s.g.K foliations on $\mathbb{CP}(3)$. In all examples the bundle $T_F$ splits. This has motivated problem 2 in x1.
x2.2 Examples. This section is devoted to describe some examples of strongly generalised Kupka foliations on CP(3). Each example will be generated by two foliations of dimension one, G1 and G2, in the sense of de\textsuperscript{e}notion 5. One of these one-dimensional foliations, say G1, will be generated by a global vector \( ^\circ \text{eld} \) S on CP(3), which in some an\textsuperscript{e} coordinate system \((x; y; z) \) \( C^3 \) \( \frac{1}{2} \) CP(3) is like in (1) \( : S = px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y} + rz \frac{\partial}{\partial z} \) where \( p; q; r \geq 2 \), \( g: cr(p; q; r) = 1 \) and \( p > q > r \). On the other hand, \( G2 \) will be of degree \( d \geq 1 \), so that the foliation will be of degree \( d \geq 1 \).

Being foliations in \( F(p; q; r; d + 1; 1) \), all the examples that we give share a geometrical pattern that we now explain. As the singular locus of the foliation is invariant by a global vector \( ^\circ \text{eld} \) in CP(3), it is globally \( ^\circ \text{xed} \) by an in\textsuperscript{f}inite group of projective automorphisms: the one given by the \( ^\circ \) ow of \( S \). Each curve in the singular locus has to be of a very special type.

Klein and Lie showed (see, e.g. \([C-LN-1]\)) that a curve \( CP(n) \) \( ^\circ \text{xed} \) by the action of an in\textsuperscript{f}inite group of projective automorphisms is rational algebraic. If it is of degree \( p \geq n \), it is obtained as an adequate linear projection of the rational normal curve \( i_p \frac{1}{2} CP(p) \), i.e. \( CP(1) \) embedded as \( i_p(s: t) \) := \( (t^p : t^{p-1} : \cdots : t^{p-1} : s^p) \). For \( n = 3 \), they showed that the projected curve could be written, after a change of coordinates, as (in the an\textsuperscript{e} open set \( E_4 \))

\[
_{p; q; r}^o(t) := (t^p; t^q; t^r)
\]

where \( p > q > r \) and 1 are positive integers. A curve so parametrized is \( ^\circ \text{xed} \) by the projective transformations \( x^\circ = @x \), \( y^\circ = @y \), \( z^\circ = @z \) that correspond to changing \( t \) by \( @t \), and \( ^\circ \text{x} \) the points \( A = (1; 0; 0; 0) \) and \( B = (0; 0; 0; 1) \). Finally, note that if the numbers \( p; q; r \) admit a greatest common divisor \( k > 1 \), then the curve (KL) is a degree \( \frac{k}{r} \) one, counted \( k \) times. One can in this case substitute the parameter \( t \) by a new parameter \( t^0 \).

Let us write \( i_{p; q; r} := \frac{1}{p+q+r} \frac{1}{2} CP(3). \) Observe that, when \( r = 1 \), \( i_{p; q; r} \) is smooth if and only if \( p = 3 \) (in this case it is the rational normal curve in \( CP(3) \)), and it has the point \( B \) as its only (cuspidal) singularity if \( p \geq 4 \). On the other hand, if \( r > 1 \), \( A \) is also a singular point of \( _{p; q; r}^o \).

Let us insist in the fact that not every cuspidal rational algebraic curve is a KL curve. In particular, not all the cuspidal rational curves with the same degree and number of cusps are projectively equivalent (see, e.g. \([E-H]\)).

Let \( t \) be the coordinate on \( C \), and consider the vector \( ^\circ \text{eld} \) on \( C \), \( t \frac{\partial}{\partial t} \). The vector \( ^\circ \text{eld} \) \( (_{p; q; r}^o)_a(t \frac{\partial}{\partial t}) \) can be extended to \( C^3 \) as \( : S = px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y} + rz \frac{\partial}{\partial z} \). On the other hand, \( (_{p; q; r}^o)_a(t \frac{\partial}{\partial t}) \), \( + r \), \( 0 \), can be extended as a polynomial vector \( ^\circ \text{eld} \) \( X \) which is \( S \)-quasi-homogeneous, if certain arithmetical relations hold among \( p; q; r \) and \( ^\circ \text{x} \). When \( r = 1 \), which is the case that we will consider in the examples, this extension can be done so that \( X \) is \( S \)-quasi-homogeneous of weight \( ^\circ \text{x} \). Thus we can de\textsuperscript{e}ne a foliation generated by the subfoliations given by \( S \) and \( X \), which will be of degree \( d \) if the foliation generated by \( X \) is of degree \( \frac{d}{r} \).

Example 1. Klein\textsuperscript{Lie} foliations with one quasi-homogeneous singularity. We give examples that extend one found in \([Ce-LN-1]\), giving origin to the so-called exceptional components. They appear in a family that we will denote as Klein\textsuperscript{Lie} (KL, for short) foliations in CP(3). KL foliations are not always s.g.K, but for each degree there is exactly one which is s.g.K, and that has just one q.h. singularity.

KL foliations in \( C^3 \) and actions of Aff (C). Recall that if \( t \) is the coordinate on \( C \), the two basic complete vector \( ^\circ \text{elds} \) on \( C \), that are the in\textsuperscript{f}nite\textsuperscript{singular} generators of the action of Aff (C), are \( t \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial t} \). As noted above, the vector \( ^\circ \text{elds} \) \( (_{p; q; r}^o)_a(t \frac{\partial}{\partial t}) \) and \( (_{p; q; r}^o)_a(\frac{\partial}{\partial t}) \), can be extended as

\[
S = px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y} + rz \frac{\partial}{\partial z}
\]
and
\[ X_\ell = p \sum_{i+j = p} \partial_{ij} z^{i}y^{j} + q^{\ell} z^{q_1} \sum_{i+j = p} \partial_{ij} \] where \( \ell \geq 1 \).

The vector fields \( S \) and \( X_\ell \) are complete, linearly independent outside the curve \( S_{p,q,l} \), and they satisfy the relation \( [S ; X_\ell] \equiv 0 \). Thus, they generate a local action of \( \text{Aff}(C) \). To define a foliation associated to it, we consider the polynomial 1-form \( \beta_{p,q,l} = i_\ell X_\ell dz \wedge dy \wedge dx \). i.e. the 1-form
\[ q(y \cdot z^{q_1} - 1)dx + p \sum_{i+j = p} \partial_{ij} z^{i+1}y^{j} + dy + pq \sum_{i+j = p} \partial_{ij} z^{i} \] The relation \( df \beta_{p,q,l} = (p + q)i_x dx \wedge dy \wedge dz \) implies that \( \beta_{p,q,l} \) is the Kupka set of the foliation represented by \( \beta_{p,q,l} \), and it has transversal type \( \gamma = i \text{ pvdu} + \text{qudv} \). Moreover, the di\text{e}omorphism
\[ \hat{A}_\ell (v; u; t) = v + p \sum_{i+j = p} \partial_{ij} z^{i} u^{j} s^i(u + s)^j ds; u + t^q; t \] which is the time \( t \) of the ow of the vector field \( X_\ell \), with initial condition \((v; u; 0)\), satisfies the relation \( \hat{A}_\ell (\beta_{p,q,l}) = i \text{ pvdu} + \text{qudv} \). Therefore, the foliation has a rational \( \ell \)rst integral
\[ H_\ell = \frac{(y \cdot z^{q_1})}{(x \cdot \hat{A}_\ell (z; y))^{q_1}} \]
where \( \hat{A}_\ell \) is a polynomial of degree \( p \) on the variable \( z \) and depending on the parameters \( \partial_{ij} \).

Now we study the extension to \( CP(3) \) of the foliations obtained above. It is given by the homogeneous 1-form \( \tau_{p,q,l} = i_\ell dz_1 + i_2 dz_2 + i_3 dz_3 + i_4 dz_4 \), obtained from \( \beta_{p,q,l} \). Note that, by means of the action of \( PGL(4; C) \) on \( \tau_{p,q,l} \), we get a family of foliations: we will refer to all of them as KL foliations in \( CP(3) \).

A natural question is, given an integer \( d \), is there a Kleinian foliations in \( CP(3) \) of degree \( d + 1 \)?

Note that the degree of the KL foliation defined by \( \tau_{p,q,l} \) is \( d + 1 = \text{max} f; i + j + 1 \) \( \partial_{ij} \neq 0 \).

Then we have
\[ !_1 = qz_{d_1}^2 z_{d_2}^1 z_{d_3}^1 z_{d_4}^1 \]
\[ !_2 = pqz_{d_1}^2 z_{d_2}^1 z_{d_3}^2 z_{d_4}^1 \]
\[ !_3 = pqz_{d_1}^2 z_{d_2}^1 z_{d_3}^2 z_{d_4}^2 \]
\[ !_4 = pq(z^1)z_{d_1}^2 z_{d_2}^1 z_{d_3}^2 z_{d_4}^2 \]

with \( 1 < q \cdot d + 1 < p \cdot qd + 1 \cdot d(d + 1) + 1 \), and one of the following possibilities holds:

1. \( q = d + 1 \), and \( i + j < d \), if \( \partial_{ij} \neq 0 \);
2. \( q = d + 1 \), and there is a unique pair \((i_0; j_0)\) with \( \partial_{i_0 j_1} \neq 0 \) and \( j_0 = d + 1 \i_0 \);
3. \( q < d \), and there is a unique pair \((i_0; j_0)\) with \( \partial_{i_0 j_1} \neq 0 \) and \( j_0 = d + 1 \).

Observe that the hyperplane \( f z_4 = 0 \) is invariant by the foliation defined by \( \tau_{p,q,l} \). Concerning its singular locus, it is the union of \( i_{p,q,l} \) and the set \( f z_4 = !_4(z_1; z_2; z_3; z_4) = 0 \) which, according to the possibilities discussed above, is:

1. \( f z_{d_1}^2 = z_4 = 0 \) \( f z_2 = z_4 = 0 \);
2. \( f z_{d_1} z_{d_2} = z_4 = 0 \) \( f z_3 = z_4 = 0 \);
3. \( f z_{d_1} z_{d_2} = z_4 = 0 \) \( f z_{d_1} z_{d_2} = z_4 = 0 \).
To study the foliation around the point \((1:0:0:0)\), we choose its affine open neighbourhood \(E_1\) and calculate the rotational of the form which represents the foliation \(\tilde{\pi}_{p,q,1} : = \frac{1}{\tilde{\pi}_{p,q,1}} E_1\)

\[
\tilde{\pi}_{p,q,1} = i \quad p(j, 1) X \quad i \quad j \quad w^{1+1}v^1 + (p_2 \quad q(v_2 + 1)u^{1+q}) \quad du \\
+ p(j, 1) \quad i \quad j \quad w^{1+1}v^1 + (p_2 \quad q(v_2 + 1)u^{1+q}) \quad dv \\
+ p(j, 1) \quad i \quad j \quad w^{1+1}v^1 + (p_2 \quad q(v_2 + 1)u^{1+q}) \quad dw:
\]

Its exterior derivative is \(d\tilde{\pi}_{p,q,1} = Q_{uw}(p;q;\ell) \quad du \quad ^w + Q_{vw}(p;q;\ell) \quad dv \quad ^v + Q_{uv}(p;q;\ell) \quad dv \quad ^w,\) where

\[
Q_{uw}(p;q;\ell) = q(p(d + 1) \quad i \quad j \quad w^{1+q}) \quad du \\
+ (p_2 \quad q(v_2 + 1)u^{1+q}) \quad dv \\
+ (p_2 \quad q(v_2 + 1)u^{1+q}) \quad dw;
\]

and the rotational is given by

\[
R_{\tilde{\pi}_{p,q,1}} = Q_{uw}(p;q;\ell) \quad \frac{\partial}{\partial u} + Q_{vw}(p;q;\ell) \quad \frac{\partial}{\partial v} + Q_{uv}(p;q;\ell) \quad \frac{\partial}{\partial w}.
\]

The only case in which the rotational above has isolated singularities is when \(q = d + 1\) and there is just one \(i \quad j \quad w\) different from zero (case 2), the one corresponding to \(i = 0\) and \(j = d\), which is 1. In that case, the K.L foliation is s.g.K. By changing to the affine coordinates \(E_2 = f(r : 1 : s : t)j(r; s; t) 2 \quad C^3\), and \(E_3 = f(r : s : 1 : t)j(r; s; t) 2 \quad C^3\), it can be shown that all points in \(CP(3)\) of \((1:0:0:0)\) are of K upka type and that \(\text{sing}(F)\) is the union of \(\pi_{p,q,1}\) with the two curves \(\frac{fz^2 + 1}{z} = \frac{q(s + 1)}{t} j(i; j) i o z^2 i o + 1\). We leave the details for the reader.

Recall that the foliation has a meromorphic first integral \(F\), which in the affine chart \(E_0\) can be written as

\[
F(x; y; z) = \frac{(y + z^q)^p}{(x + z^p h(y + z^q))^q}; \quad \text{where} \quad h(t) = \sum_{j = 0}^q x^j
\]

is the solution of \(\pi(t; 1)h(q) = \pi(t^d + h(t)).\)

In all the other cases, one can check that there is a one dimensional set of singular points on which the rotational vanish, so the corresponding K.L foliation is not s.g.K.

Finally, and motivated by the previous study, we now analyse when there is just one pair \((i; j)\) with \(j \neq 0\); that is, there is a unique determination of vector \(\pi d, X_\ell\), and of the form \(\pi_{\tilde{\pi}_{p,q,1}}\),

For this to be the case, certain relations must hold between \(p, q\) and the degree \(d + 1\):

1. if \(q = d + 1\) and \(d + 1\) divides \(p_i 1\), then \(i = 0\) and \(j = \frac{d+1}{d+1} - 1\).
2. if \(q < d + 1\) and \(p_i 1 = qd\), then \(i = 0\) and \(j = d\).

Example 2. Let us consider the curve \(\mathbb{C}^3, 2; 1\) and the extension of the vector \(\pi d\) \((\mathbb{C}^3, 2; 1)\) as \(S = 3x + 2y + z\) and the polynomial vector \(\pi d X = P + z^3 R\), where \(R = x + y + z\) is the radial vector \(\pi d\) on \(\mathbb{C}^3\), and \(P = P_1 + P_2 + P_3\), with

\[
\begin{align*}
& P_1(x; y; z) = ax^2 + byz + cy^3 \\
& P_2(x; y; z) = dxy + ax^2 + fyz \\
& P_3(x; y; z) = gxyz + hy^2 + iy^2
\end{align*}
\]
We consider this set of polynomials parametrized by \((a; b; c; d; e; f; g; h; i) \in \mathbb{C}^{9}\). It is not difficult to see that \([S; X] = 3X\), so \(X\) is a \(S\)-quasi homogeneous degree 3 polynomial vector field extending \((^o_{3;2;1})u(t^4_\theta)\). The foliations defined by \(S\) and \(X\) on \(\mathbb{C}P(3)\) generate a codimension one foliation of degree four on \(\mathbb{C}P(3)\), which will be denoted by \(F(P)\).

We take \(P\) in such a way that \(\partial i_P(\partial x ^\wedge \partial y ^\wedge \partial z)) = 0\), which is equivalent to \(\partial v(P) := P_{x} + P_{2y} + P_{3z} = 0\), or to \(2a + d + g = b + 2f + 2i = 0\). In this case, if \(P = i_S i_X (\partial x ^\wedge \partial y ^\wedge \partial z)\), then \(-P\) defines \(F(P)\) in the above chart \(E_0\). A straightforward calculation (using \(\partial v(P) = 0\)), gives \(\partial -P = i^2_{Z} (\partial x ^\wedge \partial y ^\wedge \partial z)\), where

\[
Z_P = 9P + z^3R_{ij} 6S.
\]

As the reader can check, the set

\[
A_0 = fP j2a + d + g = b + 2f + 2i = 0 \text{ and } Z_P \text{ has a nonisolated singularity at } 0 \text{ for } 2E_0 \cap \mathbb{C}^3g;
\]

is an algebraic subset of codimension three of \(\mathbb{C}^{9}\). Therefore, if \(P \not\in A_0\) then \(F(P)\) has a q.h. singularity at \(0\). Moreover, \(\text{sing}(F(P)) \setminus E_0\) contains seven integral curves of \(S\), say \(i_{j1}, \ldots, i_{j7}\), where \(i_{j6} = (y = z = 0)\), \(i_{j7} = (x = y = 0)\) and the others are generic trajectories of \(S\) of the form \(i_{j} = f(\theta^3: j^2) t j 2 C g, a; \theta j 6 0\).

Now, let us see how \(F_P\) looks like in the chart \(E_1 = f(1 : w : v : u) j(u; v; w) 2 C^3 g\). In this chart we have \(S = i_{S1}\), where

\[
(6) \quad S_1 = 3u \frac{\partial}{\partial u} + 2v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w}.
\]

Since \(X\) has a pole of order two at \((u = 0)\), the foliation \(F(P)\) is generated in this chart by \(S_1\) and \(X_1 := u^2X\). Observe that

\[
[S_1; X_1] = i_1 S[x^2] = i_1 S(x^2)X = X_1;
\]

This implies that \(X_1\) is of the same type of \(X\), that is \(X_1 = Q + mw^3R\), where \(Q = Q_1 \frac{\partial}{\partial x} + Q_2 \frac{\partial}{\partial y} + Q_3 \frac{\partial}{\partial z}\) and \(Q_1; Q_2; Q_3\) are as in (5) (by changing \(x\) \(u, y\) \(v, z\) \(w\) and the parameters \(a; \theta; i\)). In other words, the point \((1 : 0 : 0 : 0) 2 E_1\) is a q.h. singularity of \(F(P)\) for a generic \(P\). It is possible to verify, by taking other \(a\neq\) charts, that \(F(P)\) is a s.g.K foliation with two q.h. singularities, the points \(p_0 := (0 : 0 : 0 : 1) 2 E_0\) and \(p_1 := (1 : 0 : 0 : 0) 2 E_1\). Moreover, \(\text{sing}(F(P)) = \{7 = 0, 0, 1\} 2 E_0\), where \(i_0 = f(1 : w : v : u) 2 E_1\) and \(p_0 g = 0\) and the points in \(\text{sing}(F(P)) \cap \text{nf} p_0; p_1 g\) are of Kupka type. We leave the details for the reader.

Example 3. In this example we take again the curve \(^o_{3;2;1} \text{ and } S = 3x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\), as in the Example 2, and

\[
(7) \quad X = (ay^2 + bxz) \frac{\partial}{\partial x} + (cx + dyz) \frac{\partial}{\partial y} + (ey + fz^2) \frac{\partial}{\partial z};
\]

so that \([S; X] = X\).

The foliation generated by \(S\) and \(X\) on \(\mathbb{C}P(3)\) has degree three in this case. It is defined in the chart \(E_0\) by the form \(- = i_S i_X (\partial x ^\wedge \partial y ^\wedge \partial z)\). We will denote this foliation by \(F(S; X)\). If we take \(X\) in such a way that \(\partial v(X) = 0\), that is \(b + d + 2f = 0\), then \(- = i_{Z} (\partial x ^\wedge \partial y ^\wedge \partial z)\), where \(Z = 7X\). As the reader can verify, if we take \(X \not\in A\), where

\[
A = fX j X \text{ is as in (7) and } abcd ef (acf + bde) = 0 g;
\]
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then 0 2 E_0 \forall C^3 is an isolated zero of d, that is a q.h. singularity of F(S;X). For generic X \neq A, sin(F(S;X)) \setminus E_0 has three components: i_0 = (x = y = 0) and i_1, i_2, which are the closure of two trajectories of S, not contained in the coordinate planes.

If we change coordinates to the chart E_1 = f(1 : w : v : u)j(u;v;w) 2 C^3g, we find that F(S;X) is generated in E_1 by S = i_1 S_1, where S_1 is like in (6), and

\[ X_1 = uX = (bu v a u w^2) \otimes \frac{\partial}{\partial u} + (eu w + (f b v a u w^2) \otimes \frac{\partial}{\partial v} + (cu + (d b w a u w^3) \otimes \frac{\partial}{\partial w} : \]

Therefore, F(S;X) is represented in this chart by -1 = 1 S_1 i_1, (du ^ dv ^ dw). On the other hand, we have d - 1 = 1 Z_1 (du ^ dv ^ dw), where Z_1 = 8X_1 \div(X_1) ; S_1. As the reader can check, this implies that under generic assumptions on the coefficients a; b; c; d; e; f, the point 0 = p_1 2 E_1 is an isolated singularity of Z_1, so that it is a q.h. singularity of F(S;X). In this chart, the plane (u = 0) is invariant for F(S;X) and

\[ \text{sing}(F(S;X)) \setminus E_1 = \{ \text{i_1}\text{nfx} = 0\} \cup \{ \text{i_2}\text{nfx} = 0\} \cup \{ \text{i_3}\text{nfx} = 0\} \cup \{ \text{i_4}\text{nfx} = 0\} \cup \{ \text{i_5}\text{nfx} = 0\} \]

where i_3 = (u = v = 0), i_4 = (u = w = 0) and i_5 is a parabola in the plane (u = 0) of the form f(0; @^2; t) \setminus 2 C g.

We observe that the curves i_1, i_4 and i_5 meet at the point (0 : 0 : 1 : 0), which is a singularity of logarithmic type for F(S;X). It can be proved, by changing variables to other a±ne charts, that \text{sing}(F(S;X)) = \{ i_1 = 0 \} and all points in \text{sing}(F(S;X)) \text{nfx}(0 : 0 : 0 ; 1) ; (1 : 0 ; 0 ; 0) ; (0 : 1 : 0 ; 0) are of K upka type.

\[ x_2.3 \] Some remarks about the construction of the examples. In this section we discuss the possibility of constructing families of foliations s.g.K in CP(3), generated by two one-dimensional foliations, say G_1 and G_2, as in x2.2. We suppose that G_1 is the foliation defined in the a±ne chart E_0 = f(x : y : z : 1)j(x; y; z) 2 C^3g by the linear vector field S = px \otimes \frac{\partial}{\partial x} + qy \otimes \frac{\partial}{\partial y} + rz \otimes \frac{\partial}{\partial z} , where p, q, r 2 N, p, q, r > 0 and gcd(p, q, r) = 1. If p = q = r = 1, then it is possible to construct s.g.K foliations of any degree. Take a homogeneous vector field of degree d on E_0, say X, so that [S;X] = (d i_1)X. The foliation generated by S and X in CP(3) is defined on E_0 by the form - = 1 S_1 i_1 (dx ^ dy ^ dz). This type of example is considered in [C-LN] and for generic X it is s.g.K. On the other hand, in the case where the integers p, q and r are not equal, the situation is not so clear and we don't have a complete picture of all possibilities, if we x p, q, r. Nevertheless, in the case where p > q > r, the number of possible families of foliations is finite, as we will see.

Consider S as in (1) and p > q > r > 0. Let us suppose that there is a one-dimensional foliation G_2 of degree d, which in the chart E_0 is defined by a polynomial vector field X such that [S;X] = ^ X, where ^ > 0. We denote by F(S;X) the foliation on CP(3), which in the chart E_0 is generated by S and X. Observe that F(S;X) 2 F(p; q; r; d + 1; ^).

Theorem 3. If p > q > r > 0 are finite, then the set

\[ \mathcal{P} = f(d; ^) j d, 0`; 0` > 0 and F(p; q; r; d + 1; `)\]

contains a s.g.K foliation.

Proof. Observe that S has four singularities in CP(3), the points p_0 = (0 : 0 : 0 : 1) 2 E_0, p_1 = (1 : 0 : 0 : 0) 2 E_1, p_2 = (0 : 1 : 0 : 0) and p_3 = (0 : 0 : 1 : 0). The eigenvalues of S at these points are respectively (p; q; r), (i; p; q; i; p; r; i; q; i; p; i; r; q; i; r; i; r). Note that only in the rst two sets the eigenvalues have the same sign. As a consequence, the points p_2 and p_3 cannot be quasi-homogeneous singularities for a foliation F 2 F(p; q; r; d + 1; ^).
The idea is to use the formula for the multiplicity of an isolated singularity of a q.h. vector field in Remark 2. We will prove that the existence of a s.g.K foliation $F$ implies the existence of a one-dimensional foliation $G$ of degree $d$ with the following properties:

(i). $p_0$ and $p_1$ are isolated singularities of $G$.
(ii). $G$ is defined in the chart $E_0$ by a vector field $Y$ such that $[S; Y] = \lambda Y$.

Let us suppose the existence of $G$ satisfying properties (i) and (ii) and prove the theorem. Since $p_0$ is an isolated singularity for $Y$, it follows from Remark 2 that

$$G = \lambda Y,$$

Let us see how (10) implies the theorem. First of all we write (10) as a function of $\lambda$. In fact, suppose by contradiction that $\lambda = 0$. In this case, we get from Remark 2 that $q_1 = p_1$ and $r_1 = p_1 q$. We assert that $\lambda = 0$.

In fact, suppose by contradiction that $\lambda < 0$. Let $Y_1 = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C \frac{\partial}{\partial z}$. Since $p_1 = (0; 0; 0)$ is an isolated singularity of $G$, we must have $C \equiv 0$, so that there is a non-zero monomial of the form $u^p v^q w^r$ in $C$. Now, the relation $[S_1; Y_1] = \lambda Y_1$ implies that $S_1(C) = \lambda Y_1$. This contradiction implies that $\lambda = 0$.

In this case, we get from Remark 2 that

$$\lambda = \frac{\lambda}{p_1 q}.$$

But the above relation is not possible if $a, b, c, \lambda > 0$ and $p > q > r_1 > 1$. This contradiction implies that $\lambda = 0$.

In this case, we get from Remark 2 that

$$\lambda = \frac{\lambda}{p_1 q}.$$

Since $G$ has degree $d$, we must have (cf. [LN-S])

$$\lambda = \frac{\lambda}{p_1 q}.$$

Let us see how (10) implies the theorem. First of all we write (10) as a function of $\lambda$. Since $\lambda = p(d_1 - 1)$ we have

$$d_1 + d_1 = (d_1 - 1)^2 + 6(d_1 - 1) + 4 =$$

$$= \frac{\lambda}{p_1 q}.$$

Therefore, (10) is equivalent to $F(\lambda, \lambda) = 0$, where

$$F(\lambda, \lambda) = \lambda^2 q_1 r_1 (\lambda + p)(\lambda + q)(\lambda + r) + \lambda^2 q_1 r_1 (\lambda + p)(\lambda + q)(\lambda + r) + \lambda^2 q_1 r_1 G(\lambda, \lambda)$$
Note that $F(\cdot; \cdot')$ is a degree three polynomial in $(\cdot; \cdot')$ and its homogeneous term of degree three is

$$F_3(\cdot; \cdot') = p^2 q_i r_1 \cdot i + p^2 q r \cdot i q q r r_1 (\cdot + \cdot')^3;$$

Assertion. If $p > q > r > 0$, then there exists $C > 0$ (which depends only on $p; q; r$) such that $F_3(\cdot; \cdot')$, $C (\cdot + \cdot')^3$ if $\cdot; \cdot' > 0$. 

Proof. Suppose that $\cdot' > 0$, $\cdot > 0$ and set $\cdot = \cdot'.$. Then $F_3(\cdot; \cdot') = \frac{1}{3} F(y)$, where $F(y) = p^2 q_i r_1 y + p^2 q r \cdot i q q r r_1 (y + 1)^3$. Observe that $f(0) = q r (p^2 i q r_1) > 0$ and

$$1 + f(y) = p^2 q_i r_1 y^2 i q q r r_1 (y + 1)^2$$

so that $f(0) < 0$ and $f(y) = 0$ has an unique positive root $y_0 \leq \frac{p q r}{p + q r}$. As the reader can check, by calculating $f(0)$ and $f(\infty)$, the point $y_0$ is the positive minimum of $f(y)$. Since

$$f(y_0) = \frac{2 p^2 q r}{p + q r} \cdot i \frac{q + r}{2} \cdot p > 0;$$

we have $f(y), f(y_0) = \cdot > 0$ for all $y > 0$, so that $F_3(\cdot; \cdot') > \cdot^3$. Similarly, there exists $\cdot' > 0$ such that $F_3(\cdot; \cdot')$, $\cdot' > 0$ and $\cdot; \cdot' = 0$. It follows that

$$F_3(\cdot; \cdot') = \frac{1}{2} \cdot^3 + \frac{1}{2} \cdot' - \cdot', C (\cdot + \cdot')^3$$

for some $C > 0$ and $\cdot; \cdot'$. 

Now, since $F(\cdot; \cdot') \cdot F_3(\cdot; \cdot')$ is a degree two polynomial in $(\cdot; \cdot')$, there exists $\cdot > 0$ such that if $\cdot; \cdot' > 0$, $\cdot + \cdot'$, $\cdot > 0$ then $\cdot f(\cdot; \cdot') \cdot F_3(\cdot; \cdot') \cdot C (\cdot + \cdot')^3$, which implies that $F(\cdot; \cdot')$, $\cdot C (\cdot + \cdot')^3$ if $\cdot; \cdot' > 0$, $\cdot + \cdot'$. It follows that the number of pairs $(\cdot; \cdot') \cdot 2 N^2$ which are solutions of $F(\cdot; \cdot') \cdot 0$ is finite. Since $\cdot + \cdot' = p(q; 1)$, the number of pairs $(\cdot; \cdot') \cdot 2 N^2$ which are solutions of $(10)$ is also finite.

It remains to prove the existence of a foliation $G$ satisfying (i) and (ii). We will prove that there are two foliations $G_0$ and $G_1$ of degree $d$ such that:

(iii). $p$ is an isolated singularity of $G_1$, $j = 0$; 1.

(iv). $G_1$ is de- ned in the chart $E_j$ by a vector eld $z_j$ such that $[S_j; z_j] = \cdot j z_j$, where $S_0 = S$ and $\cdot = \cdot'$. 

If we have two foliations like above, then the generic foliation in the pencil $G = G_1 + \cdot G_1$ satis- es (i) and (ii), as the reader can check. Recall that $G_1$ is the foliation that in the chart $E_0$ is de- ned by $X_0 = X + \cdot x^{d_1} X_1$.

Let us construct $G_0$. Consider a foliation $F \cdot 2 F (p; q; r; d + 1; ').$ Then it has degree $d + 1$ and is de- ned in the chart $E_0$ by an integrable 1-form - such that $d - 1 = \cdot i x^2 (dx^2 dy^2 dz^2), p_0 = 0$ is an isolated singularity of $Z$ and $[S; Z] = \cdot Z$. Since $F$ has degree $d + 1$, the form - has degree $d + 2$, so that $d - 1$ dg(Z) - d + 1. If dg(Z) = d, then the foliation G(Z) on CP(3) de- ned in the chart $E_0$ by Z has degree d and we take $G_0 = G(Z)$. Let us suppose that dg(Z) = d + 1. In this case we must have d!v(Z) = 0, so that, if $Z_{d+1}$ is the homogeneous part of Z of degree d + 1, then d!v(Z_{d+1}) = 0 and [S; Z_{d+1}] = \cdot Z_{d+1}. As the reader can check, these relations imply that Z_{d+1} = g(mR_i nS), where R is the radial vector eld on $C^3$, m = \cdot p + q + r, n = d + 3 and g is a homogeneous polynomial of degree d such that S(g) = \cdot g. Let us write Z = P + g(mR_i nS), where dg(P) - d, $P = A @x + B @y + C @z$ and

$$Z = (A + (m_i n) x g) @x + (B + (m_i n) y g) @y + (C + (m_i n) z g) @z.$$
Observe that if \( \delta \) is small then 0 is an isolated singularity of \( Z + \), \( g \mathbb{R} \). Take \( \delta \) in such a way that \( m_1 \, n p + \); \( m_1 \, n q + \); \( m_1 \, n r + \). In this case, the vector \( \text{eld} \)

\[
X_0 = \frac{A}{m_1 \, n p + \} + g \frac{C}{@} + \frac{B}{m_1 \, n q + \} + g y \frac{C}{@} + \frac{A}{m_1 \, n r + \} + g z \frac{C}{@}
\]

has an isolated singularity at 0. Moreover, \([S;X_0] = \text{Horrock's splitting criterion, see }[O-S-S]\) and that the degree of \( F \) is \( \leq 1 \), \( 2 \) and \( 3 \) of \( x \). In fact, in this case if we set \( k = d; 1 \), \( 0 \), we have \( \sim \sim = 3k \) \( \sim \) and

\[
(11) \ F(\sim; 3k \ \sim) = 3[A(k)^2 \ i \ B(k) + C(k)];
\]

where \( A(k) = 3k + 4 \), \( B(k) = 12k + 9k^2 \) and \( C(k) = 7k^3 + 10k^2 \ i \ k \ i \ 4 \). On the other hand, the inequality \( F(\sim; 3k \ \sim) \cdot 0 \) implies that for a solution \( (k; \sim) \) we must have \( B^2 \ i \ 4AC \ i \ 0 \). Since

\[
B^2 \ i \ 4AC = i \ (k \ i \ 2)(k + 2)(k + 4)(3k + 4)
\]

we get that the unique possible solutions are \( k = 2 \) \( f0; 1; 2g \), that is \( d = 2 \) \( f1; 2; 3g \). If we substitute these values of \( k \) in (11) we get the following possibilities for \( \sim \)

\[
\begin{align*}
&8 \ k = 0 \Rightarrow \sim; 2 \ f0; 1g \\
&k = 1 \Rightarrow \sim; 2 \ f1; 2g \\
&k = 2 \Rightarrow \sim = 3
\end{align*}
\]

which give exactly the values of \( (d; \sim; \sim) \) of the examples.

The above result has motivated problem 1 in x.

x3 Proofs of Theorems 1 and 2

x3.1 Proof of Theorem 1. Let \( F \colon F(p; q; r; s; t) \) be a s.g.K foliation on \( \mathbb{C} \mathbb{P}(3) \). Observe that \( F \) is generated by two one-dimensional foliations of \( \mathbb{C} \mathbb{P}(3) \), say \( G_1 \) and \( G_2 \), the foliations de ned in the chart \( E_0 \) by the vector \( \text{elds} \) \( S \) and \( X \), respectively. As we have seen in Proposition 2, this implies that its tangent bundle \( T_F \) splits as the sum of two line bundles: \( T_F = L_1 \oplus L_2 \), where \( L_1 \) corresponds to the foliation \( G_1 \) and \( L_2 \) to \( G_2 \). Moreover, the Corollary of Proposition 1 implies that there exists a neighborhood \( U \) of \( F \) such that any foliation in \( U \) is s.g.K, so that its tangent bundle is well de ned.

Remark 7. Since \( S \) is a global vector \( \text{eld} \) in \( \mathbb{C} \mathbb{P}(3) \), we have that \( L_1 \) is a trivial line bundle, that is \( L_1 \colon \mathbb{C} \mathbb{P}(3) \mathcal{E} C = O(0) \). On the other hand, if \( d \) is the degree of \( G_2 \), we have \( L_2 \colon \mathcal{O}(1; d) \) (cf [Br]) and that the degree of \( F \) is \( \delta = d + 1 \).

Since \( F(d + 1; 3) \) is finite dimensional, it is su cient to prove that for any holomorphic curve \( \mathbb{C} \mathbb{T} F_t 2 F(d + 1; 3) \), such that \( 0 < \mathbb{C} \mathbb{T} F_0 = F \), then \( F_t 2 F(p; q; r; d + 1; \sim) \) for small \( \mathbb{T} \).

Let \( (F_t)_t \) be a holomorphic family of foliations on \( F(d + 1; 3) \), parametrized in an open set \( 0 < \mathbb{C} \mathbb{T} \) \( \mathbb{T} C \), where \( F_0 = F \). We take \( \mathbb{T} \) so small that for any \( t \) \( \mathbb{T} F_t \) is s.g.K and \( T_F \), is well de ned. Moreover, \( (T_F)_t \) is a holomorphic family of rank two vector bundles over \( \mathbb{C} \mathbb{P}(3) \). We will prove that \( T_F \) is isomorphic to \( T_F \) \( T_F \), if \( \mathbb{T} \) is small. To do that, we essentially use Theorem B. (Horrock's splitting criterion, see [O-S-S]) A holomorphic bundle \( E \) over \( \mathbb{C} \mathbb{P}(n) \) splits precisely when \( H^1(\mathbb{C} \mathbb{P}(n); E(k)) = 0 \) ; for \( i = 1; \ldots; n \) and \( \text{all} 2Z \).
Note that, as \( T_{F_0} \) splits, then \( H^1(CP(3); T_{F_0}(k)) = H^2(CP(3); T_{F_0}(k)) = 0 \) for every integer \( k \). But, as \( T_{F_t} \) is a holomorphic family of vector bundles over \( CP(3) \), the dimension of the vector spaces \( H^1(CP(3); T_{F_t}(k)) \) is upper semicontinuous. We conclude, by using again the splitting criterion above, that \( T_{F_t} \) splits for small \( t \).

In order to conclude that for small \( t \), it is \( T_{F_t} \) \( \neq T_{F_0} \), we make use of the well known fact (see, [S]) that the infinitesimal deformations of \( T_{F_0} = O \otimes O(1; d) \) are given by the vector space \( H^1(CP(3); \text{End} T_{F_0}) \), where \( \text{End} T_{F_0} \) is the sheaf of endomorphisms of \( T_{F_0} \). But, the dimension of that vector space is zero, as \( \text{End} T_{F_0} = T_{F_0}^2 - T_{F_0} \), where \( T_{F_0}^2 = O \otimes O(d; 1) \) is the dual bundle of \( T_{F_0} \).

Now, let \( (F_t)_{t \in \mathbb{R}} \) be a holomorphic family of foliations such that \( F = F_0 \otimes F (p; q; r; d + 1; \cdot) \) is s.g.k. It follows from Remark 7 and the results above that, if \( \mathfrak{s} \) is a small neighborhood of \( 0 \) in \( U \), then \( (F_t)_{t \in \mathfrak{s}} \) splits, \( O(0) \otimes O(1; d) \) for all \( t \in \mathfrak{s} \). On the other hand, \( (b) \) of Proposition 2, implies that \( F_t \) is generated by two foliations of dimension one, say \( G_t \) and \( G' \), then \( G_t(n) \) and \( G' \) correspond to the factor \( O(0) \) and \( G(t) \) to the factor \( O(1; d) \). As a consequence, \( G_t \) is generated by a global vector field \( \cdot \) on \( CP(3) \). Now, Proposition 1 of [2.1], implies that \( S(t) \) has a singularity whose eigenvalues, \( \lambda_1, \lambda_2, \lambda_3 \), are multiples of \( p, q, r \), so that we can suppose without loss of generality that \( \lambda_1 = p, \lambda_2 = q \) and \( \lambda_3 = r \). Consider an affine coordinate system \((u; v; w; z)\) where \( S(t) = px_{\omega} + qy_{\omega} + rz_{\omega} \). Let \(- t \) be a polynomial integrable 1-form which \( \cdot \) nishes \( F_t \) in this chart. We assert that

\[
(12) \quad L_{S(t)} - (t) = (\cdot + p + q + r) - (t).
\]

In fact, since \( G_t \) is tangent to \( F_t \), we have \( i_{S(t)} - (t) = 0 \). This implies that \( L_{S(t)} - (t) = i_{S(t)} \cdot (t) \). On the other hand, it follows from the integrability condition, \( - (t) \cdot (t) = 0 \), that \( - (t) \cdot (t) = 0 \), which implies that \( L_{S(t)} - (t) = (\cdot) - (t) \). Thus, \( \cdot : \mathbb{C}^3 \to \mathbb{C}^3 \) is holomorphic. Now, the eigenvalues of the operator \( V \) are integers, so that \( (\cdot) \) is a constant. Since \( (\cdot) - (0) = - i_{S(t)} \cdot (dx \wedge dy \wedge dz) \), where \( [S; X] = \cdot X \), we have \( L_{S(t)} - (t) = (\cdot + \text{tr}(S)) - (t) = (\cdot + p + q + r) - (t) \), which proves that \( (\cdot)(0) = (\cdot + p + q + r) - (t) \), and the assertion.

Now, let \( Z(t) \) be the vector field in \( \mathbb{C}^3 = U(t) \) defined by \( i_{Z(t)}(dx \wedge dy \wedge dz) = \cdot - (t) \). It follows from (12) that

\[
(13) \quad L_{S(t)} - (t) = (\cdot + p + q + r) - (t).
\]

This implies that \( F_t \) is a holomorphic family of foliations such that \( F \) \( \cdot \) nishes the proof of Theorem 1 as \( F(p; q; r; d + 1; \cdot) \) is an irreducible algebraic subset of \( F \). Indeed, recall from the description of the foliations in \( F(p; q; r; d + 1; \cdot) \) that in order to \( \cdot \) nish such a foliation we need choosing an affine open \( \mathbb{C}^3 \) of \( CP(3) \) (or equivalently a point in the dual projective space \( CP^N(3) \)), \( \cdot \) ring linear coordinates on it and choosing (up to multiplication by the same constant) the coe cient of the vector field \( \cdot \) on \( \mathbb{C}^3 \). This shows that there is a surjective map from a dense open subset \( U \) of \( CP^N(3) \) \& \( GL(3; C) \) \& \( \mathbb{C}^N \) onto \( F(p; q; r; d + 1; \cdot) \), for a certain \( N \). So the irreducibility of the last algebraic subset follows from that of \( U \).

Furthermore, to parametrize \( F(p; q; r; d + 1; \cdot) \), we should analyse the map above in order to detect which elements in \( F \) give rise to the same foliation. Note that for a \( \cdot \) field \( \cdot \) open, a linear change of coordinates of the form \( x^0 = \alpha x, y^0 = \beta y, z^0 = \gamma z \) takes \( S \) to \( S^0 = px^0\omega + qy^0\omega + rz^0\omega \) and \( X \to S \) quasi-homogeneous vector field \( \cdot \) on \( \mathbb{C}^3 \) of weight \( \cdot + 1 \). As the open \( \cdot \) algebraic the
coordinates \((x^0, y^0, z^0)\) and the vector \(\alpha\) elds \(S^0, X^0\) do not ne the same foliation, we should factor the group \(GL(3; C)\) by the subgroup of diagonal invertible matrices. 

For KL foliations we have the following result, extending the existence of the exceptional component in [Ce-LN1], that corresponds to the case \(d = 1\):

**Corollary.** Let \(d, 1\) be an integer. There is a 13-dimensional irreducible component

\[
F(d(d+1)+1; d+1; 1; d+1; 1)
\]

of the space \(F(d+1; 1)\) whose general point corresponds to a s.g.K Klein{Lie foliation with exactly one q.h. singularity. Moreover, this component is the closure of a \(PGL(4; C)\) orbit on \(F(d+1; 1)\).

**Proof.** It is an immediate consequence of Theorem 1, the study of KL foliations in Example 1, and the analysis of the parametrizations of the sets \(\text{Corollary }\).

Note that \(X\), the \(S\)-quasi-homogeneous vector \(\alpha\) eld of weight 0, is uniquely de ned up to the choice of the nonzero constants \(\beta\) and \(\gamma\) (we take the last coordinate, which is necessarily a constant, to be 1). The dependence locus of \(S\) and \(X\), which is the singular set of the foliation \(F\) in \(C^3\), is the Klein{Lie curve \((@d^d+1; d+1; t)\). After the linear change of coordinates given by \(x = @d^0, y = @y^0, z = @z^0\), the foliation in \(C^3\) is exactly the one described in Example 1, whose singular locus is the curve \(d^d+1; d+1; t\) = \((t^d+1); t\). The extended foliation in \(C^P(3)\) is s.g.K was studied in Example 1, it has just one q.h. singularity, an invariant hyperplane (that at in nity, \(C^P(3)\ nC^3\), and we also know its singular locus.

**x3.2 Proof of Theorem 2.** We observe that the second statement of the Theorem is a direct consequence of the rst and of Theorem 1, so that we will prove only the rst.

We will do the arguments in homogeneous coordinates. Let \(\mathcal{C} = C^{n+1}\ nC^g = C^n\) be the natural projection. Given a codimension one holomorphic foliation \(F\) on \(C^n\) of degree \(d\), then the foliation \(F^\alpha = \mathcal{C}(F, \alpha)\ nC^{n+1}\ nC^g\) extends to a foliation on \(C^{n+1}\), which can be de ned by a polynomial 1-form \(= \sum_{j=0}^{n} A_j(z) dz^j\) satifying the following properties (cf. [Ce-LN-1]):

(i) \(A_j\) is a homogeneous polynomial of degree \(d = 1\) for all \(j = 0; \ldots; n\). 
(ii) \(\sum_{j=0}^{n} A_j(z) dz^j = 0\) (integrability condition).
(iii) \(\mathcal{C}(\alpha) = \alpha\) and \(\mathcal{C}(\alpha^\beta) = \alpha^\beta\).
(v) If \(U_\alpha\) is the a±ne chart \((z_\alpha = 1)\) then \(F_jU_\alpha\) is de ned by \(- \alpha = - j\).

Moreover, if \(C^P(k) = E = C^n\), then \(C^P(k)\) is a linearly embedded k-plane, \(2 < k < n\), non-invariant for \(F\), where \(\mathcal{C}(E) = E^n\), then

Now, suppose that \(n = 3\) and that \(F\) is generated by two one dimensional foliations, say \(G^j_1\) of degree \(\gamma\), \(j = 1; 2\). We have the following:

**Lemma 1.** In the above hypothesis, let \(\gamma\) be as before. Then there exist polynomial vector \(\alpha\) elds \(X^j_1\) on \(C^4, j = 1; 2\), with the following properties:

(a) The components of \(X^j_1\) are homogeneous of degree \(\gamma\).
(b) The two-dimensional foliation \(\mathcal{C}(C^4, \alpha, \gamma)\), extends to \(C^4\) and is generated by \(X^j_1\) and the radial vector \(\alpha\) eld on \(C^4\) : 
\[
R = \sum_{j=0}^{3} \bar{z}_j @z^j_\bar{\alpha}.
\]
Proof. The existence of vector fields $X_j$, $j = 1; 2$, satisfying (a) and (b), is well known (cf. [LN-S]). Since $G_2$ and $G_2$ generate $F$, we must have $i_{X_j}X_j = 0$, $j = 1; 2$. We have also $i_R(\cdot) = 0$ (from (ii)). Let $E = i_R i_{X_1} i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$. It follows from Definition 5 and (b), that $\text{cod}(\text{sing}(E)) = 2$ and that for any $p \in \mathbb{C}^4$ we have $T_p(F^n) = \ker(E(p)) = \ker(- (p))$, where $T_p(F^n)$ denotes the tangent space to the leaf of $F^n$ through $p$. This implies that $E = i_R^*\text{sing}(E)$, where $\text{sing}(E)$ is some holomorphic function on $\mathbb{C}^4$. Since $\text{cod}(\text{sing}(E)) = 2$, $E$ extends to a holomorphic function on $\mathbb{C}^4$, which of course is a homogeneous polynomial. Now, it follows from $dg(G) = \alpha_1$, that $dg(F) = \alpha_1 + \alpha_2$, and so $dg(-) = \alpha_1 + \alpha_2 + 1 = dg(E)$. This implies that $\alpha_1$ is a constant. Now, if $X_1 = i^1 X_1$, then $- = i_R i_{X_1} i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$, which proves the Lemma.

We have the following consequences:

**Corollary 1.** Let $F$, $F^n$ and $\cdot = i_R i_{X_1} i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$ be as in Lemma 1. Then for any $p \in \mathbb{C}^4$ the sheaf of germs of holomorphic vector fields at $p$ which are tangent to $F^n$ is free and generated by the germs of $R_1$ and $X_2$ at $p$.

The proof is similar to the proof of Remark 3 of x2.1 and is left for the reader.

**Corollary 2.** Let $F$, $F^n$ and $\cdot = i_R i_{X_1} i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$ be as in Lemma 1. Let $(V_0)_{\otimes 2A}$ be a covering of $\mathbb{C}^4$ by Stein open sets and $(V_0)_{\otimes 2A}$ be an additive cocycle of holomorphic vector fields such that for any $V_0$, $X_0$ is tangent to $F^n$, that is $i_{X_j} = 0$. Then for any $A \in \otimes 2A$ there exists a holomorphic vector field $\cdot$ on $V_0$ such that $X_0$ is tangent to $F^n$ and $X_0 = X - i X_0$ on $V_\circ \setminus V = V_0 \setminus V$.

**Proof.** Let $X_1$ and $X_2$ be as in Lemma 1, so that $\cdot = i_R i_{X_1} i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$. It follows from Corollary 1 that if $V_0$, $X_0$; then there exist $f_{\otimes 2A} O(V_0)$, $j = 0; 1; 2$, such that

$$X_0 = f_{\otimes 2A} R + f_{\otimes 2A} X_1 + f_{\otimes 2A} X_2.$$ Clearly, $(f_{\otimes 2A})_{\otimes 2A}$ is an additive cocycle for $j = 0; 1; 2$. Since $H^1(\mathbb{C}^4; O) = 0$, there exist collections $(f_{\otimes 2A})_{\otimes 2A}$, where $f_{\otimes 2A} O(V_0)$, $j = 0; 1; 2$, such that $f_{\otimes 2A} = f_{\otimes 2A} R + f_{\otimes 2A} X_1 + f_{\otimes 2A} X_2$, then $X_0$ is tangent to $F^n$ and $X_0 = X - i X_0$.

Now, we consider the case in which $F_{\mathbb{C}^4}$ is s.g.K.

**Lemma 2.** Let $F$ be a codimension one foliation of degree $d$ on $\mathbb{C}P(n)$. Suppose that there exists a 3-plane $E$ like in (vi) before Lemma 1 and that $F_{\mathbb{C}^4}$ is s.g.K. Let $F^n$, $E^n$ and $\cdot$ be as before. Then, for any $p \in \mathbb{C}^n$ there exists a local coordinate system around $p$, say $(U; (t; u; \nu))$, where $t: U \to \mathbb{C}$, $\nu = (U_1; U_2; U_3): U \to C^3$ and $\nu = (v_1; \cdots; v_n): U \to C^n$. Suppose that $t(p) = 0$, $u(p) = 0$, $v(p) = 0$ and

(a) $E^n = (\nu = 0)$.
(b) $j_u = e^{d+2} j_{u_1} \otimes (u) du_1$.

In particular, $F^n j_u$ is locally equivalent to the product of a codimension one foliation on $\mathbb{C}^4$ by a non-singular foliation, say $P$, of dimension $n - 3$, which is given in this chart by $(t; u) = cte$.

**Proof.** The Lemma is a consequence of [K] and [C-LN]. First of all, observe that $L_{\mathbb{C}P(n)}(\cdot) = (d+2)$, because $\cdot$ is homogeneous of degree $d + 1$. This implies that

$$X_{\mathbb{C}P(n)}(\cdot) = e^{d+2}.$$ where $R_{\mathbb{C}P(n)}(\cdot) = e^{d+2}.$
be the hyperplane \((z_0 = 1)\) of \(C^{n+1}\). Since \(R\) is transversal to \(H\), there exists coordinate system \((t;x); V! D F C^n\), where \(V = fR_q(j)\) s 2 D; q 2 Hg such that \(R = \frac{\partial}{\partial t}\), \(H = (t = 0)\) and \(p = 0\), in this chart. It follows from (13) that

\[
(14) - (t;x) = e^{t(d+2)}j, \text{ where } j = \sum_{j=1}^{d} j(x) dx_j
\]

depends only on \(x = (x_1; \ldots; x_n)\). We can suppose also that \(E \setminus H = E^n \setminus H\) is the plane \(E_0 = (x_0 = \ldots = x_n = 0)\). Note that (v) and the hypothesis, imply that all singularities of \(! j_{E_0}\) are generalized Kupka. We have three possibilities:

(i). \( \cdot \ (p) = ! j_{E_0}(0) \neq 0\). In this case, we have \(! j_{E_0}(0) \neq 0\), that is \(F^n\) is transversal to \(E_0\) at 0. In fact, since \(! j_{E_0}(0) \neq 0\), \(F\) has a holomorphic \(^*\)rst integral in a neighborhood of 0, say \(f\), so that \(! = g df\), where \(g(0) \neq 0\). Now, \(! j_{E_0}(0) = 0\) implies that \(df j_{E_0}(0) = 0\), and so \(f j_{E_0}\) has an isolated singularity at 0, which is not possible (see Remark 4 of 2.1). As the reader can check, this implies the Lemma in this case.

(ii). \(! j_{E_0}(0) = 0\) and \(d! j_{E_0}(0) \neq 0\). In this case, 0 is a Kupka singularity of \(! j_{E_0}\) and of \(!\). The Lemma follows from the arguments in [K] or in [Me], in this case.

(iii). \(! j_{E_0}(0) = 0\), \(d! j_{E_0}(0) = 0\) and 0 is an isolated zero of \(d! j_{E_0}\). In this case, the Lemma follows from Theorem 4 of [C-LN].

Now, Lemma 2 implies that there exists an open covering \((U_0)_{\alpha \in A}\) of \(E^n\) so that with the following properties:

(vii). \(U_0 = V_0 \subseteq W_0\), where \(V_0\) is a Stein open subset of \(E^n\), and \(W_0\) is a polydisk in \(C^n; 3\).

(viii). \(F \cap U_0\) is the product of a codimension one foliation on \(V_0\) by a non-singular foliation \(P_{\alpha}\) of dimension \(n \geq 3\), transversal to \(E^n\).

We will suppose that \(E^n = (z_0 = \ldots = z_n = 0)\) and use the notation \(z = (x; y)\), where \(x = (x_1; \ldots; x_n) = (z_0; \ldots; z_3)\) and \(y = (y_1; \ldots; y_{n_1} 3) = (z_4; \ldots; z_n)\). Since \(P_{\alpha}\) is non-singular of dimension \(n \geq 3\) and transversal to \(E^n\), by taking a smaller \(U_0\) if necessary, we can suppose that it is generated by \(n \geq 3\) holomorphic vector \(^*\)elds, say \(Y_0 \subseteq \ldots; Y_{n1} 3\), of the form

\[
(15) \ Y_0(x; y) = \frac{\partial}{\partial y_j} + X_j(x; y), \text{ where } X_j(x; y) = \sum_{i=1}^{d} A_{ij}(x; y) \frac{\partial}{\partial x_i} \text{ and } A_{ij}(x; y) : 0(U_0) : \]

Lemma 3. For any \(j = 1; \ldots; n \geq 3\), there exists a constant vector \(^*\)eld \(Z_j\) on \(C^{n+1}\) of the form

\[
(16) \ Z_j = \frac{\partial}{\partial y_j} + \sum_{i=1}^{d} a_{ij} \frac{\partial}{\partial x_i}
\]

such that \(i_{Z_j} - (q) = 0\) for any \(q \in E^n\) and any \(j \geq 2\) \(f1; \ldots; n \geq 3\).

Proof. Fix \(j \geq 2\) \(f1; \ldots; n \geq 3\) and consider the covering \((U_0 = V_0 \subseteq W_0)_{\alpha \in A}\) and the vector \(^*\)elds \(Y_0\) as in (15). Consider the additive cocycle of vector \(^*\)elds \((X_{\alpha i})_{\alpha; i} \subseteq V_0\). on \(E^n\) so that \(X_{\alpha i}(x; y) = Y_0(x; y)\). Clearly, \(X_{\alpha i}\) is tangent to \(E^n j_{E_0}\) if \(V_0 \subseteq \theta\).

It follows from Corollary 2 of Lemma 1 that we can write \(X_{\alpha i} = T_{i} T_{\alpha}\), where \(T_{\alpha}\) is holomorphic on \(V_0\) and tangent to \(E^n j_{E_0}\). Since \(Y_0(x; 0) + T_{\alpha}(x) = Y_0(x; 0) + T_{\alpha}(x)\) on \(V_0\), \(T_{\alpha}\) has a holomorphic vector \(^*\)eld \(Z\) along \(E^n\) so that \(Z(x) = Y_0(x; 0) + T_{\alpha}(x)\) if \(x \in V_0\). It follows from Hartog's Theorem that we can extend \(Z\) to a vector \(^*\)eld on \(E^n\), which we shall denote by \(Z\) again. Let \(Z(x) = \sum_{k=0}^{\infty} Z^k(x)\) be Taylor series of \(Z\) at 0 2 \(E^n\), where \(Z^k(x)\) is a vector \(^*\)eld with
polynomial coefficients homogeneous of degree \( k \). Since \( Y_\partial \) is tangent to \( F \) and \( Z_\partial \) is tangent to \( F \) at \( v_\partial \), we have \( i_{Z_\partial} \mathcal{L} (q) = 0 \) for any \( q \in \mathbb{R}^n \). Now, since the coefficients of \( - \) are homogeneous of the same degree, we get that \( i_{Z_\partial} \mathcal{L} (q) = 0 \) for any \( q \in \mathbb{R}^n \). Finally, observe that \( Z_\partial \) is a constant vector field as in (16), which proves the lemma.

Let us finish the proof of the first part of Theorem 2. We will prove that there exists a linear change of variables on \( C^{n+1} \) of the form \((x; y) = L(u; v) = (u + b(v); v)\) such that

\[
\mathcal{L}^u(-) = X^i \quad \text{and} \quad j = 1 \rightarrow \text{for } j = 1 \rightarrow \text{for } j = 1 \rightarrow \text{for } j = 1 \rightarrow
\]

This clearly implies the first part of Theorem 2.

Let \( Z_j, j = 1; :::; n \) be as in (16). Consider the linear change of variables \((x; y) = L(u; v)\) as above, given by \( y = v \) and \( x_j = u_j + \sum_{i=1}^n a_{ij} v_i, j = 1; :::; 4 \). As the reader can check, we have \( L^u(Z_j) = \mathcal{O}_{ij} \) for all \( j = 1; :::; n \). Therefore, returning to the old notation, we can suppose that \( Z_j = \mathcal{O}_{ij} \).

**Assertion.** Let \((x; y) \in C^4 \mathcal{L} \mathbb{C}^{n+1} \) be a linear coordinate system such that \( E^u = \{y = 0\} \) and \( Z_j = \mathcal{O}_{ij}, j = 1; :::; n \). Then \(- = \sum_{j=1}^4 i_j (x) dx_j \) in this coordinate system.

**Proof.** Let us suppose first that \( n = 4 \), so that \( y \in C^4 \mathcal{L} \mathbb{C}^{n+1} \) and \( Z_1 = \mathcal{O}_{ij} \). Write

\[
- (x; y) = X^0 y^k - k(x)
\]

where \( \mathcal{O} \) is the degree of \(- \) and the coefficients of \(- k \) are homogeneous polynomials of degree \( \mathcal{O} \) in \( x \). We can write

\[
- k(x) = - \mathcal{O}^i_k (x) + f_k (x) dy, \quad \text{where } - \mathcal{O}^i_k (x) = X^i \quad \text{and } j = 1 \rightarrow \text{for } j = 1 \rightarrow \text{for } j = 1 \rightarrow \text{for } j = 1 \rightarrow
\]

and \( f_k, g_k \) are homogeneous polynomials of degree \( \mathcal{O} \) in \( x \). We want to prove that \(- = \mathcal{O} \). First of all, observe that \( f_0 = 0 \), because \( f_0 (x) = i_{Z_1} - (x; 0) = 0 \). Let us suppose by induction that \( - j = 0 \) for \( j = 1; :::; k \). Then \(- k = 0 \). In this case, we have

\[
- = - \mathcal{O}^1 + y^k (- \mathcal{O}^k + f_k dy) (\mod y^{k+1}) \quad \text{and } - = - \mathcal{O}^k + ky^k \mathcal{O} \quad \text{for } \mathcal{O}^k \quad \text{and } \mathcal{O} \quad \text{for } \mathcal{O}.
\]

so that, the integrability condition gives us

\[
0 = - \mathcal{O}^i - \mathcal{O}^j - \mathcal{O}^k + ky^k \mathcal{O} \quad \text{for } \mathcal{O}^k \quad \text{and } \mathcal{O} \quad \text{for } \mathcal{O}.
\]

Since \(- \mathcal{O}^0 = \mathcal{O} \), it is integrable; \(- \mathcal{O}^i \mathcal{O} = 0 \), and we get \(- \mathcal{O} \mathcal{O} \mathcal{O} = 0 \). But, the forms \(- \mathcal{O} \mathcal{O} \mathcal{O} \mathcal{O} \) do not contain terms in \( dy \), and so \(- \mathcal{O} \mathcal{O} \mathcal{O} = 0 \). This implies that \(- \mathcal{O} \mathcal{O} \mathcal{O} = 0 \), where \( \mathcal{O} \) is holomorphic, because \( \text{cod}(\text{sing}(- \mathcal{O})) = 2 \). On the other hand, the fact that the coefficients of \(- \mathcal{O} \) are homogeneous polynomials of degree \( \mathcal{O} \) in \( k \), while the coefficients of \(- \mathcal{O} \) are of degree \( \mathcal{O} > \mathcal{O} \) in \( k \), implies that \( \mathcal{O} = 0 \), and so \(- \mathcal{O} \mathcal{O} = 0 \).

Let us prove that \( f_k = 0 \). We will use the vector fields \( Y_\partial = \mathcal{O}_\partial + X_\partial \mathcal{L} \mathbb{T} \mathcal{A} \), as in (15). We can write for \((x; y) \in C^4 \mathcal{L} \mathbb{C}^{n+1} \) that

\[
Y_\partial (x; y) = Z_1 + \sum_{j=0}^k X_{\mathcal{O}^j} (x)
\]
where the vector fields $X_{\otimes i}$ contain only terms in $\delta_{i;3}$, $i = 1; \ldots; 4$. Since $i_{X_{\otimes i}} - 0 = 0$ and $i_{Z_1} - 0 = 0$, we get

$$0 \cdot i_{X_{\otimes i}}(x; y) = i_{Z_1} - (x; y) + \sum_{j=0}^{\infty} i_{X_{\otimes i}}(x) - (x; y) = y^k f_k(x) + \sum_{j=0}^{\infty} i_{X_{\otimes i}}(x) - (0)(x) \text{ (mod } y^{k+1})$$

as the reader can check. This implies that $i_{X_{\otimes i}} - 0 = 0$ for $j = 0; \ldots; k$; $k = 1$ and $f_k + i_{X_{\otimes i}} - 0 = 0$. For $V_{\otimes} \otimes \delta$; set $X_{\otimes i}(x) = X_{\otimes i}(x) \circ X_{\otimes k}(x)$. Clearly, $(X_{\otimes i})_{V_{\otimes}} \otimes \delta$ is an additive cocycle of vector fields. Moreover, $i_{X_{\otimes i}} - 0 = 0$, so that we can apply the Corollary 2 of Lemma 1 to obtain vector fields $T_{\otimes}$ on $V_{\otimes}$ such that $X_{\otimes i} = T_{\otimes} - i_{T_{\otimes}} \circ V_{\otimes} \otimes \delta$; and $i_{T_{\otimes}} - 0 = 0$ for all $\otimes \otimes$ $A$. This implies that there exists a vector field $X$ on $E^3$ for which $X_{\otimes i} = i_{X_{\otimes i} + T_{\otimes}}(x)$. By Hartog's Theorem $X$ can be extended to $E^3$. On the other hand, as the reader can check

$$(17) \ i_{X} - 0 = f_k$$

But, $f_k$ is homogeneous of degree $0_{i;3}$ and $- 0$ homogeneous of degree $\otimes 0_{i;3}$, so that $(17)$ implies that $f_k = 0$. This $- 0$ nishes the case $n = 4$.

The general case can be reduced to the above one by taking sections. In fact, since $i_{Z_1} - (x; 0) = 0$, $j = 1; \ldots; n$, we can write

$$- (x; y) = - 0(x) + \sum_{j=1}^{\infty} \left[ X_{\otimes j} f_j(x) + \sum_{i=1}^{\infty} y^{\otimes i} f_{ij}(x) dy_i \right]$$

where $\otimes (\otimes 3; \ldots; \otimes 3)$, $Y_{\otimes i} = Y_{\otimes i 3}$, $j_{\otimes} = \otimes 3 + \otimes 3 + \otimes 3 + \otimes 3 + \otimes 3$, $f_{\otimes i}$ and the coefficients of $- 0$ are homogeneous polynomials of degree $\otimes i;3$ and $- 0$ contains only terms in $dx_1; \ldots; dx_4$. Let $v = (v_1; \ldots; v_{\otimes 3})$ be a non-zero vector of $C^3_{\otimes 3}$ and consider the linear immersion $L: E^3 \subseteq C^4$ given by $L(x; w) = (x; w; v)$. We have

$$L^n(-) = - 0(x) + \sum_{k=1}^{\infty} \left[ X_{\otimes 3} f_3(x) + \sum_{l=1}^{\infty} v^{\otimes 3 l} f_{3l}(x) \otimes \otimes \right]$$

It follows from the case $n = 4$ that

$$X_{\otimes 3} f_3(x) = 0; 8 v 2 C^3_{\otimes 3}; 81 \cdot k \cdot 0 = 0; 8 3 \otimes 0 0$$

This implies that

$$- (x; y) = - 0(x) + \sum_{i;\otimes} y^{\otimes i} f_{\otimes i}(x) dy_i = \sum_{i;\otimes} y^{\otimes i} f_{\otimes i}(x) \otimes dy_i = \sum_{i;\otimes} y^{\otimes i} f_{\otimes i}(x) \otimes dy_i$$

Now, by using the integrability condition and collecting in $- \otimes d_\otimes = 0$ the coefficients of the terms containing only the factors $dx_i \otimes dx_j \otimes dy_i$, we get that

$$X_{\otimes 3} f_{\otimes 3} + f_{\otimes 3} d_\otimes = 0 \otimes dy_i = 0 \otimes dy_i = 0 \otimes dy_i = 0 \otimes dy_i = 0$$

$$X_{\otimes 3} f_{\otimes 3} - 0 = f_{\otimes 3} d_\otimes = 0; 8 i; \otimes 1 - \otimes i; 1 \cdot i; n \otimes 3$$
The last relation implies that, \( f_\frac{i}{i} = 0 \), for all \( i; \frac{j}{j} \). In fact, we have seen in the proof of Lemma 2 that \( L_R (\cdot \frac{0}{0}) = (\cdot + 1) \cdot \frac{0}{0} \), so that \( i_R (d \cdot \frac{0}{0}) = i_R (d \cdot \frac{0}{0}) + d(i_R \cdot \frac{0}{0}) = L_R (\cdot \frac{0}{0}) = (\cdot + 1) \cdot \frac{0}{0} \). Hence

\[
i_R (d \frac{i}{i} \wedge \cdot \frac{0}{0}) = i_R (f \frac{i}{i} d \cdot \frac{0}{0}) = (\cdot i \cdot j) f \frac{i}{i} = (\cdot + 1) f \frac{i}{i} = f \frac{i}{i} = 0;
\]

because \( f \frac{i}{i} \) is homogeneous of degree \( \cdot i \cdot j \). This finishes the proof of the assertion and of the Theorem.

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